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A MODIFICATION OF SUDAKOV'S LEMMA AND EFFICIENT SEQUENTIAL PLANS FOR THE ORNSTEIN-UHLENBECK PROCESS

0. Introduction. In papers [6] and [4] the authors have obtained a characterization of efficient sequential plans for some stochastic processes satisfying Sudakov's lemma (see [3], p. 55-59, and [5]). Here we prove a modification of Sudakov's lemma and give some properties of efficient sequential plans for the Ornstein-Uhlenbeck process.

1. Absolute continuity of measures generated by stopping time and some sufficient statistic.

(i) Let Ω be the space of vector-valued functions $\omega(t)$ for $t \geqslant 0$ which are right continuous. Moreover, we assume that:

$\mathscr{F}$ is the smallest σ-algebra of subsets of Ω with respect to which the functions $\omega(t)$ are measurable if $t \geqslant 0$;

$\mathscr{F}_t$ is the smallest σ-algebra of subsets of Ω with respect to which the functions $\omega(s)$ are measurable if $s \in [0, t]$;

μ_θ is a probability measure on $(\Omega, \mathscr{F})$ dependent on the real parameter $\theta \in [a, b]$.

Definition 1. A *Markov stopping time* is a random variable τ: $\Omega \to [0, \infty]$ which satisfies the following condition:

$$\{\omega\colon \tau(\omega) \leqslant t\} \in \mathscr{F}_t \quad \text{for all } t > 0.$$

(ii) Let $S(\omega, t)$ for every t be the mapping from Ω to R^l, measurable with respect to $\mathscr{F}_t$ and right continuous with respect to t, μ_θ-almost surely, for every $\theta \in [a, b]$.

LEMMA 1. *If* $\mu_\theta(\{\omega\colon 0 \leqslant \tau(\omega) < \infty\}) = 1$ *for all* $\theta \in [a, b]$, *then* $S(\omega, \tau(\omega))$ *is measurable with respect to the* σ*-algebra* $\mathscr{F}$.

For the proof see [3], p. 58.

(iii) By $\mu_{\theta,t}$ we denote the measure μ_θ defined on the σ-algebra $\mathscr{F}_t \subset \mathscr{F}$. Let us suppose that the measure $\mu_{\theta,}$ is absolutely continuous with respect to the measure $\mu_{\theta_0,t}$ and that the density function takes the form

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = g\big(t, S(\omega, t), \theta, \theta_0\big),$$

where g is a continuous function and S is a mapping satisfying (ii).

From the Fisher-Neyman theorem on factorization [1] we deduce that $S(\omega, t)$ is a sufficient statistic on the probability space $(\Omega, \mathscr{F}, \mu_{\theta,t})$ for every $t \geqslant 0$.

(iv) Let $U = [0, \infty)\times R^l = T\times R^l$, $U \ni u = \big(t(u), x(u)\big)$, where by $t(u)$ we denote the component of u which belongs to T, and by $x(u)$ – the component of u which belongs to R^l. Let $\mathscr{B}_U$ be the σ-algebra of Borel subsets of U.

On $(U, \mathscr{B}_U)$ we define, for every $A \in \mathscr{B}_U$, the measure m_θ generated by the statistic S and the stopping time τ:

$$m_\theta(A) = \mu_\theta\Big(\Big\{\omega\colon \big(\tau(\omega), S\big(\omega, \tau(\omega)\big)\big) \in A\Big\}\Big).$$

LEMMA 2. *Under assumptions* (i)-(iv) *the measure m_θ is absolutely continuous with respect to the measure m_{θ_0}, and the density function takes the form*

$$\frac{dm_\theta}{dm_{\theta_0}} = g(t, x, \theta, \theta_0), \quad t \in T,\ x \in R^l.$$

Proof. Let m_θ^n be the measure generated by the stopping time $\tau_n(\omega)$, where

$$\tau_n(\omega) = -\frac{1}{2^n}[-2^n\tau(\omega)],$$

and $[z]$ is the integer part of the number z. For every $\omega \in \Omega$ we have

$$\lim_{n\to\infty}\tau_n(\omega) = \tau(\omega) \quad \text{and} \quad \tau_n(\omega) \geqslant \tau(\omega) \quad \text{for each } n.$$

Let $Y_k^n = \{\omega\colon \tau_n(\omega) = t_k^n\} \in \mathscr{F}_{t_k^n}$, where $t_k^n = k/2^n$ $(k, n \in N)$ are the values of $\tau_n(\omega)$. We define the measure $m_{\theta,k}^n$ by

$$m_{\theta,k}^n(A) = m_\theta^n\big(A \cap \{(t, x) : t = t_k^n\}\big) \quad \text{for each } A \in \mathscr{B}_U.$$

We prove that $m_{\theta,k}^n$ is absolutely continuous with respect to $m_{\theta_0,k}^n$. Indeed,

$$\begin{aligned} m_{\theta,k}^n(A) &= m_\theta^n\big(A \cap \{(t, x) : t = t_k^n\}\big) \\ &= \mu_\theta\Big(\Big\{\omega\colon \big(\tau_n(\omega), S\big(\omega, \tau_n(\omega)\big)\big) \in A \cap \{(t, x)\colon t = t_k^n\}\Big) \\ &= \mu_\theta\Big(\big\{\omega\colon \big(t_k^n, S(\omega, t_k^n)\big) \in A\big\} \cap Y_k^n\Big) = \mu_\theta\big(\pi^{-1}(A) \cap Y_k^n\big), \end{aligned}$$

where $\pi\colon \Omega \to T\times R^l$ is defined by $\pi(\omega) = \big(t_k^n, S(\omega, t_k^n)\big)$. So we obtain

$$m_{\theta,k}^n(A) = \mu_{\theta,t_k^n}\big(\pi^{-1}(A)\cap Y_k^n\big) = \mu_{\theta,t_k^n|Y_k^n}\big(\pi^{-1}(A)\big).$$

By (iii) we have

$$\begin{aligned} m_{\theta,k}^n(A) &= \mu_{\theta,t_k^n|Y_k^n}\big(\pi^{-1}(A)\big) \\ &= \int_{\pi^{-1}(A)} g\big(t_k^n, S(\omega, t_k^n), \theta, \theta_0\big)\, d\mu_{\theta_0,t_k^n|Y_k^n} = \int_A g(t_k^n, x, \theta, \theta_0)\, dm_{\theta_0,k}^n. \end{aligned}$$

Hence the measure $m_{\theta,k}^n$ is absolutely continuous with respect to the measure $m_{\theta_0,k}^n$, and the density function takes the form

$$\frac{dm_{\theta,k}^n}{dm_{\theta_0,k}^n} = g(t_k^n, x, \theta, \theta_0).$$

Moreover, we can write

$$\begin{aligned} m_\theta^n(A) &= m_\theta^n\big(A\cap \bigcup_k \{(t,x): t = t_k^n\}\big) \\ &= \sum_k m_\theta^n\big(A\cap\{(t,x): t = t_k^n\}\big) = \sum_k m_{\theta,k}^n(A). \end{aligned}$$

Let $m_{\theta_0}^n(A) = 0$.

We know that $m_{\theta_0}^n(A) = 0$ if and only if $m_{\theta_0,k}^n(A) = 0$ for every k. By the absolute continuity of $m_{\theta,k}^n$ with respect to $m_{\theta_0,k}^n$ we have $m_{\theta,k}^n(A) = 0$ and $m_\theta^n(A) = 0$. Thus, the measure m_θ^n is absolutely continuous with respect to the measure $m_{\theta_0}^n$. Thus we have proved our lemma for the stopping time $\tau_n(\omega)$.

By (ii) and by the fact that

$$\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega) \quad \text{for each } \omega$$

we obtain the following conclusion:

$$\lim_{n\to\infty} S\big(\omega, \tau_n(\omega)\big) = S\big(\omega, \tau(\omega)\big)\ \mu_\theta\text{-almost surely}, \qquad \theta\in[a,b].$$

Let $p\colon U\to R$ be a continuous, bounded, real function defined as follows:

$$\int_U p(u)\, dm_\theta^n = \int_\Omega p\big(\tau_n(\omega)\big), S\big(\omega, \tau_n(\omega)\big)\big)\, d\mu_\theta.$$

By the Lebesgue theorem we have

$$\begin{aligned} \lim_{n\to\infty} \int_U p(u)\, dm_\theta^n &= \int_\Omega \lim_{n\to\infty} p\big(\tau_n(\omega), S\big(\omega, \tau_n(\omega)\big)\big)\, d\mu \\ &= \int_\Omega p\big(\tau(\omega), S\big(\omega, \tau(\omega)\big)\big)\, d\mu_\theta = \int_U p(u)\, dm_\theta. \end{aligned}$$

Thus the sequence m_θ^n is weakly convergent to the measure m_θ for each θ.

The thesis of our lemma is implied by the following lemma proved in [3], p. 55-59.

LEMMA 3. *Let m_n be a sequence of probability measures in R^{s+1}, weakly convergent to the measure m, and let $g(t, x)$ be a continuous non-negative function defined on R^{s+1}. Then the sequence m_n', having the density function $g(t, x)$ with respect to m_n, converges weakly to the measure m' with density $g(t, x)$ with respect to m.*

Remark 1. Denote by R^∞ the space of all real sequences with Tikhonov topology, and by $\mathscr{B}_{R^\infty}$ the σ-algebra of Borel subsets of R^∞. Let us assume, instead of (ii), that $S(\omega, t)$ is a mapping from Ω to R^∞ and that $(\mathscr{B}_{R^\infty}, \mathscr{F}_t)$ is measurable and right continuous with respect to t, μ_θ-almost surely. Then, in this case, Lemma 2 is valid.

2. Efficient sequential plans. Let $h(\theta)$ be a function of parameter θ and let $f: U \to R$, where f is $\mathscr{B}_U$-measurable. The function f is an estimator of $h(\theta)$.

Definition 2. A *sequential plan* is a pair (τ, f) containing the stopping time τ and the estimator f satisfying the following assumptions:

$$\mu_\theta(\{\omega: 0 \leqslant \tau(\omega) < \infty\}) = 1 \quad \text{for each } \theta \in [a, b],$$

$$E_\theta f = \int_U f(u) g(u, \theta, \theta_0) dm_{\theta_0}(u) = h(\theta),$$

$$E_\theta f^2 = \int_U f^2(u) g(u, \theta, \theta_0) dm_{\theta_0}(u) < \infty, \quad \theta \in [a, b].$$

(v) We also assume that

(a) the function $h(\theta)$ is differentiable and $h'(\theta) \neq 0$ on $[a, b]$;

(b) the function $g(t, x, \theta, \theta_0)$ satisfies some regularity conditions which guarantee that

$$0 < \int_U \left[\frac{\partial \ln g(t, x, \theta, \theta_0)}{\partial \theta}\right]^2 g(t, x, \theta, \theta_0) dm_{\theta_0}(u) < \infty,$$

$$\frac{\partial}{\partial \theta} \int_U f(u) g(u, \theta, \theta_0) dm_{\theta_0} = \int_U f(u) \frac{\partial \ln g(u, \theta, \theta_0)}{\partial \theta} g(u, \theta, \theta_0) dm_{\theta_0}.$$

Under assumptions (i)-(v) we can formulate the following theorem, the proof of which is analogous to that in [4].

THEOREM 1. *For each sequential plan (τ, f) satisfying the above-given assumptions the following inequality of Cramér-Rao type holds:*

(1) $$D_\theta^2 f \geqslant \frac{[h'(\theta)]^2}{\int\limits_U [\partial \ln g(u,\theta,\theta_0)/\partial\theta]^2 g(u,\theta,\theta_0)\,dm_{\theta_0}(u)}.$$

The equality in (1) *holds at a particular value of* θ *if and only if*

$$f(u)-h(\theta) = k(\theta)\,\frac{\partial \ln g\big(t(u), x(u), \theta, \theta_0\big)}{\partial\theta} \quad m_{\theta_0}\text{-almost surely}.$$

Definition 3. A sequential plan (τ, f) is called *efficient at a given value of the parameter* $\theta \in [a,b]$ if there exists an estimator f such that inequality (1) becomes an equality at θ.

Definition 4. A sequential plan (τ, f) is called *efficient* if there exists an estimator f such that inequality (1) becomes an equality for each $\theta \in [a,b]$. In this case the estimator f is *efficient* and the function $E_\theta f = h(\theta)$ is *efficiently estimable.*

3. Characterization of efficient sequential plans for the Ornstein-Uhlenbeck process. Let us consider the *Ornstein-Uhlenbeck process*, i.e. a diffusion process whose transition densities $p(y,t|x,s) = p(y,t-s|x)$, $t > s \geqslant 0$ satisfy the retrospective Kolmogorov equality

$$\frac{\partial}{\partial t}p(y,t|x) = \frac{1}{2}\,2\beta\sigma^2\,\frac{\partial^2 p(y,t|x)}{\partial x^2} - \beta(x-\theta)\frac{\partial p(y,t|x)}{\partial x}.$$

From this equality it follows that $p(y,t|x)$ is the density of the normal distribution with mean value $\theta + e^{-\beta t}(x-\theta)$ and variance $\sigma^2(1-e^{-2\beta t})$. We assume that almost all sample functions $\omega(t)$ of this process start from the point c, $c \in R$. We assume also that the parameters β and σ^2 are known ($\beta > 0$, $\sigma > 0$), but the parameter θ is unknown. Almost all sample functions of this process are continuous. Hence the process generates a measure μ_θ in the space of continuous functions.

Using Theorem 2 from [2], p. 606-608, we conclude that for every $t > 0$ the measure $\mu_{\theta,t}$ is absolutely continuous with respect to the measure $\mu_{\theta_0,t}$ for $\theta_0 = 0$, and

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = \lim_{n\to\infty}\prod_{k=0}^{n-1} q\big(t_{n,k}, \omega(t_{n,k}), t_{n,k+1}, \omega(t_{n,k+1})\big),$$

where $0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = t$ and the set $A_n = \{t_{n,k}: k = 0,1,\ldots,n-1\}$ is such that $A_n \subset A_{n+1}$ and $\bigcup_n A_n$ is a dense set in $[0,t]$. The function $q(s,x,t,y)$ is the density function of the measure $P_\theta(A,t|x,s)$ with respect to the measure $P_0(A,t|x,s)$. In our case

$$q(s,x,t,y) = \exp\left\{-\left[\frac{\theta^2(1-e^{-\beta(t-s)})^2 - 2\theta(y-xe^{-\beta(t-s)})(1-e^{-\beta(t-s)})}{2\sigma^2(1-e^{-2\beta(t-s)})}\right]\right\},$$

$$s < t.$$

Let $\exp\{-\beta(t_{n,k+1}-t_{n,k})\} = \varrho_{n,k}$. Then

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\log q\big(t_{n,k},\ \omega(t_{n,k}),\ t_{n,k+1},\ \omega(t_{n,k+1})\big)$$

$$= \lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{2\theta\big(\omega(t_{n,k+1})-\omega(t_{n,k})\varrho_{n,k}\big)(1-\varrho_{n,k})}{2\sigma^2(1-\varrho_{n,k}^2)} - \sum_{k=0}^{n-1}\frac{\theta^2(1-\varrho_{n,k})^2}{2\sigma^2(1-\varrho_{n,k}^2)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\theta}{\sigma^2}\sum_{k=0}^{n-1}\frac{\omega(t_{n,k})}{1+\varrho_{n,k-1}} - \frac{\theta}{\sigma^2}\sum_{k=1}^{n-1}\omega(t_{n,k})\frac{\varrho_{n,k}}{1+\varrho_{n,k}} + \right.$$

$$\left. + \frac{\theta}{\sigma^2}\frac{\omega(t_{n,n})}{1+\varrho_{n,n-1}} - \frac{\theta}{\sigma^2}\frac{\omega(t_{n,0})\varrho_{n,0}}{1+\varrho_{n,0}} - \frac{\theta^2}{2\sigma^2}\sum_{k=0}^{n-1}\frac{1-\varrho_{n,k}}{1+\varrho_{n,k}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\theta}{\sigma^2}\sum_{k=1}^{n-1}\omega(t_{n,k})\left(\frac{\beta}{4}(t_{n,k+1}-t_{n,k}) + \frac{\beta}{4}(t_{n,k}-t_{n,k-1}) + \right.\right.$$

$$\left. + o(t_{n,k+1}-t_{n,k}) + o(t_{n,k}-t_{n,k-1})\right) + \frac{\theta}{\sigma^2}\frac{\omega(t)}{1+\varrho_{n,n-1}} -$$

$$\left. - \frac{\theta}{\sigma^2}c\,\frac{\varrho_{n,0}}{1+\varrho_{n,0}} - \frac{\theta^2}{2\sigma^2}\frac{\beta}{2}\sum_{k=0}^{n-1}\big((t_{n,k+1}-t_{n,k}) + o(t_{n,k+1}-t_{n,k})\big)\right]$$

$$= \frac{\beta\theta}{2\sigma^2}\int_0^t \omega(s)\,ds + \frac{\theta}{2\sigma^2}\omega(t) - \frac{\theta}{2\sigma^2}c - \frac{\beta\theta^2}{4\sigma^2}t$$

$$= \frac{\theta}{2\sigma^2}\left[\left(\omega(t)+\beta\int_0^t \omega(s)\,ds\right) - \left(c+\frac{\beta\theta}{2}t\right)\right].$$

Thus

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = \exp\left\{\frac{\theta}{2\sigma^2}\left[\left(\omega(t)+\beta\int_0^t\omega(s)\,ds\right) - \left(c+\frac{\beta\theta}{2}t\right)\right]\right\}.$$

Let

$$S(\omega,t) = \omega(t)+\beta\int_0^t \omega(s)\,ds.$$

From Lemma 2 we infer that the measure m_θ is absolutely continuous with respect to the measure m_{θ_0}, and the density function takes the form

$$\frac{dm_\theta}{dm_{\theta_0}} = g\big(t(u), x(u),\ \theta,\ \theta_0\big) = \exp\left\{\frac{\theta}{2\sigma^2}\left[x(u) - \left(c+\frac{\beta\theta}{2}t(u)\right)\right]\right\},$$

$$\frac{\partial \log g\big(t(u), x(u),\ \theta,\ \theta_0\big)}{\partial\theta} = \frac{1}{2\sigma^2}\,[x(u)-c-\beta\theta t(u)].$$

For a given sequential plan (τ, f) satisfying the additional assumptions (v), inequality (1) takes the form

$$\mathrm{D}_\theta^2 f \geqslant \frac{4\sigma^4[h'(\theta)]^2}{\int\limits_U [x(u)-\beta\theta t(u)-c]^2 dm_\theta(u)} = \frac{4\sigma^4[h'(\theta)]^2}{\mathrm{E}_\theta[S_\tau-\beta\theta\tau-c]^2}. \tag{2}$$

The estimator f is efficient for $h(\theta)$ at the point θ if and only if

$$f(u) = k(\theta)[x(u)-\beta\theta t(u)-c]+h(\theta) \quad m_0\text{-almost surely}, \tag{3}$$

where $k(\theta) \neq 0$.

Let $\varphi(u, \theta)$ be a function defined on $U\times[a, b]$. We assume that φ is $\mathscr{B}_U$-measurable and m_θ-integrable at every $\theta\in[a, b]$. Moreover, we suppose that

$$\frac{\partial}{\partial\theta}\int\limits_U \varphi(u, \theta)m_\theta(du) = \frac{\partial}{\partial\theta}\int\limits_U \varphi(u, \theta)g(u, \theta)m_0(du)$$
$$= \int\limits_U \frac{\partial}{\partial\theta}[\varphi(u, \theta)g(u, \theta)]m_0(du).$$

Then, after differentiating the function under the integral sign in the formula

$$\mathrm{E}_\theta\varphi(\tau, S_\tau, \theta) = \int\limits_U \varphi(u, \theta)\exp\left\{\frac{\theta}{2\sigma^2}\left[x(u)-\left(c+\frac{\beta\theta}{2}t(u)\right)\right]\right\}m_0(du)$$

with respect to the parameter θ, we obtain the following equation:

$$\frac{1}{2\sigma^2}\mathrm{E}_\theta[S_\tau-\beta\theta\tau-c]\varphi(\tau, S_\tau, \theta) = \mathrm{E}'_\theta\varphi(\tau, S_\tau, \theta)-\mathrm{E}_\theta\frac{\partial}{\partial\theta}\varphi(\tau, S_\tau, \theta). \tag{4}$$

If we put $\varphi(\tau, S_\tau, \theta) = 1$ in formula (4), we obtain the first Wald identity

$$\mathrm{E}_\theta(S_\tau-\beta\theta\tau-c) = 0, \tag{5}$$

whence

$$\mathrm{E}_\theta S_\tau = \beta\theta\mathrm{E}_\theta\tau+c. \tag{6}$$

If we put

$$\varphi(\tau, S_\tau, \theta) = \frac{1}{2\sigma^2}[S_\tau-\beta\theta\tau-c],$$

then from (4) we get the second Wald identity

$$\mathrm{E}_\theta[S_\tau-\beta\theta\tau-c]^2 = 2\beta\sigma^2\mathrm{E}_\theta\tau. \tag{7}$$

Now, let $\varphi(\tau, S_\tau, \theta) = \tau$. Then we have

$$\mathrm{E}_\theta\tau[S_\tau-\beta\theta\tau-c] = 2\sigma^2\mathrm{E}'_\theta\tau, \tag{8}$$

whence

$$\mathrm{E}_\theta\tau S_\tau = \beta\theta\mathrm{E}_\theta\tau^2+c\mathrm{E}_\theta\tau+2\sigma^2\mathrm{E}'_\theta\tau. \tag{9}$$

Taking into account equalities (5), (7) and (8), we obtain

$$\mathrm{E}_\theta S_\tau^2 = \beta^2\theta^2\mathrm{E}_\theta\tau^2+2\beta\theta c\mathrm{E}_\theta\tau+2\beta\sigma^2 c\mathrm{E}_\theta\tau+4\beta\theta\sigma^2\mathrm{E}'_\theta\tau+c^2$$

and

$$\mathrm{D}_\theta^2 S_\tau = \beta^2\theta^2\mathrm{D}_\theta^2\tau+2\beta\sigma^2\mathrm{E}_\theta\tau+4\beta\theta\sigma^2\mathrm{E}'_\theta\tau. \tag{10}$$

With further restrictions on τ and S_τ one can obtain the Wald identities of higher order and other equations connecting moments of τ and S_τ.

Remark 2. Using (7) we can write inequality (2) in the form

$$\mathrm{D}_\theta^2 f \geqslant \frac{2\sigma^2[h'(\theta)]^2}{\beta\mathrm{E}_\theta\tau}.$$

Now we prove the following

THEOREM 2. *If the sequential plan* (τ, f) *is efficient, then there exist constants* a_1, a_2, a_3 $(a_1^2+a_2^2\neq 0)$ *for which*

$$a_1x(u)+a_2t(u)+a_3 = 0 \quad m_0\text{-almost surely}. \tag{11}$$

Proof. The estimator f is efficient at the points θ_1 and θ_2 $(\theta_1\neq\theta_2)$. Hence

$$\left.\begin{aligned} f(u) &= k(\theta_1)[x(u)-\beta\theta_1 t(u)-c]+h(\theta_1),\\ f(u) &= k(\theta_2)[x(u)-\beta\theta_2 t(u)-c]+h(\theta_2),\end{aligned}\right\} m_0\text{-almost surely},$$

where $k(\theta_1)\neq 0$ and $k(\theta_2)\neq 0$. We subtract one equality from the other and obtain

$$\big(k(\theta_1)-k(\theta_2)\big)x(u)+\beta\big(k(\theta_2)\theta_2-k(\theta_1)\theta_1\big)t(u)+$$
$$+c\big(k(\theta_2)-k(\theta_1)\big)+h(\theta_1)-h(\theta_2) = 0 \quad m_0\text{-almost surely},$$

which completes the proof.

Let $\big(\tau, f(\tau, S_\tau)\big)$ be an efficient plan at the point θ_1. Then

$$f(\tau, S_\tau) = k(\theta_1)[S_\tau-\beta\theta_1\tau-c]+h(\theta_1),$$
$$\mathrm{E}_\theta f(\tau, S_\tau) = k(\theta_1)\mathrm{E}_\theta S_\tau-\beta\theta_1 k(\theta_1)\mathrm{E}_\theta\tau-k(\theta_1)c+h(\theta_1),$$
$$h(\theta) = k(\theta_1)\beta\theta\mathrm{E}_\theta\tau+k(\theta_1)c-k(\theta_1)\beta\theta_1\mathrm{E}_\theta\tau-k(\theta_1)c+h(\theta_1).$$

So, if the function $h(\theta)$ is efficiently estimable at the point θ_1, then there exists some constant k_1 for which

$$h(\theta) = k_1\beta(\theta-\theta_1)\mathrm{E}_\theta\tau + h(\theta_1).$$

Let us suppose that the function $h(\theta)$ is efficiently estimable at different points θ_1 and θ_2. Then we have

$$h(\theta) = k_1\beta(\theta-\theta_1)\mathrm{E}_\theta\tau + h(\theta_1),$$
$$h(\theta) = k_2\beta(\theta-\theta_2)\mathrm{E}_\theta\tau + h(\theta_2).$$

Hence, if $h(\theta)$ is an efficiently estimable function, then there exist constants a_1, a_2, b_1, b_2 for which

$$h(\theta) = \frac{a_1\theta + a_2}{b_1\theta + b_2}.$$

Definition 5. A sequential plan (τ, f), where τ is equal, with probability 1, to a constant $T > 0$, is called a *fixed-time plan.*

Definition 6. A sequential plan (τ, f), where τ is equal, with probability 1, to the first attaining time of the line $x(u) = x_0$, is called an *inverse plan.*

Definition 7. A sequential plan (τ, f), where τ is equal, with probability 1, to the first attaining time of the line $x(u) = at(u) + s$ $(a \neq 0,\ s \neq 0)$ is an *oblique plan.*

For a fixed-time plan (τ, f) we have

$$\mathrm{E}_\theta\tau = T \quad \text{and} \quad \mathrm{D}_\theta^2\tau = 0. \tag{12}$$

Let the estimator $f(T, S_T)$ be efficient in this plan. Then it is efficient at some value θ_1 and, by (3), we have

$$f(T, S_T) = k(\theta_1)(S_T - \beta\theta_1 T - c) + h(\theta_1).$$

So, if the estimator $f(T, S_T)$ is efficient, then there exist constants c_1 and c_2 for which

$$f(T, S_T) = c_1 S_T + c_2.$$

By (6) and (12) we conclude that

$$h(\theta) = \mathrm{E}_\theta f(T, S_T) = c_1\mathrm{E}_\theta S_T + c_2 = c_1\beta\theta T + cc_1 + c_2$$

is the only efficiently estimable function for this plan. For example, by (6), (11) and (12), the estimator

$$f(T, S_T) = \frac{1}{\beta T}(S_T - c)$$

is efficient for the function $h(\theta) = \theta$.

LEMMA 4. *If $x_0 > c$, $\theta > 0$ or $x_0 < c$, $\theta < 0$, then for an inverse plan we have*

$$\mu_\theta(\{\omega: 0 < \tau(\omega) < \infty\}) = 1.$$

Proof. Let us consider the first case, the second is analogous. We have

$$\mu_\theta(\{\omega: 0 < \tau(\omega) < \infty\}) \geqslant \mu_\theta(\bigcup_{t>0} \{\omega: S_t(\omega) > x_0\}) \geqslant \mu_\theta(\{\omega: S_t(\omega) > x_0\}).$$

The statistic

$$S_t(\omega) = \omega(t) + \beta \int_0^t \omega(s)\,ds$$

has the normal distribution with $ES_t = \beta\theta t + c$ and $D_\theta^2 S = 2\beta\sigma^2 t$. Thus

$$\mu_\theta(\{\omega: S_t(\omega) > x_0\}) = \int_{x_0}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\beta\sigma^2 t}} \exp\left\{-\frac{(y-\beta\theta t - c)^2}{4\beta\sigma^2 t}\right\} dy$$

$$= \int_{l_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du, \quad \text{where } l_1 = \frac{x_0 - \beta\theta t - c}{\sqrt{2\beta\sigma^2 t}},$$

$$1 \geqslant \mu_\theta(\{\omega: 0 < \tau(\omega) < \infty\}) \geqslant \mu_\theta(\bigcup_{t>0} \{\omega: S_t(\omega) > x_0\})$$

$$\geqslant \lim_{t\to\infty} \mu_\theta(\{\omega: S_t(\omega) > x_0\}) = 1,$$

which completes the proof.

Let us suppose that for the inverse plan (τ, f) the assumptions of Lemma 4 are satisfied and $E_\theta \tau^2 < \infty$ for $\theta \in [a, b]$. Then we have, with probability 1, $S_\tau = x_0$. From (6) we obtain the equality

$$E_\theta \tau = \frac{x_0 - c}{\beta\theta}. \tag{13}$$

By (10) we have

$$D_\theta^2 \tau = \frac{2\sigma^2(x_0 - c)}{\beta^2\theta^3}. \tag{14}$$

Let the estimator $f(\tau, S_\tau)$ be efficient in this plan. Then it is efficient at some value θ_1 and, by (3), we have

$$f(\tau, x_0) = k(\theta_1)(x_0 - \beta\theta_1\tau - c) + h(\theta_1).$$

So, if the estimator $f(\tau, S_\tau)$ is efficient, then there exist some constants c_1, c_2 for which

$$f(\tau, x_0) = c_1\tau + c_2$$

and, by (13),

$$h(\theta) = \mathrm{E}_\theta f(\tau, x_0) = c_1 \mathrm{E}_\theta \tau + c_2$$

is the only efficiently estimable function for this plan. For example, by (11), (13) and (14) the estimator

$$f(\tau, S_\tau) = \frac{\beta}{x_0 - c}\tau$$

is efficient for the function $h(\theta) = 1/\theta$.

LEMMA 5. *If $s > c$, $\theta > a/\beta$ or $s < c$, $\theta < a/\beta$, then for an oblique plan we have*

$$\mu_\theta(\{\omega:\ 0 < \tau(\omega) < \infty\}) = 1.$$

Proof. In the first case (the second being analogous) we have

$$\mu_\theta(\{\omega:\ 0 < \tau(\omega) < \infty\}) \geqslant \mu_\theta\Big(\bigcup_{t>0} \{\omega:\ S_t(\omega) > at+s\}\Big)$$
$$\geqslant \mu_\theta(\{\omega:\ S_t(\omega) > at+s\}),$$

$$\mu_\theta(\{\omega:\ S_t(\omega) > at+s\}) = \int\limits_{at+s}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\beta\sigma^2 t}} \exp\left\{-\frac{(y-\beta\theta t - c)^2}{4\beta\sigma^2 t}\right\} dy$$

$$= \int\limits_{l_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du, \quad \text{where } l_1 = \frac{s-c+at-\beta\theta t}{\sqrt{2\beta\sigma^2 t}},$$

$$1 \geqslant \mu_\theta(\{\omega:\ 0 < \tau(\omega) < \infty\}) \geqslant \mu_\theta\Big(\bigcup_{t>0} \{\omega:\ S_t(\omega) > at+s\}\Big)$$
$$\geqslant \lim_{t\to\infty} \mu_\theta(\{\omega:\ S_t(\omega) > at+s\}) = 1,$$

which completes the proof.

Let us suppose that for the oblique plan (τ, f) the assumptions of Lemma 5 are satisfied and $\mathrm{E}_\theta \tau^2 < \infty$ for $\theta \in [a, b]$. Then we have, with probability 1, $S_\tau = a\tau + s$. From (6) we obtain the equality

$$\mathrm{E}_\theta \tau = \frac{s-c}{\beta\theta - a} \tag{15}$$

and, by (10),

$$\mathrm{D}_\theta^2 \tau = \frac{2\sigma^2\beta(s-c)}{(\beta\theta - a)^3}. \tag{16}$$

If $f(\tau, S_\tau)$ is an efficient estimator in this plan, then there exist some constants c_1, c_2 for which

$$f(\tau, S_\tau) = c_1\tau + c_2.$$

By (16),

$$h(\theta) = \mathrm{E}_\theta f(\tau, S_\tau) = c_1 \frac{s-c}{\beta\theta-a} + c_2$$

is the only efficiently estimable function for this plan. For example, by (11), (15) and (16) the estimator

$$f(\tau, S_\tau) = \frac{1}{s-c}\tau$$

is efficient for the function $h(\theta) = 1/(\beta\theta - a)$.

4. Sequential plans, efficient at a given value θ_1**.** We have proved that the sequential plan $(\tau, f(\tau, S_\tau))$ is efficient at a given value θ_1 if and only if

$$f(\tau, S_\tau) = k(\theta_1)[S_\tau - \beta\theta_1\tau - c] + h(\theta_1),$$

and if the function $h(\theta)$ is efficiently estimable at θ_1, then it takes the form

$$h(\theta) = k_1(\theta-\theta_1)\mathrm{E}_\theta\tau + h(\theta_1),$$

where k_1 is some constant.

Let us denote by D_0 the class of all sequential plans, efficient at θ_1, for the function $h(\theta)$. Assume that the plan $(\tau, f(\tau, S_\tau))$ belongs to D_0. We have

$$\begin{aligned} \mathrm{D}_\theta^2 f(\tau, S_\tau) &= \mathrm{D}_\theta^2[k(\theta_1)(S_\tau - \beta\theta_1\tau - c) + h(\theta_1)] \\ &= [k(\theta_1)]^2(\mathrm{D}_\theta^2 S_\tau + \beta^2\theta_1^2\mathrm{D}_\theta^2\tau - 2\beta\theta_1\mathrm{E}_\theta\tau S_\tau + 2\beta\theta_1\mathrm{E}_\theta\tau\mathrm{E}_\theta S_\tau).\end{aligned}$$

By (8), (9) and (10) we obtain

$$\begin{aligned} \mathrm{D}_\theta^2 f(\tau, S_\tau) &= [k(\theta_1)]^2[\beta^2(\theta-\theta_1)^2\mathrm{D}_\theta^2\tau + 2\beta\sigma^2\mathrm{E}_\theta\tau + 4\beta\sigma^2(\theta-\theta_1)\mathrm{E}_\theta'\tau] \\ &= A + C\mathrm{D}_\theta^2\tau.\end{aligned}$$

For all plans $(\tau, f(\tau, S_\tau))$ belonging to D_0 for which $\mathrm{E}_\theta\tau$ is the same, the constants A and C are the same. In this case we can say that the plan τ belonging to D_0 for which $\mathrm{D}_\theta^2\tau$ is smaller is better at θ.

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MODYFIKACJA LEMATU SUDAKOWA ORAZ EFEKTYWNE PLANY SEKWENCYJNE DLA PROCESU ORNSTEINA-UHLENBECKA

STRESZCZENIE

Niech Ω będzie przestrzenią funkcji prawostronnie ciągłych $\omega(t)$ $(t \geqslant 0)$ o wartościach w R^k. Załóżmy, że:

$\mathscr{F}$ jest najmniejszym σ-ciałem podzbiorów Ω, względem którego funkcje $\omega(t)$ są mierzalne, gdy $t \geqslant 0$;

$\mathscr{F}_t$ jest najmniejszym σ-ciałem podzbiorów Ω, względem którego funkcje $\omega(s)$ są mierzalne, gdy $s \in [0, t]$;

μ_θ jest miarą probabilistyczną na $(\Omega, \mathscr{F})$, zależną od rzeczywistego parametru $\theta \in [a, b]$;

$\mu_{\theta,t}$ jest miarą μ_θ obciętą do σ-ciała $\mathscr{F}_t$.

Zakładamy ponadto, że miara $\mu_{\theta,t}$ dla każdego $t > 0$ jest absolutnie ciągła względem $\mu_{\theta_0,t}$ oraz że

$$\frac{d\mu_{\theta,t}}{d\mu_{\theta_0,t}} = g(t, S(\omega, t), \theta, \theta_0),$$

gdzie g jest funkcją ciągłą, a $S(\omega, t)$ jest odwzorowaniem z Ω w R^l, mierzalnym względem $\mathscr{F}_t$ dla każdego t i prawostronnie ciągłym względem t, μ_θ-prawie wszędzie dla każdego $\theta \in [a, b]$.

Niech zmienna losowa τ: $\Omega \to [0, \infty]$ będzie markowskim czasem zatrzymania, a więc spełnione są następujące warunki:

$$\{\omega: \tau(\omega) \leqslant t\} \in \mathscr{F}_t$$

oraz

$$\mu_\theta(\{\omega: 0 \leqslant \tau(\omega) < \infty\}) = 1 \quad \text{dla } \theta \in [a, b].$$

Niech $U = [0, \infty) \times R^l = T \times R^l$, $U \ni u = (t(u), x(u))$, gdzie $t(u) \in T$ oraz $x(u) \in R^l$. Niech $\mathscr{B}_U$ oznacza σ-ciało podzbiorów borelowskich zbioru U. Na $(U, \mathscr{B}_U)$ określamy miarę m_θ generowaną przez statystykę (τ, S_τ):

$$m_\theta(A) = \mu_\theta\Big(\big\{\omega: \big(\tau(\omega), S(\tau(\omega))\big) \in A\big\}\Big) \quad \text{dla } A \in \mathscr{B}_U.$$

Przy podanych założeniach udowodniono modyfikację lematu Sudakowa, stwierdzającą, że *miara* m_θ *jest absolutnie ciągła względem miary* m_{θ_0} *oraz*

$$\frac{dm_\theta}{dm_{\theta_0}} = g\big(t(u), x(u), \theta, \theta_0\big).$$

Rozważmy proces Ornsteina-Uhlenbecka, tzn. proces dyfuzji, którego stacjonarne gęstości przejścia $p(x, t, y)$ spełniają równanie

$$\frac{\partial}{\partial t} p(x, t, y) = \frac{1}{2} 2\beta\sigma^2 \frac{\partial^2 p(x, t, y)}{\partial x^2} - \beta(x-\theta)\frac{\partial p(x, t, y)}{\partial x}, \quad \beta > 0, \sigma > 0.$$

Zakładamy, że prawie wszystkie realizacje tego procesu przyjmują w chwili $t = 0$ ustaloną wartość c. Dla tego procesu funkcja

$$S_t(\omega) = \omega(t) + \beta \int_0^t \omega(s)\, ds$$

jest statystyką dostateczną parametru θ.

Dalsze rozważania dotyczą efektywnych planów sekwencyjnych dla różniczkowalnej funkcji $h(\theta)$ nie znanego parametru θ. Udowodniono, że efektywne plany sekwencyjne spełniają równanie

$$a_1 x(u) + a_2 t(u) + a_3 = 0, \quad a_1^2 + a_2^2 \neq 0, \ m_{\theta_0}\text{-prawie wszędzie}.$$

Funkcja efektywnie estymowalna ma postać

$$h(\theta) = \frac{a_1\theta + a_2}{b_1\theta + b_2}.$$

Pokazano, że plany stałe, odwrotne i ukośne są efektywne, oraz podano postać funkcji efektywnie estymowalnych dla tych planów.

Rozważano również plany efektywne w ustalonym punkcie θ_1. Udowodniono, że wśród planów efektywnych w punkcie θ_1, dla których $E_\theta\tau$ jest identyczne, ten jest lepszy, dla którego wartość $D_\theta^2\tau$ jest mniejsza.