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ON POLYHEDRAL GAMES

0. Introduction. We call the triplet $(\mathscr{A}_1, \mathscr{A}_2, C)$ a *polyhedral game* (PG), where $\mathscr{A}_1 = \{x | xA_1 \geqslant a_1\}$ and $\mathscr{A}_2 = \{y | A_2 y \leqslant a_2\}$ are non-void polyhedral sets in the Euclidean spaces R^m and R^n, respectively, and where C is a matrix of size $m \times n$.

Let us define

$$v_1 = \sup_{\mathscr{A}_2} \inf_{\mathscr{A}_1} xCy \quad \text{and} \quad v_2 = \inf_{\mathscr{A}_1} \sup_{\mathscr{A}_2} xCy.$$

Using similar arguments as in the case of common matrix games, it is easy to see (cf. [4]) that v_1 is the optimal value of the following linear programming (LP) problem (the LP-problem related to the maximizing player):

$$\left.\begin{aligned} &\forall_i: && -\sum_k (\bar{x}_i C \bar{y}_k)\lambda_k - \sum_l (\bar{x}_i C \bar{\bar{y}}_l)\mu_l + v \leqslant 0 \\ &\forall_j: && -\sum_k (\bar{\bar{x}}_j C \bar{y}_k)\lambda_k - \sum_l (\bar{\bar{x}}_j C \bar{\bar{y}}_l)\mu_l \leqslant 0 \\ & && \sum_k \lambda_k = 1 \end{aligned}\right\} v \to \max,$$

$$\lambda_k, \mu_l \geqslant 0,$$

where $\bar{x}_i$, $\bar{\bar{x}}_j$ and $\bar{y}_k$, $\bar{\bar{y}}_l$ are the extremal elements of $\mathscr{A}_1^{\Delta}$, $\mathscr{A}_1^{<}$ and $\mathscr{A}_2^{\Delta}$, $\mathscr{A}_2^{<}$, respectively, i.e. the extremal elements of the components in the canonical decomposition of $\mathscr{A}_1$ and $\mathscr{A}_2$ (cf. [1]). The LP-problem related in a similar manner to the other player is a dual of the above one. So, applying the duality theorem, we have one of the following possibilities:

$$-\infty < v_1 = v_2 < +\infty, \tag{0.1}$$

$$-\infty = v_1 = v_2, \tag{0.2}$$

$$v_1 = v_2 = +\infty, \tag{0.3}$$

$$-\infty = v_1, v_2 = +\infty. \tag{0.4}$$

The algorithm to be presented in Section 1 is essentially the building up and the solution of the above LP-problem. Section 2 consists of some remarks on the algorithm and in Section 3 we give two possibilities of reducing an LP-problem to the solution of a PG. These give two decomposition procedures for LP-problems. The detailed discussion of the computational aspects of these procedures and some experience will be given in another paper.

From the formulation it is always clear which symbol is a row or a column vector, which one is a scalar, etc.

1. The algorithm.

1.1. Let $x_1 \in \mathscr{A}_1^{\Delta}$ be an arbitrary extremal element and let $\varrho = 1$.

1.2. Let us suppose the extremal elements $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{\varrho-\varrho'} \in \mathscr{A}_1^{\Delta}$ and $\bar{\bar{x}}_1, \bar{\bar{x}}_2, \ldots, \bar{\bar{x}}_{\varrho'} \in \mathscr{A}_1^{<}$ to be known and let us consider the following LP-problem:

$$\left.\begin{array}{l} -\bar{x}_1 Cy + v \leqslant 0 \\ \cdots\cdots\cdots\cdots \\ -\bar{x}_{\varrho-\varrho'} Cy + v \leqslant 0 \\ -\bar{\bar{x}}_1 Cy \leqslant 0 \\ \cdots\cdots\cdots\cdots \\ -\bar{\bar{x}}_{\varrho'} Cy \leqslant 0 \\ A_2 y \leqslant a_2 \end{array}\right\} v \to \max$$

If this problem has no solution, the procedure is completed and we have case (0.2) or case (0.4). If this problem is not bounded, let y_ϱ be an extremal element of $\mathscr{A}_2^{<}$ for which $\bar{x}_1 Cy_\varrho > 0, \ldots, \bar{\bar{x}}_1 Cy_\varrho \geqslant 0, \ldots$, with at least one strict inequality, and let be $v_1^{(\varrho)} = +\infty$. If the problem has a solution, let y_ϱ be an extremal solution, and let $v_1^{(\varrho)}$ be the optimal value of this program.

1.3. Let us consider the LP-problem

$$xA_1 \geqslant a_1\} \; xCy_\varrho \to \min.$$

If this problem is unbounded, let $v_2^{(\varrho)} = -\infty$ and let $x_{\varrho+1} \in \mathscr{A}_1^{<}$ be such an extremal element that $x_{\varrho+1} Cy_\varrho < 0$. Otherwise, let $x_{\varrho+1} \in \mathscr{A}_1^{\Delta}$ be an extremal solution and let $v_2^{(\varrho)}$ be the optimal value of this program.

1.4. If $v_1^{(\varrho)} = +\infty$ and $v_2^{(\varrho)} > 0$, the procedure is completed and we have case (0.3) or case (0.4). If $v_1^{(\varrho)} = v_2^{(\varrho)}$, the procedure is completed and we have case (0.1). Otherwise, let us make the substitutions: $\bar{x}_{\varrho-\varrho'+1} = x_{\varrho+1}$ if $x_{\varrho+1} \in \mathscr{A}_1^{\Delta}$ or $\bar{\bar{x}}_{\varrho'+1} = x_{\varrho+1}$ if $x_{\varrho+1} \in \mathscr{A}_1^{<}$, and $\varrho = \varrho'+1$.

Now let us continue from 1.2.

The procedure may be validated in the following way:

Ad 1.2. If the LP-problem in 1.2 has no feasible solution, then for any $y \in \mathscr{A}_2$ we have $\overline{\overline{x}}_j Cy < 0$ for some $1 \leqslant j \leqslant \varrho - \varrho'$. So

$$\inf_{\mathscr{A}_1} xCy = -\infty \quad \text{for any } y \in \mathscr{A}_2, \text{ i.e. } v_1 = -\infty.$$

Ad 1.4. If $v_2^{(\varrho)} > 0$, then for any $x \in \mathscr{A}_1$ we have $xCy_\varrho > 0$. From $v_1^{(\varrho)} = +\infty$ it follows that $y_\varrho \in \mathscr{A}_2^{<}$, thus

$$\sup_{\mathscr{A}_2} xCy = +\infty \quad \text{for any } x \in \mathscr{A}_1, \text{ i.e. } v_2 = +\infty.$$

If $v_1^{(\varrho)} = v_2^{(\varrho)}$, let $\tilde{x} = tX$, where the matrix X is composed from the row-vectors $\overline{x}_1, \ldots, \overline{x}_{\varrho-\varrho'}, \overline{\overline{x}}_1, \ldots, \overline{\overline{x}}_{\varrho'}$ and where t is the optimal solution of the dual of the LP-problem in 1.2 (which now exists), and let $\tilde{y} = y_\varrho$. Then from 1.3 we have

$$\min_{\mathscr{A}_1} xCy = v_2^{(\varrho)}$$

and from the duality theorem we obtain

$$\tilde{x}C\tilde{y} = v_1^{(\varrho)},$$

and, for any $y \in \mathscr{A}_2$, $\tilde{x}Cy \leqslant v_1^{(\varrho)}$, i.e.

$$\max_{\mathscr{A}_2} \tilde{x}Cy = v_1^{(\varrho)}.$$

Since the number of extremal elements of $\mathscr{A}_1$ and $\mathscr{A}_2$ is finite and a repeated appearance of any extremal elements is followed by such a case where the procedure is completed, the procedure is finite.

2. Some remarks.

2.1. Obviously,

$$v_1^{(\varrho)} \geqslant v_1^{(\varrho+1)} \geqslant \ldots \geqslant v_1 \geqslant v_2^{(\varrho)}.$$

2.2. If we introduce a more sophisticated rule for the selection of $x_{\varrho+1}$ in 1.3, passing to step $\varrho+1$ of the iteration procedure, we may drop all the constraints of the LP-problem in 1.2 which were not binding ones in step ϱ (if there were any).

2.3. If we do not assume $\mathscr{A}_1$ or $\mathscr{A}_2$ to be non-void, then applying the usual definition there is a further possibility: $v_1 = +\infty$ and $v_2 = -\infty$. In this case one has to substitute 1.1 by solving an LP-problem to obtain x_1, and if the LP-problems in 1.2 and 1,3 have no solutions, the possibility introduced just now may also be valid.

2.4. One may modify the above algorithm in the following way. If $v_1^{(\varrho)} < +\infty$, then let $x_{\varrho+1}$ be such an extremal element for which $x_{\varrho+1}Cy_\varrho < v_1^{(\varrho)}$ (if it exists) and $v_2^{(\varrho)} = x_{\varrho+1}Cy_\varrho$.

2.5. Passing to step $\varrho+1$, one may add to the LP-problem in 1.2 the constraint (besides that one corresponding to $x_{\varrho+1}$) $-xCy+v \leqslant 0$, where $x \in \mathscr{A}_1$ is arbitrary (or other constraints of this type).

3. Reducing a large scale LP-problem to solving a PG. Let us consider the LP-problem

$$\sup(cx \mid Ax = b, x \geqslant 0).$$

If $A = [A_1, A_2, \ldots, A_N]$, $c = [c_1, c_2, \ldots, c_N]$, and $x = [x_1, x_2, \ldots, x_N]$ is a partition of the parameters and variables of the LP-problem, then

$$\begin{aligned}\sup(cx \mid Ax = b, x \geqslant 0) &= \sup\Big(\sum_i c_i x_i \Big| \sum_i A_i x_i = b,\ \forall i\colon x_i \geqslant 0\Big)\\ &= \sup\Big(\sum_i \sup(c_i x_i \mid A_i x_i = b_i, x_i \geqslant 0) \Big| \sum_i b_i = b\Big)\\ &= \sup\Big(\sum_i \inf(p_i b_i \mid p_i A_i \geqslant c_i) \Big| \sum_i b_i = b\Big) = \sup_{\mathscr{B}}\inf_{\mathscr{P}} pb,\end{aligned}$$

where

$$\mathscr{B} = \Big\{b = (\ldots, b_i, \ldots) \Big| \sum_i b_i = b\Big\}$$

and

$$\mathscr{P} = \{p = (\ldots, p_i, \ldots) \mid \forall i\colon p_i A_i \geqslant c_i\}.$$

This way of reduction of an LP-problem to a PG is due to Lipták [2]. The procedure suggested by him for solving this PG is based on the so-called *fictitious playing* [3]. We may briefly compare this method and our algorithm which can be obtained by applying the procedure of Section 2. Although in our case one has to solve "coordinating problems" (cf. [2]) of larger size, this procedure is finite, which is a very definite advantage because of the rather slow convergence of fictitious playing.

Another possible reduction which may be more advantageous, depending on the structure of matrix A, is the following.

Let $A = [A_1, A_2]$, $c = [c_1, c_2]$ and $x = [x_1, x_2]$ be a partition of the parameters and variables. Then we have

$$\begin{aligned}\sup(cx \mid Ax = b, x \geqslant 0) &= \sup(c_1x_1 + c_2x_2 \mid A_1x_1 + A_2x_2 = b, x_1, x_2 \geqslant 0)\\ &= \sup\big(c_1x_1 + \sup(c_2x_2 \mid A_2x_2 = b - A_1x_1, x_2 \geqslant 0) \mid x_1 \geqslant 0\big)\\ &= \sup\big(c_1x_1 + \inf(p(b - A_1x_1) \mid pA_2 \geqslant c_2) \mid x_1 \geqslant 0\big) = \sup_{\mathscr{X}}\inf_{\mathscr{P}} pCx,\end{aligned}$$

where $\mathscr{X} = \{x = (x_1, 0) \mid x_1 \geqslant 0\}$, $\mathscr{P} = \{p = (p, 0) \mid pA_2 \geqslant c_2\}$ and

$$C = \begin{bmatrix} 0 & c_1 \\ b & -A_1 \end{bmatrix}.$$

For example in the case

$$A_2 = \begin{bmatrix} A_{21} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{2N} \end{bmatrix}$$

$\mathscr{P}$ is the Cartesian product of polyhedral sets of smaller size.

References

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O GRACH WIELOŚCIENNYCH

STRESZCZENIE

Nota podaje związek gier wielościennych (polyhedral games) z programami liniowymi. Proponuje się pewien algorytm rozwiązywania tych gier i porównuje z podobnym algorytmem, podanym przez Liptáka [2]. Nota ma charakter teoretyczny i nie zawiera wyników praktycznych stosowania proponowanego algorytmu.
