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ON GALERKIN APPROXIMATIONS OF PARABOLIC EQUATIONS IN TIME DEPENDENT DOMAINS

1. Introduction. Galerkin approximations of initial-boundary value problems for parabolic equations of second order have been studied by many authors (see [3] and [7]–[9], where other references are given). However, as far as we know, only the case of a constant domain was considered. The aim of the present paper is to study the case where the domain is varying in time.

By means of a diffeomorphism we transform our problem to an equivalent one, posed in the cylindrical space-time domain and connected with a coercive Dirichlet bilinear form. Now the standard Galerkin semidiscretization leads to a system of ordinary differential equations in time. This system is proposed to be solved by the Galerkin method as well. In this manner we obtain approximate solutions which are continuous (or more smooth) with respect to time.

In Section 7 we propose another method of obtaining approximants continuous in time, which in the particular case of the finite element method is connected with the triangulation of the space-time domain. Under some additional assumptions concerning the domain, this method was presented in the author's previous paper [5].

We give some estimates of the error of the proposed approximate methods.

2. Basic notation and assumptions. For x, $y \in R^N$ we denote by $|x|$ the Euclidean norm and by $\langle x, y\rangle$ the scalar product. If $x = (x_1, \ldots, x_N)$, then $x' = (x_1, \ldots, x_{N-1})$. The derivation is written as follows.

$$D_{x_j} = \frac{\partial}{\partial x_j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ (α_j non-negative integers), $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The derivative with respect to time is written as u_t or $\dot{u}$ in the one-dimensional case. All the derivations are understood in the weak (distributional) sense. All the considered functions are real-valued.

Let $\Omega \subset R^n$ be an arbitrary domain. By $C^\infty(\Omega)$ we mean the set of all functions u such that $u = v|_\Omega$, $v \in C_0^\infty(R^n)$. We put

$\|\cdot\|_{0,\Omega}$ for the norm in $L^2(\Omega)$,

$(\cdot, \cdot)_\Omega$ for the scalar product in $L^2(\Omega)$.

Let $X_{(t)}$ $(0 \leqslant t \leqslant T)$ be a family of linear normed spaces. By $L^2(0, T; X_{(t)})$ we denote the set of all functions $[0, T] \ni t \to u(t) \in X_{(t)}$ such that

$$\int_0^T \|u(t)\|_{X_{(t)}}^2 dt < \infty .$$

If $X_{(t)} = R^N$, this space is denoted by $L^{2,N}(0, T)$ and considered with the scalar product

$$(u, v) = \int_0^T \langle u(t), v(t) \rangle dt$$

and the norm $\|u\|_0 = (u, u)^{1/2}$. For fixed $T > 0$ we put $\Delta_T = \Omega \times (0, T)$. Given an $(N \times N)$-matrix C, we put

$$|C| = \sup_{|x|=1} |Cx| \qquad (x \in R^N)$$

for its spectral norm.

3. Some auxiliary definitions and lemmas. For the convenience of the reader we give in this section some definitions and lemmas used in the sequel.

By a *Sobolev space* $H_k(\Omega)$ $(k = 0, 1, \ldots)$ we mean the set of all functions $u \in L^2(\Omega)$ such that $D^\alpha u = L^2(\Omega)$ for $|\alpha| \leqslant k$. Obviously, $H_0(\Omega) = L^2(\Omega)$. It is known [1] that $H_k(\Omega)$ equipped with the norm

$$\|u\|_{k,\Omega} = \Big(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_{0,\Omega}^2\Big)^{1/2}$$

corresponding to the inner product

$$(u, v)_{k,\Omega} = \sum_{|\alpha| \leqslant k} (D^\alpha u, D^\alpha v)_\Omega$$

is a Hilbert space. We need in the sequel also the seminorm

$$|u|_{k,\Omega}^2 = \sum_{|\alpha| = k} \|D^\alpha u\|_{0,\Omega}^2 .$$

Considering the one-dimensional case with $\Omega = (0, T)$ we denote by $H_k^N(0, T)$ the product of N copies of $H_k(0, T)$, equipped with the norm

$$\|u\|_k = \Big(\sum_{j=0}^k \|D_x^j u\|_0^2\Big)^{1/2} .$$

We put

$$H_1^{0,N}(0, T) = \{u \in H_1^N(0, T)\colon\ u(T) = 0\}$$

(it is known that u is a continuous vector function, so the value of $u(T)$ is well defined). The closure of $C_0^\infty(\Omega)$ in $H_k(\Omega)$ is denoted by $\mathring{H}_k(\Omega)$.

From now on we suppose that Ω is bounded. We say that Ω has the *segment property* if there exists a finite covering $\partial\Omega \subset \bigcup U_j$ with open sets U_j and the corresponding set of vectors $\eta_j \in R^n$ such that $(\bar{\Omega} \cap U_j) + t\eta_j \subset \Omega$ for each j and $t \in (0, 1)$. It is obvious that Δ_T has the segment property if Ω has this property. In the sequel c_T denotes a positive constant which may depend on T.

LEMMA 1. *If Ω has the segment property, then the set $C^\infty(\bar{\Omega})$ is dense in $H_k(\Omega)$.*

The proof may be found in [1]. Using quite similar arguments one may prove a little more general lemma:

LEMMA 2. *Suppose that*

(i) Ω has the segment property,

(ii) $D^\alpha u \in L^2(\Omega)$ for α belonging to a finite set A of multi-indices.

Then there is a sequence $\{\varphi_\nu\} \subset C_0^\infty(R^n)$ such that

$$\lim_{\nu \to \infty} D^\alpha \varphi_\nu = D^\alpha u \quad \text{in } L^2(\Omega) \text{ for } \alpha \in A.$$

LEMMA 3. *Suppose that Ω has the segment property. Then for each fixed $\tau \in [0, T]$ there exists a linear continuous mapping $S\colon H_1(\Delta_T) \mapsto L^2(\Omega)$ such that $Su = u(\cdot, \tau)$ for $u \in C^\infty(R^{n+1})$.*

Proof. We suppose $\tau < T$; the case $\tau = T$ may be treated similarly. Let $\varphi \in C^\infty(R)$, $\varphi = 0$ in a neighbourhood of T, and $\varphi = 1$ in a neighbourhood of τ. Then for $u \in C^\infty(R^{n+1})$ we have

$$\int_\tau^T D_t(\varphi u)\,dt = -u(y, \tau),$$

so

$$|u(y, \tau)|^2 \leqslant T \int_0^T |D_t(\varphi u)|^2\,dt,$$

and therefore

$$\int_\Omega |u(y, \tau)|^2\,dy \leqslant c_T \|u\|_{1,\Delta_T}^2.$$

This means that S is continuous when considered on the set $C^\infty(\bar{\Delta}_T)$ which is dense in $H_1(\Delta_T)$ by Lemma 1. Thus our assertion follows.

In the sequel we write $u(\cdot, \tau)$ instead of Su. Each function $u(x, t)$ may be considered as a vector-valued mapping $t \mapsto u(\cdot, t)$. If $u \in L^2(0, T; H_k(\Omega))$, then it is in $L^2(\Delta_T)$, and one may consider its distributional derivative u_t.

LEMMA 4. *Let Ω have the segment property and let $u, u_t \in L^2(0, T; H_k(\Omega))$. Then*

$$(1) \qquad \sup_{0 \leqslant s \leqslant T} \|u(\cdot, s)\|_{k,\Omega}^2 \leqslant c_T \Big(\int_0^T \|u(\cdot, t)\|_{k,\Omega}^2 dt + \int_0^T \|u_t(\cdot, t)\|_{k,\Omega}^2 dt\Big).$$

Proof. By Lemma 2 it is sufficient to prove (1) for smooth u. For $0 \leqslant t < s \leqslant T$ we have

$$u(x, s) = u(x, t) + \int_t^s u_\tau(x, \tau)\, d\tau.$$

Thus, applying the Schwarz inequality, we obtain

$$|u(x, s)|^2 < 2|u(x, t)|^2 + 2T \int_0^T |u_\tau(x, \tau)|^2 d\tau.$$

Integrating with respect to $(x, t) \in \Delta_T$ yields (1) for $k = 0$. The case of an arbitrary k may be treated similarly with u replaced by $D_x^\alpha u$ for $|\alpha| \leqslant k$.

LEMMA 5. *Let Φ be a diffeomorphism defined in a neighbourhood of $\bar{\Omega}$ and let us put $\Psi = \Phi^{-1}$. Then*

(i) *the operator $u \mapsto u \circ \Psi$ is a linear continuous mapping of $H_1(\Omega)$ onto $H_1(\Phi(\Omega))$;*

(ii) *if $x = \Phi(y)$, then the chain rule*

$$(2) \qquad D_{x_j}(u \circ \Psi) = \sum_{k=1}^{n} (D_{x_j} \Psi_k)(D_{y_k} u) \circ \Psi \qquad (j = 1, \ldots, n)$$

holds for any $u \in H_1(\Omega)$.

The proof of (i) may be found in [1]. Part (ii) follows immediately if we notice that formula (2) is valid for smooth u and that any function $u \in H_1(\Omega)$ is a limit in the $\|\cdot\|_{1,\Omega}$-norm of a sequence $\{u_\nu\} \subset C^\infty(\Omega)$.

LEMMA 6. *Suppose that Ω has the segment property. Then for any $u, v \in H_1(\Delta_T)$ we have*

$$(3) \qquad \int_0^T (u_t, v)_\Omega dt = -\int_0^T (u, v_t)_\Omega dt + \big(u(\cdot, t), v(\cdot, t)\big)_\Omega\Big|_{t=0}^{t=T}.$$

Proof. In view of Fubini's theorem we may change the order of integration. Then for smooth u and v formula (3) is obtained after integrating by parts in the interval $[0, T]$. For arbitrary $u, v \in H_1(\Delta_T)$ we obtain (3) passing to the limit according to Lemmas 1 and 3.

We deal in the sequel with finite element approximations. Using the notation of [2] we suppose that $\{T_h\}$ is a family of triangulations of the considered domain Ω (which is assumed to be a polyhedron) and that the following conditions hold:

(f_1) the family $\{T_h\}$ is regular;

(f_2) (K, P_K, Σ_K) with $K \in \bigcup_h T_h$ is the family of finite elements of class C^0;

(f_3) each (K, P_K, Σ_K) is affinely equivalent to a pattern finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$;

(f_4) $P_r \subset \hat{P} \subset H_1(\hat{K})$;

(f_5) the set $\hat{\Sigma}$ is defined by means of the derivations D^α with $\alpha \in A$ and s is the maximal order of such α (so $s = 0$ in the case of finite elements of Lagrange type).

We use in the sequel the interpolation theorem (see [2], Theorem 3.2.1) in the following form:

THEOREM A. *We suppose that* (f_1)–(f_5) *hold and that* $\varphi \in H_{r+1}(\Omega)$ *with* $r+1 > n/2+s$. *Then*

$$\|\varphi - \Pi_h \varphi\|_{1,\Omega} \leqslant \eta h^r |\varphi|_{r+1,\Omega},$$

where Π_h *denotes the interpolation operator and* η *is a positive constant depending on the family* $\{T_h\}$ *and on the element* $(\hat{K}, \hat{P}, \hat{\Sigma})$.

4. Weak formulation of the exact problem. We suppose from now on that $\Omega \subset R^h$ is a bounded domain having the segment property. Particularly, Ω may be a polygon for $n = 2$, a polyhedron for $n = 3$ and, more generally, any bounded domain with Lipschitz-continuous boundary.

We deal in the sequel with a family of diffeomorphisms $F_{(t)}$. Putting $F_{(t)}^{-1} = G_{(t)}$ and $x_r = F_r(y, t)$, $y_r = G_r(x, t)$ $(r = 1, \ldots, n)$ for $x = F_{(t)}(y)$ we assume the following:

(a_1) there are a neighbourhood Ω_0 of $\bar{\Omega}$ and a number $\delta > 0$ such that $F_{(t)}\colon \Omega_0 \mapsto R^n$ is a diffeomorphism of class C^2 for $-\delta < t < T+\delta$;

(a_2) F_0 is the identical mapping;

(a_3) $D_{y_j} F_r$, $D_{y_j} D_{y_k} F_r$, and $D_t F_r$ are continuous in $\Omega_0 \times (-\delta, T+\delta)$.

The Jacobi matrix of $F_{(t)}$ is denoted by $F'_{(t)}$. It is known that $G'_{(t)} = [F'_{(t)}]^{-1}$. We consider our initial-boundary value problem in the time-space domain

$$D_T = \{(x, t)\colon x \in \Omega_{(t)},\ 0 < t < T\},$$

where $\Omega_{(t)} = F_{(t)}(\Omega)$. Obviously, the mapping Φ defined by

$$t = s, \qquad x = F_{(s)}(y) \tag{4}$$

is a diffeomorphism in a neighbourhood of $\bar{\Delta}_T$, which maps Δ_T onto D_T. For any function h defined in D_T we put

$$\tilde{h}(y, s) = h(x, t),$$

where (x, t) and (y, s) are connected by (4). Then, by Lemma 5, the mapping $\Phi^*\colon h \mapsto \tilde{h}$ maps $H_1(D_T)$ onto $H_1(\Delta_T)$ and the following formulas of derivation hold:

$$(5) \qquad h_t = \sum_{j=1}^{n} (D_{y_j}\tilde{h})(D_t G_j)+\tilde{h}_s, \quad D_{x_r} h = \sum_{k=1}^{n} (D_{y_k}\tilde{h})(D_{x_r} G_k).$$

For any $h \in H_1(D_T)$ and $\tau \in [0, T]$ we set $h(\cdot, \tau) = \tilde{h}(\cdot, \tau)$ (see Lemma 3). In further calculations we write t instead of s. This is justified by the first identity in (4).

It follows from (a_1) and (a_3) that the Jacobian $J = \det F'_{(t)}$ is a continuous non-vanishing function in $\bar{\Delta}_T$, so it may be assumed without loss of generality that

(a_4) $m_1 \leqslant J(y, t) \leqslant m_2$ for $(y, t) \in \Delta_T$ with some positive m_1 and m_2.

We consider the operator A in the divergent form

$$A = -\sum_{j,k=1}^{n} D_k a_{jk}(x, t) D_j + \sum_{j=1}^{n} a_j(x, t) D_j + a(x, t)$$

assuming the following:

(a_5) the coefficients a_{jk}, a_j and a are bounded in D_T;

(a_6) there is a constant $c > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(x, t)\xi_j \xi_k \geqslant c|\xi|^2$$

for $(x, t) \in D_T$, $\xi \in R^n$.

To describe the boundary conditions we introduce the closed subspace V of $H_1(\Omega)$ satisfying

(a_7) $C_0^\infty(\Omega) \subset V \subset H_1(\Omega)$;

(a_8) $\alpha v \in V$ for each $\alpha \in C^1(\Omega)$, $v \in V$.

Moreover, we put

$$(6) \qquad V_{(t)} = \{v \circ F_t^{-1} \colon v \in V\}.$$

We write

$$H(W) = \{u \in L^2(0, T; W) \colon u_t \in L^2(\Delta_T)\}$$

for any linear subspace $W \subset H_1(\Omega)$ and

$$H(V_{(t)}) = \{u \in L^2(0, T; V_{(t)}) \colon u_t \in L^2(D_T)\}.$$

Obviously, $H(W) \subset H_1(\Delta_T)$, $H(V_{(t)}) \subset H_1(D_T)$, and the following lemma, similar to Lemma 5, is easily obtained:

LEMMA 7. *There are positive constants c_1 and c_2 depending on the family $F_{(t)}$ such that*

$$c_1 \|\tilde{v}\|_{j,\Omega} \leqslant \|v\|_{j,\Omega_{(t)}} \leqslant c_2 \|\tilde{v}\|_{j,\Omega}$$

for any $v \in H_1(\Omega_{(t)})$, $t \in [0, T]$, and $j = 0, 1$.

In the sequel we need also the space

$$H^0(V) = \{u \in H(V):\ u(\cdot, T) = 0\}.$$

For fixed $t \in (0, T)$ let us introduce the Dirichlet bilinear form corresponding to the operator A as

$$a(t; u, v) = \sum_{j,k=1}^{n} (a_{jk} D_j u, D_k v)_{\Omega(t)} + \sum_{j=1}^{n} (a_j D_j u, v)_{\Omega(t)} + (au, v)_{\Omega(t)}.$$

We formulate our initial-boundary value problem as follows:

(P_1) Given $f \in L^2(D_T)$ and $u_0 \in L^2(\Omega)$, find a $u \in H(V_{(t)})$ such that

(i) the identity

$$(u_t, v)_{\Omega(t)} + a(t; u, v) = (f, v)_{\Omega(t)} \tag{7}$$

holds for any $v \in V_{(t)}$ and almost all $t \in (0, T)$,

(ii) $u(\cdot, 0) = u_0$.

It follows from (7) that the solution of problem (P_1) satisfies the equation $Auu_t = f$ in D_T. Concerning the boundary conditions let us consider some examples.

Example 1. Put $V = \mathring{H}_1(\Omega)$; then $V_{(t)} = \mathring{H}_1(\Omega_{(t)})$. Therefore u satisfies the homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega(t)} = 0 \quad (0 < t < T)$$

in some generalized sense.

Example 2. If $V = H_1(\Omega)$, then $V_{(t)} = H_1(\Omega_{(t)})$. Integrating by parts in (1) we obtain now, for sufficiently smooth u and $\partial\Omega_{(t)}$, the natural boundary condition

$$\partial^{\mathrm{tr}} u|_{\partial\Omega(t)} = 0 \quad (0 < t < T),$$

where ∂^{tr} denotes the conormal derivation. Particularly, for $A = -\Delta$ this is the homogeneous Neumann condition

$$\left.\frac{\partial u}{\partial \nu_{(t)}}\right|_{\partial\Omega(t)} = 0 \quad (0 < t < T),$$

where $\nu_{(t)}$ is the unit vector normal to $\partial\Omega_{(t)}$.

By means of the mapping $F_{(t)}$ problem (P_1) may be transformed to an equivalent one in the cylindrical domain Δ_T. Using namely formulas (5) and the identity

$$(D_{y_q} \tilde{v}) J = D_{y_q}(\tilde{v}J) - (\tilde{v}J) J^{-1} D_{y_q} J$$

we can write (7) as

$$(\tilde{u}_t, \tilde{v}J)_\Omega + b(t; \tilde{u}, \tilde{v}J) = (\tilde{f}, \tilde{v}J)_\Omega,$$

where

$$b(t; v, w) = \sum_{p,q=1}^{n} \int_{\Omega} b_{pq}(D_{y_p} v)(D_{y_q} w)\, dy + \\ + \sum_{p=1}^{n} \int_{\Omega} b_p(D_{y_p} v) w\, dy + \int_{\Omega} bvw\, dy$$

with

$$b_{pq} = \sum_{j,k=1}^{n} \tilde{a}_{jk}(D_{x_j} G_p)(D_{x_k} G_q),$$

$$b_p = D_t G_p + \sum_{j=1}^{n} \tilde{a}_j (D_{x_j} G_p) + J^{-1} \sum_{q=1}^{n} b_{pq}(D_{y_q} J),$$

$$b = \tilde{a}.$$

According to (6) and the assumption (a_8) the function $\tilde{v}J$ belongs to V if and only if $v \in V_{(t)}$. Thus problem (P_1) may be reformulated equivalently (with $g = \tilde{f}$) as

(P_2) Given $g \in L^2(\Delta_T)$ and $u_0 = L^2(\Omega)$, find a $\tilde{u} \in H(V)$ such that
(iii) the identity

$$(\tilde{u}_t, v)_\Omega + b(t; \tilde{u}, v) = (g, v)_\Omega \tag{8}$$

holds for all $v \in V$ and almost all $t \in (0, T)$,
(iv) $u(\cdot, 0) = u_0$.

For an arbitrary $w \in H(V)$ and fixed t let us put $v = w(\cdot, t)$ in identity (8). Integrating both sides over the interval $[0, T]$ and applying Lemma 6 we obtain

$$B(\tilde{u}, w) = l_{g,u_0}(w), \tag{9}$$

where

$$B(\tilde{u}, w) = \int_0^T b(t; \tilde{u}, w)\, dt - \int_0^T (\tilde{u}, w_t)\, dt + (\tilde{u}(\cdot, T), w(\cdot, T))_\Omega$$

and

$$l_{g,u_0}(w) = \int_0^T (g, w)_\Omega dt + (u_0, w(\cdot, 0))_\Omega.$$

Now we can state another formulation of problem (P_2), namely:

(P_3) Given $g \in L^2(\Delta_T)$ and $u_0 \in L^2(\Omega)$, find a $\tilde{u} \in H(V)$ such that identity (9) holds for all $w \in H^0(V)$.

PROPOSITION 1. *Problems (P_j) $(j = 1, 2, 3)$ are equivalent when $g = \tilde{f}$. If u is a solution, then (9) holds for all $w \in H(V)$.*

Proof. The equivalence of (P_1) and (P_2) has been established. We also have just shown that any solution $\tilde{u}$ of (P_2) satisfies (9) for all $w\in H(V)$, particularly for $w\in H^0(V)$. To prove the converse statement let us put $w(y, t) = v(y)\varphi(t)$ in (9) with arbitrary $v\in V$, $\varphi\in C_0^\infty(0, T)$. Integrating by parts, in view of Lemma 7, we obtain

$$\int_0^T b(t;\tilde{u}, v)\varphi(t)\,dt+\int_0^T (\tilde{u}_t, v)_\Omega\,\varphi(t)\,dt = \int_0^T (g, v)_\Omega\varphi(t)\,dt$$

and this yields (8) for almost all $t\in(0, T)$. Since we infer from (8) that (9) holds for arbitrary $w\in H(V)$, we can put now $w(y, t) = v(y)\psi(t)$ with $v\in V$, $\psi\in C^1([0, T])$, $\psi(0) = 1$. Integrating by parts we obtain

$$\int_0^T b(t;\tilde{u}, v)\psi(t)\,dt+\int_0^T (\tilde{u}_t, v)_\Omega\psi(t)\,dt+(\tilde{u}(\cdot, 0), v)_\Omega$$

$$= \int_0^T (g, v)_\Omega\psi(t)\,dt+(u_0, v)_\Omega;$$

thus in view of (8) we get

$$(\tilde{u}(\cdot, 0), v)_\Omega = (u_0, v)_\Omega$$

for arbitrary $v\in V$. As V is dense in $L^2(\Omega)$, this yields the initial condition (iv).

PROPOSITION 2. *There are constants $d>0$ and $\lambda_0\geqslant 0$ such that*

$$b(t; v, v)\geqslant d\,\|v\|_{1,\Omega}^2-\lambda_0\,\|v\|_{0,\Omega}^2 \tag{10}$$

for any $v\in V$, $t\in(0, T)$.

Proof. For any $\eta\in R^n$ and $(y, t)\in\Delta_T$ we have

$$\sum_{p,q=1}^{n} b_{pq}\eta_p\eta_q = \sum_{j,k=1}^{n} \tilde{a}_{jk}\,\xi_j\,\xi_k, \tag{11}$$

where $\xi = \eta G'_{(t)}$ and $\eta = \xi F'_{(t)}$. It follows from (11), in view of (a_6), that

$$\sum_{p,q=1}^{n} b_{p,q}\eta_p\eta_q = c_1\,|\eta|^2$$

with $c_1 = c\sup_{\Delta_T} |F'_{(t)}|^{-2}$. Therefore

$$\sum_{p,q=1}^{n} \int_\Omega b_{p,q}(D_{y_p}v)(D_{y_q}v)\,dy\geqslant c_1\,|v|_{1,\Omega}^2.$$

On the other hand,

$$\Big|\sum_{p=1}^{n} \int_\Omega b_p(D_{y_p}v)\,v\,dy+\int_\Omega bv^2\,dy\Big|\leqslant M\Big(\sum_{p=1}^{n} \int_\Omega |D_{y_p}v|\,|v|\,dy+\int_\Omega v^2\,dy\Big)$$

with a positive M such that

$$M \geqslant \max(\sup_{\Delta_T} |b_p|, \sup_{\Delta_T} |b|).$$

Using the inequality

$$|D_{y_p} v|\,|v| \leqslant \frac{\varepsilon^2}{2}|D_{y_p} v|^2 + \frac{1}{2\varepsilon^2}|v|^2$$

and taking $\varepsilon = (c_1 M^{-1})^{1/2}$ we obtain (10) with

$$d = \frac{c_1}{2} \quad \text{and} \quad \lambda_0 = \left(1 + \frac{Mn}{2c_1}\right)M + \frac{c_1}{2}.$$

It is essential for further considerations that the bilinear form $b(t; \cdot, \cdot)$ is positive definite on V. This may be obtained by a simple change of the unknown function, namely $\tilde{u} = \tilde{w}\exp\{\lambda_0 t\}$. Then it is easy to verify that $\tilde{u}$ is a solution of problem (P_2) iff $\tilde{w}$ is a solution of the same problem with g replaced by $g\exp\{-\lambda_0 t\}$ and with the underlying bilinear Dirichlet form

$$b_1(t; \tilde{w}, v) = b(t; \tilde{w}, v) + \lambda_0(\tilde{w}, v),$$

where, in view of Proposition 2,

$$b_1(t; v, v) \geqslant d\,\|v\|_{1,\Omega}^2$$

for $v \in V$. In the sequel we assume that the above change of the unknown function has been done, and therefore

(a_9) Proposition 2 holds with $\lambda_0 = 0$.

Let us put now, for $w \in H_1(\Delta_T)$,

$$[w] = \Big(\int_0^T \|w\|_{1,\Omega}^2 dt + \|w(\cdot, 0)\|_{0,\Omega}^2 + \|w(\cdot, T)\|_{0,\Omega}^2\Big)^{1/2}.$$

Obviously, $[w] = c_T \|w\|_{1,\Delta_T}$ and, integrating by parts with respect to t (see Lemma 6), we get

PROPOSITION 3. *For any $w \in H(V)$ we have*

$$B(w, w) \geqslant d\,[w]^2.$$

Using Lemma 3 we can easily prove

PROPOSITION 4. *For any $w, v \in H(V)$ we have*

$$|B(w, v)| \leqslant c\,[w]\,\|v\|_{1,\Delta_T}$$

with a positive constant c depending on the upper bounds of the coefficients of the bilinear form b.

5. Galerkin approximations in the space variable. We define the Galerkin approximation of (P_2) in the usual way (see [3], where other references are given). Let $V_h \subset V$ be a finite-dimensional space with the basis $\{v_j\}_{j=1}^{N_h}$; then the approximate problem is formulated as follows:

($Q_{2,h}$) Find a $\tilde{U} \in H(V_h)$ such that the identities

$$(\tilde{U}_t, v)_\Omega + b(t; \tilde{U}, v) = (g, v)_\Omega$$

and

$$(\tilde{U}(\cdot, 0), v)_\Omega = (u_0, v)_\Omega$$

hold for all $v \in V_h$, $t \in (0, T)$.

Using the decomposition of the approximate solution

$$\tilde{U}(y, t) = \sum_{j=1}^{N_h} \alpha_j(t) v_j(y) \tag{12}$$

we can reduce problem ($Q_{2,h}$) to the following one:

($Q_{3,h}$) Find $\alpha \in H_1^{N_h}(0, T)$ such that

$$C\dot{\alpha} + B(t)\alpha = \beta(t), \tag{13}$$

$$C\alpha(0) = \gamma, \tag{14}$$

where

$$C_{kj} = (v_j, v_k)_\Omega, \quad B_{kj}(t) = b(t; v_j, v_k), \tag{15}$$
$$\beta_k(t) = (g(\cdot, t), v_k)_\Omega, \quad \gamma_k = (u_0, v_k)_\Omega \quad (j, k = 1, \ldots, N_h).$$

We have namely

PROPOSITION 5. *Both problems ($Q_{2,h}$) and ($Q_{3,h}$) have at most one solution. The function U is a solution of ($Q_{2,h}$) if and only if α solves ($Q_{3,h}$).*

Proof. The uniqueness of the solution $\tilde{U}$ may be seen from Proposition 3 after integrating by parts and using Lemma 6. It is also easily seen that $\tilde{U}$ solves ($Q_{2,h}$) if α solves ($Q_{3,h}$), and so our assertion follows.

THEOREM 1. *Let us consider problem ($Q_{3,h}$) with arbitrary C, $B(t)$, $\beta(t)$, and γ satisfying the following assumptions:*

(i) $C = C^{\mathrm{tr}}$;

(ii) $B_0 = \sup_{[0,T]} |B(t)| < \infty$;

(iii) *there is a positive constant $\varkappa$ such that, for any $\xi \in R^{N_h}$ and $t \in [0, T]$,*

$$\langle B(t)\xi, \xi\rangle \geqslant 2\varkappa|\xi|^2, \quad \langle C\xi, \xi\rangle \geqslant 2\varkappa|\xi|^2;$$

(iv) $L^{2,N_h}(0, T)$, $\gamma = R^{N_h}$.

Then ($Q_{3,h}$) has a unique solution.

Remark. Using (a_9) it is easy to check that the above assumptions are satisfied by C, $B(t)$, $\beta(t)$, and γ given by formulas (15).

Proof. For simplicity we write N instead of N_h and omit the interval $(0, T)$ in the notation of the considered spaces. We begin with an equivalent formulation of the considered problem. Multiplying (13) in $L^{2,N}$ by a vector-valued function $\varphi \in H_1^N$ we get, after integrating by parts,

$$d(\alpha, \varphi) = l_{\beta,\gamma}(\gamma) \tag{16}$$

with

$$d(\alpha, \varphi) = (B\alpha, \varphi) - (C\alpha, \dot{\varphi}) + \langle C\alpha(T), \varphi(T)\rangle,$$

$$l_{\beta,\gamma}(\varphi) = (\beta, \varphi) + \langle \gamma, \varphi(0)\rangle.$$

Let us formulate now the problem

($Q_{4,h}$) Find an $\alpha \in L^{2,N}$ such that (16) holds for any $\varphi \in H_1^{0,N}$.

LEMMA 8. *Problems* ($Q_{3,h}$) *and* ($Q_{4,h}$) *are equivalent.*

Proof of the lemma. We have just shown that the solution α of ($Q_{3,h}$) solves also ($Q_{4,h}$). To prove the converse implication notice first that putting $\varphi_j = \delta_{js}\psi$ with an arbitrary $\psi \in C_0^\infty(0, T)$ and $j, s = 1, \ldots, N$ in (16) we see that the solution α of ($Q_{4,h}$) satisfies (13). Therefore, $\alpha \in H_1^N$ and we can perform integration by parts in the form $d(\alpha, \varphi)$ obtaining

$$d(\alpha, \varphi) = (\beta, \varphi) + \langle C\alpha(0), \varphi(0)\rangle \tag{17}$$

for any $\varphi \in H_1^N$. Comparing (17) with (16) we get

$$\langle C\alpha(0), \varphi(0)\rangle = \langle \gamma, \varphi(0)\rangle$$

for $\varphi \in H_1^{0,N}$, and so (14) holds.

Notice that we have also shown the following

LEMMA 9. *If* α *solves* ($Q_{4,h}$), *then* (16) *holds for any* $\varphi \in H_1^N$.

LEMMA 10. *For any* $\varphi \in H_1^N$ *we have*

$$d(\varphi, \varphi) \geqslant \varkappa |||\varphi|||^2,$$

where $|||\varphi||| = \left(\|\varphi\|_0^2 + |\varphi(0)|^2 + |\varphi(T)|^2\right)^{1/2}$.

The lemma follows from (iii) if we notice that integrating by parts and using (i) we get

$$(C\varphi, \dot{\varphi}) = -(C\varphi, \dot{\varphi}) + \langle C\varphi, \varphi\rangle|_0^T$$

and, consequently,

$$(C\varphi, \dot{\varphi}) = \tfrac{1}{2}\langle C\varphi, \varphi\rangle|_0^T.$$

To prove the theorem it is sufficient now to consider problem ($Q_{4,h}$). The uniqueness follows from Lemmas 8–10. To prove the existence of the solution we use the method given in [4]. For fixed $\varphi \in H_1^{0,N}$ the linear functional $v \mapsto d(v, \varphi)$ is continuous over $L^{2,N}$, so it may be written in the form

(18) $$d(v,\varphi)=(v,S\varphi)\quad (v\in L^{2,N}(0,T))$$

with $S\colon H_1^{0,N}\mapsto L^{2,N}$. Suppose that for some $v_0\in L^{2,N}$ we have $(v_0,S\varphi)=0$ identically in φ. According to (18) this means that v_0 solves problem $(Q_{4,h})$ with vanishing data, and therefore, by the above-proved uniqueness, $v_0=0$. Thus $\operatorname{Im} S$ is dense in $L^{2,N}$. In view of Lemma 10, from (18) we get

$$\varkappa|||\varphi|||^2\leqslant(\varphi,S\varphi)\leqslant|||\varphi|||\cdot\|S\varphi\|_0,$$

so

(19) $$|||\varphi|||\leqslant\varkappa^{-1}\|S\varphi\|_0.$$

As obviously

(20) $$|l_{\beta,\gamma}(\varphi)|\leqslant(\|\beta\|_0+|\gamma|)\,|||\varphi|||,$$

the mapping $l\colon S\varphi\mapsto L_{\beta,\gamma}(\varphi)$ is a continuous linear functional over a dense subset in $L^{2,N}$ and it may be therefore extended, by continuity, to the whole space. According to the Riesz theorem there is a $u\in L^{2,N}$ satisfying $l(\psi)=(u,\psi)$ for all $\psi\in L^{2,N}$. Putting $\psi=S\varphi$ we see that u is the desired solution of $(Q_{4,h})$.

PROPOSITION 5. *Under the assumptions of Theorem* 1, *if* $B_{kj}\in C^l[0,T]$ *and* $\beta\in H_l^N$, *then* $\alpha\in H_{l+1}^N$ *and*

(21) $$\|\alpha\|_{l+1}\leqslant\mu_l(\|\beta\|_l+|\gamma|)$$

with some positive constant μ_l *depending on* $\sup\limits_{[0,T]}|B^{(m)}|$ $(m=0,1,\ldots,l)$, $|C|$, $\varkappa$, *and* l.

Proof. We have

$$\|\alpha\|_0=\sup_{\varphi\in H_1^{0,N}}\frac{|(\alpha,S\varphi)|}{\|S\varphi\|_0},$$

which is equivalent to

$$\|\alpha\|_0=\sup_{\varphi\in H_1^{0,N}}\frac{|l_{\beta,\gamma}(\varphi)|}{\|S\varphi\|_0}.$$

Thus it follows from (19) and (20) that

(22) $$\|\alpha\|_0\leqslant\varkappa^{-1}(\|\beta\|_0+|\gamma|).$$

Since $\dot\alpha=C^{-1}(\beta-B\alpha)$ by (13), we have

(23) $$\|\dot\alpha\|_0\leqslant|C^{-1}|(\|\beta\|_0+\sup_{[0,T]}|B|\,\|\alpha\|_0),$$

and for $l=0$ we get (21) from (22) and (23). The general case may be easily derived by induction.

It follows from Theorem 1 and Proposition 5 that problem $(Q_{2,h})$ has a unique solution. Going back to our basic problem (P_1), we define its approximate solution U putting

$$U(x, t) = \tilde{U}(y, s),$$

where (x, t) and (y, s) are connected by (4). Using formula (12) we can write this in terms of basis functions

$$U(x, t) = \sum_{j=1}^{N_h} \alpha_j(t) p_j(x, t),$$

where $p_j(x, t) = v_j(y)$, and the coefficients α_j are obtained by solving $(Q_{3,h})$ or, equivalently, $(Q_{4,h})$. Obviously, $p_j(\cdot, t) \in V_{(t)}$ according to (6).

We have the following theorem concerning the error $e = u - U$:

THEOREM 2. *There is a constant* $\delta > 0$ *(depending on* T, $\sup|a_{jk}|$, $\sup|a_j|$, $\sup|a|$, *the constant* c *in* (a_6), *and the family* $F_{(t)}$*) such that*

$$\sup_{0 \leq t \leq T} \|e(\cdot, t)\|^2_{0,\Omega_{(t)}} + \int_0^T \|e(\cdot, t)\|^2_{1,\Omega_{(t)}} dt \tag{24}$$

$$\leq \delta \Big(\sup_{0 \leq t \leq T} \|\tilde{u}(\cdot, t) - v(\cdot, t)\|^2_{0,\Omega} + \int_0^T \|\tilde{u}(\cdot, t) - v(\cdot, t)\|^2_{1,\Omega} dt + $$

$$+ \int_0^T \|\tilde{u}_t(\cdot, t) - v_t(\cdot, t)\|^2_{0,\Omega} dt \Big)$$

for any $v \in H(V_h)$.

Proof. Replacing e by $\tilde{e} = \tilde{u} - \tilde{U}$ and $\Omega_{(t)}$ by Ω on the left-hand side of inequality (24), we can prove it using the same arguments as in [3], Theorem 3.1. Now (24) follows from Lemma 7. Note that both sides are finite according to Lemma 4.

We consider now more closely the case where Ω is a polyhedron and V_h a finite element space. In Theorem 3 below we denote by $\partial/\partial t$ or d/dt the derivation in the classical sense.

THEOREM 3. *Let* V_h *be a finite element space connected with the triangulation* T_h *of* Ω *and let* (f_1)–(f_5) *hold. We suppose also that*

(i) $\tilde{u}$, $\tilde{u}_t \in L^2(0, T; H_{r+1}(\Omega))$ *with* $r+1 > n/2 + s$,

(ii) $(\partial/\partial t) D_y^\alpha \tilde{u}$ *exists and is continuous in* $\bar{\Delta}_T$ *for* $\alpha \in A$.

Then

$$\sup_{0 \leq t \leq T} \|e(\cdot, t)\|^2_{0,\Omega_{(t)}} + \int_0^T \|e(\cdot, t)\|^2_{1,\Omega_{(t)}} dt \tag{25}$$

$$\leq \delta \eta h^r \Big(\sup_{0 \leq t \leq T} |\tilde{u}(\cdot, t)|^2_{r+1,\Omega} + \int_0^T |\tilde{u}(\cdot, t)|^2_{r+1,\Omega} dt + \int_0^T |\tilde{u}_t(\cdot, t)|^2_{r+1,\Omega} dt \Big)^{1/2}$$

with η *as in Theorem* A.

Proof. Using Sobolev's imbedding theorem (see [1]) we conclude from (i) and Lemma 4 that $D_y^\alpha \tilde{u}$ is continuous in $\bar{\Delta}_T$ for $|\alpha| \leqslant s$, particularly for $\alpha \in A$. Therefore, $D_y^\alpha \tilde{u}(\mathring{y}, t)$ is a well-defined function of $t \in [0, T]$ for any fixed $\mathring{y} \in \bar{\Omega}$. It follows from (ii) that

$$\frac{\partial}{\partial t} D_y^\alpha \tilde{u} = D_t D_y^\alpha \tilde{u} \quad \text{in } \Delta_T,$$

and therefore

$$D_t\big(D_y^\alpha \tilde{u}(\mathring{y}, t)\big) = (D_t D_y^\alpha \tilde{u})(\mathring{y}, t) \tag{26}$$

for $\alpha \in A$, $\mathring{y} \in \bar{\Omega}$, $t \in [0, T]$. To obtain (25) we put

$$v(\cdot, t) = \Pi_h \tilde{u}(\cdot, t),$$

where Π_h is the interpolation operator corresponding to the finite element space V_h. By (26) there is $(\Pi_h v)_t = \Pi_h v_t$ and our assertion follows from Theorem A.

6. Galerkin approximations in time. We have shown in the preceding section that the approximate solution of problem (P_1) may be reduced to a standard Galerkin approximation of problem (P_2) in the cylindrical time-space domain or, equivalently, to the solution of the initial value problem for a system of ordinary differential equations $(Q_{3,h})$. There are many papers dealing with discretization of this system by various difference methods (see [3] and [6]–[9], where other references are given). In this paper we try to discuss the Galerkin approximations of problem $(Q_{3,h})$. It is clear that in this manner we obtain approximate solutions of $(Q_{2,h})$ continuous (or more smooth) in time and, consequently, solutions of our basic problem (P_1).

In the sequel we suppose that the space V_h has been chosen and we omit the index h to simplify the formulas. Of course, all constants in the obtained estimates depend on the fixed system of basis functions $\{v_j\}_{j=1}^{N_h}$.

By Lemmas 8–10 it is sufficient to consider the problem

$(Q_{5,h})$ Find an $\alpha \in H_1^N(0, T)$ such that (16) holds for any $\varphi \in H_1^N(0, T)$.

We assume that conditions (i)–(iv) of Theorem 1 hold and choose the finite-dimensional subspace X_τ of $H_1^N(0, T)$ with the basis $\{q_m\}_{m=1}^{M_\tau}$. Now the approximate problem to $(Q_{5,h})$ is formulated as follows:

$(Q_{h,\tau}^*)$ Find an $\alpha^* \in X_\tau$ such that

$$d(\alpha^*, \varphi) = l_{\beta,\gamma}(\varphi) \tag{27}$$

holds for any $\varphi \in X_\tau$.

Putting $e^* = \alpha - \alpha^*$ we have the following estimate:

Theorem 4. *There is a positive constant* $\mu = \max(B_0, |C|)$ *such that*

$$|||e^*||| \leqslant \mu \varkappa^{-1} \inf_{\varphi \in X_\tau} \|\alpha - \varphi\|_1. \tag{28}$$

Proof. It follows from (16) and (27) that for any $\varphi \in X_\tau$ we have $d(e^*, \varphi) = 0$, and therefore

$$d(e^*, e^*) = d(e^*, \alpha - \varphi).$$

As for $\psi \in H_1^N(0, T)$ we have

$$d(e^*, \psi) \leqslant \mu |||e^*||| \cdot ||\psi||_1,$$

so (28) follows from Lemma 10.

Concerning the finite element approximations it is easy to prove the following

THEOREM 5. *Let T_τ be a triangulation of the segment $[0, T]$ and let (f_1)–(f_5) hold with h replaced by τ and r replaced by l. Denoting by Y_τ the corresponding finite element space we put $X_\tau = (Y_\tau)^N$ and suppose that*

(i) *$l \geqslant s$ (so there is no restriction in the case of finite elements of Lagrange type);*

(ii) *$\beta \in H_l^N(0, T)$, $B_{kj} \in C^l[0, T]$.*

Then $|||e^*||| \leqslant \omega\tau^l$ with $\omega = \mu\eta\,\mu_l \varkappa^{-1}(||\beta||_l + |\gamma|)$, η as in Theorem A.

Proof. It follows from Proposition 5 that $\alpha \in H_{l+1}^N$, and therefore, by Theorem A,

$$||\alpha - \Pi_\tau^* ||_1 \leqslant \eta\tau^l ||\alpha^{(l+1)}||_0, \tag{29}$$

where Π_τ^* denotes the interpolation operator connected with the space Y_τ (we put $\Pi_\tau^* \alpha = (\Pi_\tau^* \alpha_1, \ldots, \Pi_\tau^* \alpha_N)$). Thus our assertion is easily obtained if we take $\varphi = \Pi_\tau^* \alpha$ in (28) and use Proposition 5 together with (29).

Going back to problem (P_1) let us put

$$U^*(x, t) = \sum_{j=1}^{N_h} \alpha_j^*(t) p_j(x, t),$$

where α^* solves $(Q_{h,\tau})$. Then

$$U - U^* = \sum_{j=1}^{N_h} (\alpha_j - \alpha_j^*) p_j(x, t) = \sum_{j=1}^{N_h} (\alpha_j - \alpha_j^*) v_j(y)$$

with $x = F_{(t)}(y)$ and, using Lemma 5, we get

$$\int_0^T ||U - U^*||_{0,\Omega_{(t)}}^2 dt \leqslant c ||\alpha - \alpha^*||_0^2 \sum_{j=1}^{N_h} ||v_j||_{1,\Omega}^2 \tag{30}$$

with the positive constant c depending on the family $F_{(t)}$. The right-hand side of (30) may now be estimated by means of Theorem 5.

Using the decomposition in terms of basis functions

$$\alpha^* = \sum_{m=1}^{M_\tau} \bar{\alpha}_m q_m,$$

we see that problem $(Q^*_{h,\tau})$ is equivalent to the system of linear equations

$$\sum_{m=1}^{M_\tau} \bar{\alpha}_m d(q_m, q_k) = l_{\beta,\gamma}(q_k) \quad (k = 1, \ldots, M_\tau).$$

It follows from Lemma 10 that the matrix $[d(q_m, q_k)]$ is positive definite, and therefore non-singular, so problem $(Q^*_{h,\tau})$ is uniquely solvable. Note that the matrix $[d(q_m, q_k)]$ is sparse when X_τ is constructed by the finite element method; particularly, it is three-diagonal if X_τ consists of sectionally linear splines.

7. Galerkin approximations in space and time. Using formulation (P_3) of our initial-boundary value problem we are led to simultaneous Galerkin approximations in space- and time-variables. Let namely Z_h be a linear finite-dimensional subspace of $H(V)$ with the basis $\{\tilde{z}_j\}_{j=1}^{P_h}$. We formulate the approximate problem as follows:

(R_h) Find a function $\tilde{W} \in Z_h$ such that

$$B(\tilde{W}, w) = l_{g,u_0}(w) \tag{31}$$

holds for all $w \in Z_h$.

Using the decomposition of $\tilde{W}$ in terms of basis functions

$$\tilde{W}(y, t) = \sum_{j=1}^{P_h} \xi_j \tilde{z}_j(y, t)$$

we see that (31) is equivalent to the linear algebraic system

$$\sum_{j=1}^{P_h} \xi_j B(\tilde{z}_j, \tilde{z}_k) = l_{g,u_0}(\tilde{z}_k) \quad (k = 1, \ldots, P_h).$$

Its matrix is positive definite according to Proposition 3, so it is non-singular, and thus problem (R_h) has a unique solution. Going back to problem (P_1) we define its approximate solution as $W(x, t) = W(y, t)$ or, equivalently, by the formula

$$W(x, t) = \sum_{j=1}^{P_h} \xi_j z_j(x, t),$$

where $z_j(x, t) = \tilde{z}_j(y, t)$ with $x = F_{(t)}(y)$.

To estimate the error of the approximation $E = u - W$ let us put, for $v \in H_1(D_T)$,

$$[v]_{D_T} = \Big(\int_0^T \|v\|^2_{1,\Omega(t)} dt + \|v(\cdot, 0)\|^2_{0,\Omega(t)} + \|v(\cdot, T)\|^2_{0,\Omega(t)}\Big)^{1/2}.$$

THEOREM 6. *There is a positive constant* γ *(depending on* T, $\sup|a_{jk}|$, $\sup|a_j|$, $\sup|a|$, *the constant* c *in* (a$_6$), *and the family* $F_{(t)}$*) such that*

$$[E]_{D_T} \leqslant \gamma \inf_{z\in Z_h} \|\tilde{u}-z\|_{1,\Delta_T}.$$

Proof. It follows from (31) and (9), in view of Proposition 1, that

$$B(\tilde{u}-\tilde{W}, z) = 0 \quad \text{for } z \in Z_h.$$

Therefore

$$B(\tilde{E}, \tilde{E}) = B(\tilde{u}-\tilde{W}, \tilde{u}-z)$$

and, using Propositions 3 and 4, we obtain

$$[\tilde{E}] \leqslant cd^{-1} \inf_{z\in Z_h} \|\tilde{u}-z\|_{1,\Delta_T}.$$

Thus our assertion follows from Lemma 7.

Let us consider now the case of finite element approximations assuming Ω to be a polyhedron.

THEOREM 7. *Let* Z_h *be a finite element space connected with the triangulation* T_h *of* Δ_T *and let* (f$_1$)–(f$_5$) *hold. We suppose that* $\tilde{u} \in H_{r+1}(\Delta_T)$ *with* $r+1 > n/2+s$. *Then*

$$[E]_{D_T} \leqslant \gamma\eta h^r |\tilde{u}|_{r+1,\Delta_T}$$

with η *as in Theorem* A.

The theorem follows immediately from Theorem A and Theorem 6 if we put $z = \Pi_h \tilde{u}$.

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