ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVI, 1 (1977)

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A BISECTION METHOD FOR THE TRAVELING SALESMAN PROBLEM

The note deals with a new primal method for the traveling salesman problem. The fundamental step of the method is very similar to that of Netter [7] but some modifications with the idea of bisection cause the presented algorithm to be faster than Netter's algorithm.

1. Introduction. The *traveling salesman problem* (TSP) is stated as follows. Let $D_d = \langle X, U; d\rangle$ be the digraph with arc weights, where X is the set of vertices, $U \subseteq X \times X$, and d is a real function, $d\colon U \to \boldsymbol{R}_+ \cup \{0\}$. Let $|X| = n$. Then determine a minimum Hamiltonian circuit of the network D_d.

The TSP is one of the problems for which an efficient algorithm has not been presented yet (the efficiency in the sense of Edmonds, see [4]). This short note deals with a new method for the TSP.

2. The method. Let t^* denote the length of a minimum Hamiltonian circuit of the network D_d. There exist some lower and upper bounds to t^* and methods for examination of the inequality $t^* \leqslant t$ for any real number t. Let t_1 and t_u denote the lower and upper bounds, respectively, to the optimal solution of the TSP, i.e., $t_1 \leqslant t^* \leqslant t_u$. Then the following is the representation of our proposed algorithm:

Step 1. $t \leftarrow t_1 + 0.5\,(t_u - t_1)$.

Step 2. $t_u \leftarrow t'$ if $t^* \leqslant t$, and $t_1 \leftarrow t$ otherwise, where t' ($t^* \leqslant t' \leqslant t$) is the cost of the Hamiltonian circuit obtained as the result of examination of the inequality $t^* \leqslant t$.

Repeat steps 1 and 2 until a required accuracy is attained, i.e., until $t_u - t_1 \leqslant \varepsilon$, where ε is a fixed small number.

The number of iterations of steps 1 and 2 for the TSP, with an integer function d and integer variables t_1, t_u and t, is not greater than $\log_2(t_u - t_1)$, where t_1 and t_u are the initial bounds to the optimal solution t^*.

The efficiency of the algorithm depends on the efficiencies attained at the following stages:

(i) computation of an initial value of t_l;
(ii) computation of an initial value of t_u;
(iii) examination of the inequality $t^* \leqslant t$.

As regards (i), the optimal solution of the assignment problem (AP) for the matrix $D = (d_{ij})$ is the lower bound to the optimal solution of the TSP for the function $d([i,j]) = d_{ij}$ $(i,j = 1, 2, \ldots, n)$. A better lower bound can be obtained by Christofides' method [1], resulting in the order of difficulty of this stage of the algorithm being n^3.

With respect to stage (ii), the length of an arbitrary Hamiltonian circuit of the network D_d, particularly the length of the Hamiltonian circuit obtained by the nearest-city method, is the upper bound to the optimal solution of the TSP. At this stage, the order of difficulty is n^3.

In regard to (iii), the inequality $t^* \leqslant t$ can be examined by Netter's method [7]. Let us define, for a fixed number t (step 2 of the algorithm), the network $D_{d,v} = \langle X, U; d, v\rangle$ derived from the network D_d and such that

$$v: X \to \boldsymbol{R}_- \cup \{0\} \quad \text{and} \quad \sum_{i=1}^{n} t_i = -t, \quad \text{where } v(i) = t_i \ (i = 1, 2, \ldots, n).$$

Let the length of a circuit $[i_1, i_2, \ldots, i_k, i_l]$ in the network $D_{d,v}$ be defined as

$$\sum_{j=1}^{k} d_{i_j i_{j+1}} + \sum_{j=1}^{k} t_{i_j}, \quad \text{where } i_{k+1} = i_l.$$

It is easy to see that the inequality $t^* \leqslant t$ is satisfied for the network D_d if and only if there exists a non-positive Hamiltonian circuit in the network $D_{d,u}$. To detect the circuit we can apply the direct method for finding a non-positive Hamiltonian circuit in the network $D_{d,v}$ (see [3] or [7]). Unfortunately, the direct method is not efficient in the accepted sense. Murty [6] has presented another method for examination of the inequallity $t^* \leqslant t$ which also is inefficient.

3. Computational experience. The algorithm was tested on the ODRA 1204 computer, and the computation time was found to depend upon a partition of $-t$ into $t_1, t_2, \ldots, t_n$ and very little upon values of the lower and upper bounds to the optimal solution t^*. Also, the algorithm was compared with Netter's algorithm.

Two different ways of the partition of $-t$ were tested. According to Netter's suggestion, the first was formed from an n-tuple (different for different t) $t_1', t_2', \ldots, t_n'$ of a sequence of uniformly distributed random numbers and

$$t_i = \frac{-t_i' t}{\sum_{j=1}^{n} t_j'},$$

and in the second a generic element t_i was in the direct proportion to

$$-\sum_{j=1,j\neq i}^{n}\frac{d_{ij}}{n-1}.$$

The times (in sec.) of computations by the algorithms for some problems are shown in Table 1.

TABLE 1

n	Netter's algorithm		The bisection method	Problem source
	the first partition of $-t$	the second partition of $-t$	the second partition of $-t$	
10	32	23	17	[5]
10	30	10	11	Problem obtained from the previous one by symmetrization of the matrix D.
10	457	214	167	[2]

Computational time of the new algorithm has been considerably saved due to the new method of generating t and to the partition of it. The reason of the latter is that the second way of partitioning takes values of matrix elements into consideration more than the first one.

The time of examination of the inequality $t^* \leqslant t$ for both algorithms was in the inverse proportion to $|t-t^*|$.

References

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Received on 25. 9. 1976

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METODA POŁOWIENIA DLA ROZWIĄZYWANIA ZAGADNIENIA KOMIWOJAŻERA

STRESZCZENIE

W pracy przedstawiono modyfikację metody Nettera dla rozwiązywania zagadnienia komiwojażera. Zasadniczy krok algorytmu polega na wyznaczaniu kolejnego przybliżenia przez połowienie przedziału zawierającego szukane rozwiązanie. Obie metody zostały porównane na kilku przykładach. Zaproponowana modyfikacja wpłynęła na zmniejszenie czasu obliczeń na maszynie cyfrowej.
