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ON LIMIT DISTRIBUTION THEOREMS FOR SUMS OF A RANDOM NUMBER OF RANDOM VARIABLES APPEARING IN THE STUDY OF RAREFACTION OF A RECURRENT PROCESS

1. Introduction. In [1] it is observed that the limit laws for processes obtained by the rarefaction of a renewal process can be expressed as limiting distribution theorems for sums of a random number of independent random variables. Taking this into account one can also prove limiting distribution theorems concerning the rarefaction of the process $\sum_{k=1}^{n} X_k$ $(n = 1, 2, \ldots)$, where X_k are independent and identically distributed random variables which can take on arbitrary (not only non-negative) values. In spite of the fact that the intuitive meaning of the rarefaction is lost in this case, some investigations concerning the rarefaction of the mentioned process lead to interesting results (see, e.g., [8] and [12]).

Let us begin with a look at the model of rarefaction described in [9] (see also [12]). Let $F(x)$ be the distribution function of the time interval between consecutive renewal points of a recurrent process for which

$$\mu = \int_0^\infty x\,dF(x) < \infty.$$

The time intervals between consecutive renewal points $T_0 \equiv 0 \leqslant T_1 \leqslant T_2 \leqslant \ldots$ are independent and commonly distributed random variables $X_1, X_2, \ldots, X_n, \ldots$ We rarefy the process $T_0 \equiv 0 \leqslant T_1 \leqslant T_2 \leqslant \ldots$ in such a way that every its event, independently one of another, be maintained with probability q and cancelled with probability $1-q$ for $0 < q < 1$. After the first rarefaction the time distance between the consecutive renewal points of the new process is

$$S_{\nu_1} = X_1 + X_2 + \ldots + X_{\nu_1}, \tag{1}$$

where $\nu_1 - 1$ is the number of cancelled events. The random variable ν_1

is independent of every X_n and its distribution is equal to

$$P[\nu_1 = k] = q(1-q)^{k-1} \quad (k = 1, 2, \ldots).$$

The time distance between consecutive events after the n-th rarefaction is

$$S_{\nu_n} = X_1 + X_2 + \ldots + X_{\nu_n},$$

where ν_n is also independent of summands and its distribution is equal to

$$P[\nu_n = k] = q^n(1-q^n)^{k-1} \quad (k = 1, 2, \ldots).$$

The result of [9] asserts that if $\mu > 0$, then

$$\lim_{n\to\infty} P[q^n S_{\nu_n} < x] = \begin{cases} 1-\exp\left(\frac{-x}{\mu}\right) & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [2] and [4], all possible limiting distributions for type (1) and the domains of attraction of limiting distributions in any cases are given for $0 < q < 1$. The above-mentioned results can also be obtained, by a simple transformation, in the case where the mathematical expectation is negative. As it has been proved in [8] and [12], the situation is different in the case where the mean value is 0.

The aim of this note is to extend some limit theorems concerning the rarefaction of a recurrent process to the case of non-identically distributed random variables. Moreover, we are studying the rarefaction of a recurrent process allowing the number of operations of the rarefaction to be a random variable. The obtained results generalize theorems contained in [8], [9] and [12].

2. Rarefaction of a process having only finite mean values. In what follows we need the following theorems, being extensions of the results given in [7].

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with finite expectations $\mu_n = EX_n$ such that*

$$n^{-1}\sum_{k=1}^{n}\mu_k \to \mu \quad \textit{as } n\to\infty,$$

where μ is finite. Moreover, we suppose that $\{X_n, n \geqslant 1\}$ satisfies the weak law of large numbers.

Let further $\{\nu_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that ν_n does not depend on $X_1, X_2, \ldots$ and, for a sequence $\{a_n, n \geqslant 1\}$, $a_n\to\infty$ as $n\to\infty$,

$$\lim_{n\to\infty} P\left[\frac{\nu_n}{a_n} < x\right] = G(x) \quad (G(+0) = 0)$$

exists, where $G(x)$ is a distribution function.

Then

$$\lim_{n\to\infty} P\left[\frac{S_{\nu_n}}{a_n} < x\right] = \begin{cases} G\left(\dfrac{x}{\mu}\right) & \text{if } \mu > 0, \\ D(x) & \text{if } \mu = 0, \\ 1 - G\left(-\dfrac{x}{\mu}+0\right) & \text{if } \mu < 0, \end{cases}$$

where $D(x) = 0$ *if* $x \leqslant 0$, *and* $D(x) = 1$ *if* $x > 0$.

Proof. It is enough to prove that

$$\frac{S_{\nu_n}}{\nu_n} \xrightarrow{P} \mu \quad \text{as } n \to \infty.$$

By the assumption, $\nu_n, X_1, X_2, \ldots$ are independent for every n. Therefore, for every n, $k_0 > 0$ and $\varepsilon > 0$ we have

$$P\left[\left|\frac{S_{\nu_n}}{\nu_n} - \mu\right| \geqslant \varepsilon\right] \leqslant P[\nu_n < k_0] + \sum_{k=k_0}^{\infty} P\left[\left|\frac{S_k}{k} - \mu\right| \geqslant \varepsilon\right] P[\nu_n = k].$$

Since $k^{-1}S_k \xrightarrow{P} \mu$ as $k \to \infty$, we can choose, for any given $\delta > 0$, the value k_0, in the second probability statement of the previous formula, such that

$$P\left[\left|\frac{S_k}{k} - \mu\right| \geqslant \varepsilon\right] < \frac{\delta}{2} \quad \text{for every } k \geqslant k_0.$$

Thus

$$\sum_{k=k_0}^{\infty} P\left[\left|\frac{S_k}{k} - \mu\right| \geqslant \varepsilon\right] P[\nu_n = k] < \frac{\delta}{2} \quad \text{for every } n.$$

Now for this value of k_0 choose the value of n so large that

$$P[\nu_n < k_0] < \frac{\delta}{2}.$$

This can be done, for we assume $\nu_n \xrightarrow{P} \infty$ as $n \to \infty$. Thus we have proved that

$$\frac{S_{\nu_n}}{\nu_n} \xrightarrow{P} \mu \quad \text{as } n \to \infty.$$

Now, since

$$P\left[\frac{S_{\nu_n}}{a_n} < x\right] = P\left[\frac{S_{\nu_n}}{\nu_n}\frac{\nu_n}{a_n} < x\right],$$

we see that Theorem 1 follows from Cramer's lemma.

To prove the next theorem we need the following

LEMMA 1 (Heyde [3]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with finite expectations $\mu_n = \mathrm{E}X_n$ such that*

$$n^{-1}\sum_{k=1}^{n}\mu_k \to \mu \quad \textit{as } n\to\infty,$$

where $\mu \geqslant 0$ is finite. If $n^{-1}S_n \xrightarrow{\mathrm{P}} \mu$ as $n\to\infty$, then

$$n^{-1}\max_{1\leqslant k\leqslant n} S_k \xrightarrow{\mathrm{P}} \mu \quad \textit{and} \quad n^{-1}\max_{1\leqslant k\leqslant n}|S_k| \xrightarrow{\mathrm{P}} \mu \quad \textit{as } n\to\infty.$$

Using this lemma we can prove a stronger result than Theorem 1:

THEOREM 1′. *Under the assumption of Theorem 1 with $\mu \geqslant 0$ we have*

$$\lim_{n\to\infty}\mathrm{P}[a_n^{-1}\max_{1\leqslant k\leqslant \nu_n} S_k < x] = \lim_{n\to\infty}\mathrm{P}[a_n^{-1}\max_{1\leqslant k\leqslant \nu_n}|S_k| < x] = \begin{cases} G\left(\dfrac{x}{\mu}\right) & \textit{if } \mu > 0, \\ D(x) & \textit{if } \mu = 0. \end{cases}$$

Proof. By Lemma 1, we have

$$\frac{\max\limits_{1\leqslant k\leqslant n} S_k}{n} \xrightarrow{\mathrm{P}} \mu \quad \text{and} \quad \frac{\max\limits_{1\leqslant k\leqslant n}|S_k|}{n} \xrightarrow{\mathrm{P}} \mu \quad \text{as } n\to\infty.$$

Hence, supposing that $\nu_n \xrightarrow{\mathrm{P}} \infty$ as $n\to\infty$, we can prove, in the same way as for S_{ν_n}/ν_n, that

$$\frac{\max\limits_{1\leqslant k\leqslant \nu_n} S_k}{\nu_n} \xrightarrow{\mathrm{P}} \mu \quad \text{and} \quad \frac{\max\limits_{1\leqslant k\leqslant \nu_n}|S_k|}{\nu_n} \xrightarrow{\mathrm{P}} \mu \quad \text{as } n\to\infty.$$

Now, as in the proof of Theorem 1, it is enough to use Cramer's lemma.

By Theorems 1 and 1′ we obtain the following theorem and corollary which are a strengthening and an extension of Rényi's result [9].

THEOREM 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with finite expectations $\mu_n = \mathrm{E}X_n$ such that*

$$n^{-1}\sum_{k=1}^{n}\mu_k \to \mu \quad \textit{as } n\to\infty,$$

where μ is finite and strictly positive. Moreover, we suppose that $n^{-1}S_n \xrightarrow{\mathrm{P}} \mu$ as $n\to\infty$. If $\{\nu_n, n \geqslant 1\}$ is a sequence of geometrically distributed random variables with parameter q^n $(0 < q < 1,\ n = 1, 2, \ldots)$ and such that, for *every* $n \geqslant 1$, ν_n *is independent of the sequence* $\{X_n, n \geqslant 1\}$, *then*

$$\lim_{n\to\infty} \mathrm{P}[q^n S_{\nu_n} < x] = \lim_{n\to\infty} \mathrm{P}[q^n \max_{1\leqslant k\leqslant \nu_n} S_k < x] = \lim_{n\to\infty} \mathrm{P}[q^n \max_{1\leqslant k\leqslant \nu_n} |S_k| < x] = \begin{cases} 1-\exp\left(-\dfrac{x}{\mu}\right) & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Indeed, by Theorems 1 and 1′ it is enough to observe that

$$\lim_{n\to\infty} \mathrm{P}[q^n \nu_n < x] = 1-\exp\left(-\frac{x}{\mu}\right).$$

COROLLARY 1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent, identically distributed random variables with finite expectation* $\mathrm{E}X_1 = \mu > 0$, *and let* $\{\nu_n, n \geqslant 1\}$ *be a sequence of geometrically distributed random variables with parameter* q^n $(0 < q < 1,\ n = 1, 2, \ldots)$ *and such that, for every* $n \geqslant 1$, ν_n *is independent of the sequence* $\{X_n, n \geqslant 1\}$. *Then* (2) *holds.*

It is obvious that the random variables of Corollary 1 satisfy the assumptions of Theorem 2. Therefore, we have (2).

Following the investigations in [6] and [8], we consider also a more general rarefaction when ν_n is an arbitrary positive integer-valued random variable (see the definitions of ν_1 in Section 1) for which $1 < a = 1/q = \mathrm{E}\nu_1 < \infty$ and $0 < \sigma^2\nu_1 < \infty$. If $f(z)$ for $|z| < 1$ is the generating function of a random variable, then ν_n is the n-th iteration of $f(z)$, i.e. it is $f(z)$. For such random variables we have

LEMMA 2 (Mogyoródi [8]). *Suppose that* $0 < \sigma^2\nu_1 < \infty$. *Then*

$$\lim_{n\to\infty} \mathrm{P}[q^n \nu_n < x] = G(x) \tag{3}$$

exists, where $G(x)$ *is a distribution function with mean value* 1 *and variance* $\sigma^2\nu_1/(a^2-a)$. *Furthermore,* $G(x)$ *has a probability density function, and the limiting distribution* (3) *belongs to the class of limiting distributions for Galton-Watson processes with* $a > 1$.

Using Lemma 2 we can obtain an extension of a result given in [6] (see also [8]).

THEOREM 3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables with finite expectations* $\mu_n = \mathrm{E}X_n$ *such that*

$$n^{-1}\sum_{k=1}^{n} \mu_k \to \mu \quad \text{as } n\to\infty,$$

where $\mu \neq 0$ *is finite. Moreover, suppose that* $n^{-1}S_n \xrightarrow{\mathrm{P}} \mu$ *as* $n\to\infty$. *If*

$\{\nu_n, n \geqslant 1\}$ is the above-given sequence with $0 < \sigma^2 \nu_1 < \infty$, then

$$(4) \qquad \lim_{n\to\infty} P[q^n S_{\nu_n} < x] = \lim_{n\to\infty} P[q^n \max_{1\leqslant k\leqslant \nu_n} S_k < x]$$

$$= \lim_{n\to\infty} P[q^n \max_{1\leqslant k\leqslant \nu_n} |S_k| < x] = H_\mu(x) = G\left(\frac{x}{\mu}\right) \quad \textit{if } \mu > 0$$

and

$$(5) \qquad \lim_{n\to\infty} P[q^n S_{\nu_n} < x] = H_\mu(x) = G\left(-\frac{x}{\mu}\right) \quad \textit{if } \mu < 0.$$

Assertions (4) and (5) follow immediately from Theorems 1 and 1′

3. Rarefaction of a process with mean values equal to zero and finite variances.

THEOREM 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $\mathrm{E}X_n = 0$ $(n = 1, 2, \ldots)$ and finite variances satisfying Lindeberg's condition and such that*

$$(6) \qquad \lim_{n\to\infty} \frac{s_n^2}{n} = \frac{2}{\lambda^2}, \quad \textit{where } s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \textit{ and } \lambda > 0.$$

If $\{\nu_n, n \geqslant 1\}$ is a sequence of geometrically distributed random variables with parameter q^n $(0 < q < 1,\ n = 1, 2, \ldots)$ and such that, for every n, ν_n is independent of the sequence $\{X_n, n \geqslant 1\}$, then

$$(7) \qquad \lim_{n\to\infty} P[\sqrt{q^n} S_{\nu_n} < x] = \int_{-\infty}^{x} \frac{\lambda}{2} e^{-\lambda|y|/2} dy.$$

Proof. Let $\{Y_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables with $\mathrm{E}Y_1 = 0$ and $\sigma^2 Y_1 = 2/\lambda^2$. In what follows we put

$$S_n' = \sum_{k=1}^{n} Y_n.$$

It is known [12] that

$$\lim_{n\to\infty} P[\sqrt{q^n} S_{\nu_n}' < x] = \int_{-\infty}^{x} \frac{\lambda}{2} e^{-\lambda|y|} dy.$$

Further on, let for any given $\varepsilon > 0$

$$X_k^* = X_k I[|X_k| < m\varepsilon], \qquad X_k^{**} = X_k I[|X_k| \geqslant m\varepsilon] \quad (k = 1, 2, \ldots, m),$$

where $I[A]$ denotes the indicator of A, and let us set

$$S_m^* = \sum_{k=1}^{m} X_k^* \quad \text{and} \quad s_m^{*2} = \sum_{k=1}^{m} \sigma^2 X_k^*.$$

Finally, let us put

$$\varphi_k(t) = \mathrm{E}\exp(itX_k), \quad \varphi_k^*(t) = \mathrm{E}\exp(itX_k^*), \quad \varphi_k^{**}(t) = \mathrm{E}\exp(itX_k^{**}).$$

Now it is easy to see that

$$(8) \quad |\varphi_{\sqrt{q^n}S_{\nu_n}}(t) - \varphi_{\sqrt{q^n}S'_{\nu_n}}(t)| \leqslant \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} |\varphi_{S_m}(\sqrt{q^n}t) - \varphi_{S_m^*}(\sqrt{q^n}t)| +$$

$$+ \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} |\varphi_{S_m^*}(\sqrt{q^n}t) - \varphi_{S'_m}(\sqrt{q^n}t)|$$

$$= \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \Big| \prod_{k=1}^{m} [\varphi^*(\sqrt{q^n}t) - \varphi^{**}(\sqrt{q^n}t)] - \prod_{k=1}^{m} \varphi^*(\sqrt{q^n}t) \Big| +$$

$$+ \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \frac{q^n t^2}{2} \left| s_m^{*2} - m\frac{2}{\lambda^2} + o(s_m^{*2}) - o\left(m\frac{2}{\lambda^2}\right) \right|.$$

Using the identity

$$\prod_{k=1}^{m} (u_k + v_k) - \prod_{k=1}^{m} u_k = \sum_{j=1}^{n} v_j \Big(\prod_{k<j} u_k\Big)\Big(\prod_{j<k} (u_k + v_k)\Big)$$

(where an empty product is to be replaced by 1) to estimate the first series in the last equality, we obtain

$$\sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \Big| \prod_{k=1}^{m} [\varphi^*(\sqrt{q^n}t) + \varphi^{**}(\sqrt{q^n}t)] - \prod_{k=1}^{m} (\sqrt{q^n}t) \Big|$$

$$\leqslant \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \sum_{k=1}^{m} \mathrm{P}[|X_k| \geqslant \varepsilon m]$$

$$\leqslant \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \frac{1}{m\varepsilon} \sum_{k=1}^{m} \int_{|x| \geqslant \varepsilon m} x^2 dF_k(x).$$

Now Lindeberg's condition and (6) allow us to choose, for any $\delta > 0$, an m_0 such that the series in the last formula is bounded by

$$\frac{2}{\lambda^2}\mathrm{P}[\nu_n \leqslant m_0] + \frac{\delta}{4}\mathrm{P}[\nu_n > m_0] = \frac{2}{\lambda^2}[1-(1-q^n)^{m_0+1}] + \frac{\delta}{4}(1-q^n)^{m_0+1}.$$

Hence, for sufficiently large n, we have

$$\sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} \Big| \prod_{k=1}^{m} [\varphi^*(\sqrt{q^n}t) + \varphi^{**}(\sqrt{q^n}t)] - \prod_{k=1}^{m} \varphi^*(\sqrt{q^n}t) \Big| < \frac{\delta}{2}.$$

Now we estimate the second term on the right-hand side of (8). We can observe that for any $\eta > 0$ there is m_1 such that

$$\sum_{m=1}^{\infty} q^n(1-q^n)^{m-1}\frac{q^n t^2}{2}\left|s_m^{*2} - m\frac{2}{\lambda^2} + o(s_m^{*2}) - o\left(m\frac{2}{\lambda^2}\right)\right|$$

$$\leqslant \sum_{m=1}^{\infty} q^n(1-q^n)^{m-1} q^n t^2 m \left|\frac{s_m^2}{m} - \frac{2}{\lambda^2} - \frac{1}{m}\sum_{k=1}^{m}\int\limits_{|x|\geqslant \varepsilon m} x^2 dF_k(x)\right|$$

$$\leqslant q^n t^2 m_1 \max_{1\leqslant m\leqslant m_1}\left|\frac{s_m^2}{m} - \frac{2}{\lambda^2} - \frac{1}{m}\sum_{k=1}^{m}\int\limits_{|x|\geqslant \varepsilon m} x^2 dF_k(x)\right| \mathrm{P}[\nu_n < m_1] +$$

$$+ q^n t^2 \frac{\eta}{2}\sum_{m=m_1}^{\infty} q^n(1-q^n)^{m-1} m \leqslant t^2\eta.$$

Thus, by (8), for $|t| < \sqrt{\delta/2\eta}$ and sufficiently large n, we have

$$|\varphi_{\sqrt{q^n}S_{\nu_n}}(t) - \varphi_{\sqrt{q^n}S'_{\nu_n}}(t)| < \delta.$$

Hence, our assertion follows from the fact that [12]

$$\lim_{n\to\infty}\varphi_{\sqrt{q^n}S'_{\nu_n}}(t) = \frac{\lambda^2}{\lambda^2 + t^2}$$

and that for $\eta > 0$ we can take an arbitrary small number.

Now consider the class of random variables satisfying the assumptions of Theorem 4.

Definition. The sequence $\{X_n, n \geqslant 1\}$ of independent random variables is said to satisfy *condition* (A) if there exist some positive constants x_0, C_0 and a random variable X such that

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{P}[|X_k| \geqslant x_0] \leqslant C_0\mathrm{P}[|X| \geqslant x_0].$$

It is easy to prove

LEMMA 3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $\mathrm{E}X_n = 0$ and finite variances $\sigma_n^2 = \sigma^2 X_n$. If $\{X_n, n \geqslant 1\}$ satisfies condition* (A) *with a random variable X such that $\mathrm{E}X = 0$, $\sigma^2 X = \sigma^2 = 2/\lambda^2 < \infty$, $\lambda > 0$, then for any given $\varepsilon > 0$*

$$\frac{1}{n}\sum_{k=1}^{n}\int\limits_{|x|\geqslant \varepsilon n} x^2 dF_k(x) \to 0 \quad \textit{as } n\to\infty.$$

Using Lemma 3 we can obtain the following

THEOREM 5. *Let the assumptions of Lemma 3 be satisfied. If, moreover,*

$$\lim_{n\to\infty}\frac{s_n^2}{n}=\frac{2}{\lambda^2}$$

and if $\{\nu_n, n\geqslant 1\}$ *is a sequence of random variables from Theorem 4, then* (7) *holds.*

It is enough to observe that, under the assumptions of Theorem 5, the sequence $\{X_n, n\geqslant 1\}$ satisfies Lindeberg's condition.

4. More general rarefaction process.

THEOREM 6. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of independent random variables with* $\mathrm{E}X_n=0$ *and finite variances* $\sigma_n^2=\sigma^2X_n$ *satisfying Lindeberg's condition and such that*

$$\lim_{n\to\infty}\frac{s_n^2}{n}=2\sigma^2<\infty. \tag{9}$$

Suppose that $\{\nu_n, n\geqslant 1\}$ *is a sequence of positive integer-valued random variables independent of* $\{X_n, n\geqslant 1\}$ *and such that the probability generating distribution function* $f_n(z)$ *of* ν_n $(n=1,2,\ldots)$ *is the n-th iteration of* $f(z)$, *where, for* $|z|<1$ *and* $1<f'(1)=1/q$, $f''(1)<\infty$ *is the probability generating distribution function of* ν_1.

Then

$$\lim_{n\to\infty}\mathrm{P}[\sqrt{q^n}S_{\nu_n}<x]=\int_0^\infty\Phi\left(\frac{x}{\sigma\sqrt{2z}}\right)dG(z), \tag{10}$$

where $\Phi(x)$ *is the standard normal distribution function, and* $G(x)$ *is the distribution function given by* (3).

Proof. By the assumptions concerning ν_n and by Lemma 2, we see that

$$\lim_{n\to\infty}\mathrm{P}[q^n\nu_n<x]=G(x)$$

exists, where $G(x)$ is a distribution function with mean value 1 and variance $\sigma^2\nu_1/(a^2-a)$.

Moreover, by Lindeberg's condition we have

$$\lim_{n\to\infty}\mathrm{P}\left[\frac{S_n}{\sqrt{s_n^2}}<x\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{y^2}{2}\right)dy.$$

Hence, by (9) and Cramer's lemma, we also have

$$\lim_{n\to\infty}\mathrm{P}\left[\frac{S_n}{\sigma\sqrt{2n}}<x\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{y^2}{2}\right)dy.$$

Now, using the random central limit theorem [10], we obtain

$$\lim_{n\to\infty} \mathrm{P}\left[\frac{S_{\nu_n}}{\sigma\sqrt{2\nu_n}} < x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy.$$

To use the methods of the proof of Dobrushin's lemma [1], let us denote by U and V random variables with distribution functions $G(x)$ and $\Phi(x)$, respectively. Then we can write

$$\sqrt{q^n} S_{\nu_n} = \sqrt{q^n \nu_n}\,\sigma\sqrt{2}\,\frac{S_{\nu_n}}{\sigma\sqrt{2\nu_n}} = (\sqrt{q^n \nu_n} - U)\,\sigma\sqrt{2}\left(\frac{S_{\nu_n}}{\sigma\sqrt{2\nu_n}} - V\right) +$$
$$+(\sqrt{q^n \nu_n} - U)\frac{\sqrt{2}}{\lambda} + \left(\frac{S_{\nu_n}}{\sigma\sqrt{2\nu_n}} - V\right) + \sigma\sqrt{2}\,UV.$$

Now the proof is based on the fact that if a sequence of distribution functions $H_n(z)$ for $n = 1, 2, \ldots$ is weakly convergent to $H(z)$, then a sequence $\{Z_n, n \geqslant 1\}$ of random variables can be constructed so that $\mathrm{P}[Z_n < z] = H_n(z)$ and Z_n converges in probability to a random variable Z for which $\mathrm{P}[Z < z] = H(z)$. By this fact and the construction given in [12] (details are omitted because they need only insignificant changes), we have

$$\lim_{n\to\infty} \mathrm{P}[\sqrt{q^n} S_{\nu_n} < x] = \mathrm{P}[\sqrt{2}\sigma UV < x].$$

But, it has been proved that [12]

$$\mathrm{P}[\sqrt{2}\sigma UV < x] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi\left(\frac{x}{\sigma\sqrt{2z}}\right) dG(z)$$

and this completes the proof.

Using Lemma 3 we can obtain the following generalization of Theorem 3 from [12]:

THEOREM 7. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $\mathrm{E}X_n = 0$ and finite variances $\sigma^2 X_n = \sigma_n^2$ such that*

$$\lim_{n\to\infty} \frac{s_n^2}{n} = 2\sigma^2 < \infty.$$

Moreover, suppose that $\{X_n, n \geqslant 1\}$ satisfies condition (A) *with a random variable X such that $\mathrm{E}X = 0$ and $\sigma^2 X = 2\sigma^2$.*

If $\{\nu_n, n \geqslant 1\}$ is a sequence from Theorem 6, *then* (10) *holds.*

5. The number of operations of rarefaction is a random variable. Now we investigate the rarefaction of a recurrent process in the case where the number of operations of rarefaction is a random variable.

By the previous results and the well-known Toeplitz's lemma [5], p. 238, or by results of [11], the following theorems can easily be proved.

THEOREM 8. *Let* $\{X_n, n \geqslant 1\}$ *and* $\{\nu_n, n \geqslant 1\}$ *be sequences of random variables such as in Theorem* 2. *If* $\{\lambda_m, m \geqslant 1\}$ *is a sequence of positive integer-valued random variables such that* λ_m *is independent of* $\{X_n, n \geqslant 1\}$ *and of* $\{\nu_n, n \geqslant 1\}$, *and* $\lambda_m \xrightarrow{P} \infty$ *as* $m \to \infty$, *then*

$$(11) \qquad \lim_{m\to\infty} P[q^{\lambda_m} S_{\nu_{\lambda_m}} < x] = \lim_{m\to\infty} P[q^{\lambda_m} \max_{1\leqslant k\leqslant \nu_{\lambda_m}} S_k < x]$$
$$= \lim_{m\to\infty} P[q^{n} \max_{1\leqslant k\leqslant \nu_{\lambda_m}} |S_k| < x]$$
$$= \begin{cases} 1-\exp\left(-\dfrac{x}{\mu}\right) & \text{for } x > 0, \\ 0 & \text{otherwise}. \end{cases}$$

COROLLARY 2. *If* $\{X_n, n \geqslant 1\}$ *and* $\{\nu_n, n \geqslant 1\}$ *are sequences of random variables such as in Corollary* 1, *and* $\{\lambda_m, m \geqslant 1\}$ *is a sequence of random variables satisfying the assumptions of Theorem* 8, *then* (11) *holds.*

THEOREM 9. *If* $\{X_n, n \geqslant 1\}$ *and* $\{\nu_n, n \geqslant 1\}$ *are sequences of random variables such as in Theorem* 3 *and* $\{\lambda_m, m \geqslant 1\}$ *is a sequence of random variables satisfying the assumptions of Theorem* 8, *then*

$$\lim_{m\to\infty} P[q^{\lambda_m} S_{\nu_{\lambda_m}} < x] = \lim_{m\to\infty} P[q^{\lambda_m} \max_{1\leqslant k\leqslant \nu_{\lambda_m}} S_{\nu_{\lambda_m}} < x]$$
$$= \lim_{m\to\infty} P[q^{\lambda_m} \max_{1\leqslant k\leqslant \nu_{\lambda_m}} |S_k| < x]$$
$$= \begin{cases} G\left(\dfrac{x}{\mu}\right) & \text{if } \mu > 0, \\ 1-G\left(-\dfrac{x}{\mu}\right) & \text{if } \mu < 0. \end{cases}$$

In a similar way we can change Theorems 4, 5, 6 and 7 for the case where the number of operations of rarefaction is a random variable.

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O TWIERDZENIACH GRANICZNYCH DLA SUM O LOSOWEJ LICZBIE ZMIENNYCH LOSOWYCH WYSTĘPUJĄCYCH W BADANIU ROZRZEDZONEGO PROCESU REKURENCYJNEGO

STRESZCZENIE

W pracy podano twierdzenia o granicznym rozkładzie procesu, uzyskanego w wyniku rozrzedzania procesu $T_0 \equiv 0 \leqslant T_1 \leqslant T_2 \leqslant \ldots$, gdy zmienne losowe $X_i = T_i - T_{i-1}$ $(i = 1, 2, \ldots)$ mają różne rozkłady. Twierdzenia tej pracy rozszerzają lub wzmacniają niektóre wyniki zawarte w [8], [9] i [12].