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QUEUEING SYSTEMS WITH A RESERVE SERVICE CHANNEL

1. Introduction. Applying queueing theory to practical problems one often comes across two-channel systems in which one of the channels is treated as the reserve one, e.g. due to running costs. Thus for example there may be a reserve service counter at the post-office, a reserve ticket-office at the railway station or a reserve computer in the computing centre. Operating the system with a reserve channel may be carried on in two ways:

1) a specified channel (*main channel*) is always available and the operation of the reserve channel depends on the size of the queue, it is switched on when the number of customers in the system is large enough and is switched off when the queue size becomes small. Such a system may be called *asymmetric*;

2) one of the channels is always available and if the number of customers in the system exceeds a given level, the second channel begins servicing. If both are switched on and the number of customers in the system drops below a given level the channel which actually completed the service is being switched off. Such a system may be called *symmetric*. In this paper we try to answer some questions concerning the operation of these models.

2. Steady-state probabilities in the asymmetrical model. Let us consider a stochastic service system with a Poisson input with constant arrival rate λ. There are two service channels: the main channel, always available, and the reserve service channel which is available only when switched on. Switching on occurs at the moment t at which the *queue size* (it is the number of customers in the system) exceeds N and switching off occurs at the moment t' when the reserve channel completes a service and the queue size does not exceed n ($1 \leqslant n \leqslant N$). There is no limitation of the capacity of the waiting room. Service times in both channels are identically distributed, independent random variables with the distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x}, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

Denote by $N(t)$ the queue size at the moment t, i.e. $N(t)$ is the number of customers waiting or being served at the moment t. Further, define a two-dimensional process

$$Z(t) = \{N(t), a(t)\},$$

where $a(t) = 0$ if at the moment t the reserve channel is switched off and $a(t) = 1$ if at the moment t the reserve channel is switched on. There is a one-to-one correspondence between the process $Z(t)$ and the one-dimensional process $M(t) = Na(t) + N(t)$. We say that *the system is in the state E_i at the moment* t if $M(t) = i$, $i = 0, 1, \ldots$ Under the specified conditions, $M(t)$ is a homogeneous Markov process. In Fig. 1 we show the graph of intermediate transitions between the states E_i.

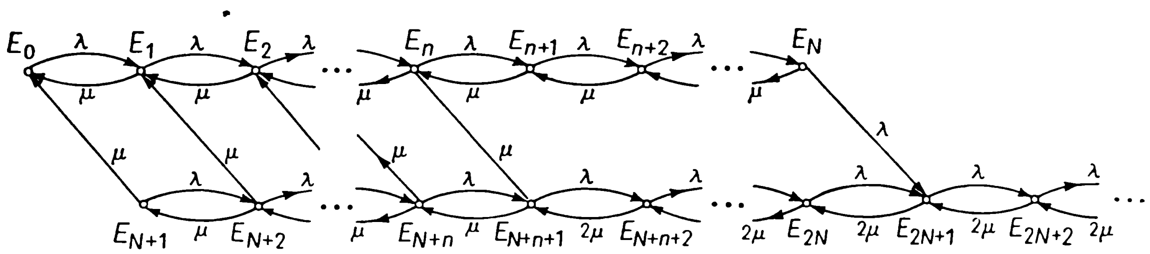

Fig. 1

Denote by $A = (a_{ij})$ the transition density matrix. Its elements are of the form

$$\begin{aligned}
&a_{i,i+1} = \lambda, && \text{for } i \neq N, \\
&a_{N,2N+1} = \lambda, && \\
&a_{i,i-1} = \mu, && \text{for } i = 1, \ldots, N, N+2, \ldots, N+n+1, \\
(1) \quad &a_{i,i-1} = 2\mu, && \text{for } i = N+n+2, N+n+3, \ldots, \\
&a_{N+i,i-1} = \mu, && \text{for } i = 1, 2, \ldots, n+1, \\
&a_{i,i} = -\sum_j a_{i,j}, && \text{for } i = 0, 1, \ldots, \\
&a_{i,j} = 0, && \text{for all other cases.}
\end{aligned}$$

We shall prove here only the first of formulae (1) for $0 \leqslant i \leqslant N$ (other formulae may be proved in a similar way). There is

$$a_{i,i+1} = \lim_{h\to 0} \frac{P\{M(h) = i+1 \mid M(0) = i\}}{h}$$

$$= \lim_{h\to 0} \frac{\big(\lambda h + o(h)\big)\big(1-\mu h + o(h)\big) + o(h)}{h} = \lambda .$$

Since $a_{i,i} < \infty$, $i = 0, 1, \ldots$, and the convergence

$$\lim_{h\to 0} \frac{P\{M(h) = k \mid M(0) = j\}}{h} = a_{j,k}$$

is uniform with respect to j, the probabilities $P_i(t) = \{PM(t) = i\}$ have to obey the system of differential equations (see [3], p. 292)

$$\frac{d}{dt} P(t) = A^T P(t), \tag{2}$$

where $P(t) = \big(P_0(t), P_1(t), \ldots\big)^T$.

We shall restrict ourselves to the steady-state probabilities only which we will denote by

$$\lim_{t\to\infty} P_i(t) = \begin{cases} Q_i & \text{for } i = 0, 1, \ldots, N, \\ P_{i-N} & \text{for } i = N+1, N+2, \ldots \end{cases}$$

Q_i is the limiting probability of the state when i customers are present in the system and the reserve channel is switched off, while P_i is the limiting probability of the same number of customers in the system and the reserve channel to be switched on. For the steady-state case the system (2) becomes

$$\begin{array}{lll}
(3,\mathrm{a}) & 0 = -\varrho Q_0 + Q_1 + P_1, & \\
(3,\mathrm{b}) & 0 = \varrho Q_{i-1} - (1+\varrho) Q_i + Q_{i+1} + P_{i+1}, & 1 \leqslant i \leqslant n, \\
(3,\mathrm{c}) & 0 = \varrho Q_{i-1} - (1+\varrho) Q_i + Q_{i+1}, & n+1 \leqslant i \leqslant N, \\
(3,\mathrm{d}) & 0 = -(1+\varrho) P_1 + P_2, & \\
(3,\mathrm{e}) & 0 = \varrho P_{i-1} - (2+\varrho) P_i + P_{i+1}, & 2 \leqslant i \leqslant n, \\
(3,\mathrm{f}) & 0 = \varrho P_{i-1} - (2+\varrho) P_i + 2P_{i+1}, & n+1 \leqslant i \leqslant N, \\
(3,\mathrm{g}) & 0 = \varrho (Q_N + P_N) - (2+\varrho) P_{N+1} + 2P_{N+2}, & \\
(3,\mathrm{h}) & 0 = \varrho P_{i+1} - (2+\varrho) P_i + 2P_{i+1}, & i \geqslant N+2,
\end{array}$$

where $\varrho = \lambda/\mu$ and $Q_i = 0$ for $i > N$.

Now we shall prove the following

THEOREM 1. *If* $\varrho < 2$, *then the solution of the system* (3, a)-(3, h) *satisfies the reccurent formulae*

(i) $$P_{k+1} = \sum_{j=1}^{k} P_j + \varrho P_k, \qquad 1 \leqslant k \leqslant n,$$

(ii) $$P_{k+1} = \frac{\varrho}{2} P_k + \frac{1}{2} \sum_{j=1}^{n+1} P_j, \qquad n+1 \leqslant k \leqslant N,$$

(iii) $$Q_{k+1} = \varrho Q_k - \sum_{j=1}^{k+1} P_j, \qquad 0 \leqslant k \leqslant n,$$

(iv) $$Q_{k+1} = \varrho Q_k - \sum_{j=1}^{n+1} P_j, \qquad n+1 \leqslant k \leqslant N-1,$$

(v) $$P_{k+1} = \left(\frac{\varrho}{2}\right)^{k-N} P_{N+1}, \qquad k \geqslant N+1.$$

Proof. To have (i) add to equation (3, d) the first $k-1$ equations of (3, e).

To obtain (ii) add to equation (3, d) all equations of (3, e) and the first $k-n$ equations of (3, f).

To achieve (iii) add to equation (3, a) the first $k-1$ equations of (3, b).

To have (iv) add to equation (3, a) all equations of (3, b) and the first $k-n-1$ equations of (3, c).

To obtain (v), first add all equations of (3, a), (3, b), (3, c). This yields

$$Q_N = \frac{1}{\varrho} \sum_{j=1}^{n+1} P_j.$$

Therefrom and from the already proved (ii) we obtain

$$P_{N+1} - \frac{\varrho}{2}(Q_N + P_N) = \frac{\varrho}{2} P_N + \frac{1}{2} \sum_{j=1}^{n+1} P_j - \frac{\varrho}{2} Q_N - \frac{\varrho}{2} P_N$$
$$= \frac{1}{2} \sum_{j=1}^{n+1} P_j - \frac{\varrho}{2} \cdot \frac{1}{\varrho} \sum_{j=1}^{n+1} P_j = 0.$$

Since, according to (3, g),

$$P_{N+2} = \frac{\varrho}{2} P_{N+1} + P_{N+1} - \frac{\varrho}{2}(Q_N + P_N),$$

we have

$$P_{N+2} = \frac{\varrho}{2} P_{N+1}.$$

Applying the induction principle, equation (v) may be easily obtained for all $k \geqslant N+1$.

Recurrent formulae in this theorem enable the construction of a simple algorithm for computing steady-state probabilities, the expected length of queue etc.

3. Distributions of idle periods and busy periods of the reserve channel in the asymmetrical case. Let $x_0, x_1, x_2, \ldots$ be the sequence of consecutive moments of switching on the reserve channel and $y_1, y_2, y_3, \ldots$ the sequence of moments of switching it off. There is, of course, $0 < x_0 < y_1 < x_1 < y_2 < \ldots$ Define two sequences of random variables

$$Y_i = y_i - x_{i-1},$$

$$X_i = x_i - y_i,$$

for $i = 1, 2, \ldots$

According to the definition of $a(t)$, x_i and y_i are discontinuity points of $a(t)$. X_i are the lengths of idle periods of the reserve channel, Y_i are the lengths of busy periods of the reserve channel. X_i are independent random variables with the common distribution function $F(t) = P\{X_i < t\}$, $i > 0$.

Similarly, Y_i are independent identically distributed random variables. In the sequel we use symbols X and Y to denote an arbitrary idle period and busy period for the reserve channel.

To find the distribution function $F(t)$ we introduce a homogeneous random walk $U(t)$ defined on the subset $\{E_0, E_1, \ldots, E_N, E_{2N+1}\}$ of the set of states of the process $M(t)$. E_{2N+1} will be an absorbing state and the transition densities will be the following: $q_{i+1,i} = \mu$, $q_{i,i+1} = \lambda$ for $i = 0, 1, \ldots, N-1$, $q_{N,2N+1} = \lambda$, $q_{ij} = 0$ for $j \neq 2N+1$ and $|i-j| \geqslant 2$.

LEMMA 1. *For the defined random walk with the initial condition* $P(U(0) = i) = \overline{p}_i$, *where* $\sum_{i=0}^{n} \overline{p}_i = 1$ *and* $\overline{p}_k = 0$ *for* $k > n$, *the distribution function of the time until absorption in the state* E_{2N+1} *is of the form*

$$g_{2N+1}(t) = 1 - \sum_{i=1}^{N+1} A_i e^{-a_i t}, \tag{4}$$

where $\sum_{i=1}^{N+1} A_i = 1$ *and* $a_i > 0$, $i = 1, 2, \ldots, N+1$.

Proof. Let us write $g_i(t) = P\{U(t) = i, \ i\} = 0, 1, \ldots, N, 2N+1$. By an argument similar to that presented by Gnedenko, Belayev, Solovyev ([4], § 6.4), one can prove that there holds the following system of equations:

$$\begin{aligned}
g_0'(t) &= -\lambda g_0(t) + \mu g_1(t), \\
g_k'(t) &= \lambda g_{k-1}(t) - (\lambda+\mu) g_k(t) + \mu g_{k+1}(t), \qquad k = 1, \ldots, N-1, \\
g_N'(t) &= \lambda g_{N-1}(t) - (\lambda+\mu) g_N(t), \\
g_{2N+1}'(t) &= \lambda g_N(t).
\end{aligned}$$

This system may be solved by using the Laplace transform

$$a_i(s) = \int_0^\infty e^{-st} g_i(t)\,dt,$$

which yields

$$a_{2N+1}(s) = \frac{\Delta_{2N+1}(s)}{\Delta(s)},$$

where

$$\Delta(s) = s\left[s^n + \sum_{k=1}^{N+1} \left(\lambda + (n-k) \sum_{i=0}^{k} \lambda^{k-i}\mu^i\right) s^{N-k+1}\right]$$

and

$$\Delta_{2N+1}(s) = \begin{vmatrix}
\lambda+s & -\mu & 0 & & 0 & \overline{p}_0 \\
-\lambda & \lambda+\mu+s & -\mu & & 0 & \overline{p}_1 \\
0 & -\lambda & \lambda+\mu+s & \ldots & 0 & \overline{p}_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \ldots & \lambda+\mu+s & \overline{p}_N \\
0 & 0 & 0 & & -\lambda & 0
\end{vmatrix}.$$

$a_{2N+1}(s)$ is a rational function, so it may be brougth (see [4]) to the form

$$a_{2N+1}(s) = \sum_{i=0}^{N+1} \frac{A_i}{s+a_i}, \tag{5}$$

where $a_0 = 0$, $a_i > 0$ for $i > 0$, $a_i \neq a_j$ for $i \neq j$ and

$$A_i = \frac{\Delta_{2N+1}(-a_i)}{\Delta'(-a_i)}.$$

It is easy to see that

$$A_0 = \frac{\Delta_{2N+1}(0)}{\Delta'(0)} = 1.$$

Applying the inverse Laplace transform to (5) we obtain (4).

Denote by p_i $(0 \leqslant i \leqslant n)$ the probabilities that at the moment of switching off the reserve channel the number of customers in the system is equal to i. If $\lambda < 2\mu$; then $\sum_{i=0}^{n} p_i = 1$ since the reserve channel may be switched off only when the number of units is not greater than n. The following theorem may be proved:

THEOREM 2. *The distribution function* $g_{2N+1}(t)$ *defined by* (4) *in lemma* 1 *is the distribution function of idle periods for the reserve channel in the asymmetrical model.*

Proof. Let us assume that the moment t' is the starting point for an idle period of the reserve channel and there are customers in the system at that moment (i.e. $M(t'+0) = N+k+1$, $M(t') = k$). From that moment until the next switching on of the reserve channel the process $M(t)$ is equivalent to the random walk $U(t-t')$ starting from the state E_k. The absorption in the state E_{2N+1} corresponds to the next switching on of the reserve channel, so the absorbing time is equivalent to the length of the idle period. The probability of the initial state E_k (i.e. $M(t') = k$) is equal to p_k. Applying lemma 1 for $\overline{p}_k = p_k$, we obtain $F(t) = g_{2N+1}(t)$.

Remark. p_i may be computed using the methods of Markov chains. Let us consider a homogeneous random walk V_i in discrete time on all states of the process $M(t)$ with transition probabilities

$$p_{ij} = \frac{q_{ij}}{\sum_j q_{ij}},$$

where $q_{i,i+1} = \lambda$ for $i \geqslant N+1$, $q_{i+1,i} = \mu$ for $N+1 \leqslant i \leqslant N+n$, $q_{i+1,i} = 2\mu$ for $i \geqslant N+n+1$, $q_{N+i+1,i} = \mu$ for $0 \leqslant i \leqslant n$, $q_{ij} = 0$ for the remaining cases. If $\lambda < 2\mu$, we may restrict our considerations only to the random walk defined on the states $E_0, E_1, \ldots, E_n, E_{N+1}, E_{N+2}, \ldots, E_{N+n+2}$, where E_{N+n+2} is a reflecting state. We obtain a finite Markov chain for which the steady-state solution may be found by solving the finite system of algebraic equations ([1], XIV, § 7).

Without knowing the distribution function $F(t)$ we can compute the expected lengths of idle periods and busy periods of the reserve channel:

$$\mathrm{E}X = \sum_{j=0}^{n} p_j \mathrm{E}\big(X \mid M(0) = j\big),$$

where $\mathrm{E}(X \mid M(0) = j)$ is the conditional expected value of X given the state E_j at the beginning of an idle period. $\mathrm{E}(X \mid M(0) = j)$ may be computed applying a method similar to that used by Gnedenko et al. [3]:

$$\mathrm{E}\big(X \mid M(0) = j\big) = \frac{1-\varrho^{N-j+1}}{\lambda(1-\varrho)^2\varrho^N} - \frac{(N-j+1)\varrho}{\lambda(1-\varrho)}, \quad 0 \leqslant j \leqslant n. \tag{6}$$

Since $\mathrm{E}X/(\mathrm{E}X+\mathrm{E}Y)$ is the steady-state probability of the reserve channel to be switched off, which is equal to

$$\lim_{t\to\infty} P\{a(t) = 0\} = \sum_{j=0}^{N} Q_j,$$

we may calculate also the expected length of busy period of the reserve channel:

$$\mathrm{E}Y = \frac{1-\sum_{j=0}^{N} Q_j}{\sum_{j=0}^{N} Q_j}\,\mathrm{E}X.$$

4. Theory of the symmetrical system. In the symmetrical case we have the system with two channels, each of them is either switched on or switched off. If only one of the channels is switched on, the switching on of the other one occurs as soon as the number of customers in the system exceeds N. If both are switched on and the number of customers in the system drops to the level n, the channel which actually completed the service is being switched off. Here we assume also a Poisson input with arrival rate λ and the distribution function $H(x) = 1-e^{-\mu x}$ of service times in both channels.

Let us define

$$Z_s(t) = \{N(t), b(t)\},$$

where $b(t) = 0$ if at the moment t only one channel is switched on and $b(t) = 1$ if at the moment t both channels are switched on. Similarly as in section 2, we define $M_s(t) = Nb(t)+N(t)-n$. The steady-state probabilities

$$\lim_{t\to\infty} P\{M_s(t) = i\} = \begin{cases} Q_i, & i \leqslant N, \\ P_{i-N+n}, & i > N \end{cases}$$

are of the form:

(i) $$Q_k = Q_0 \varrho^k, \quad 0 \leqslant k \leqslant n,$$

(ii) $$Q_k = Q_0 \varrho^k \frac{1-\varrho^{N-k+1}}{1-\varrho^{N-n+1}}, \quad n+1 \leqslant k \leqslant N,$$

(iii) $$P_k = Q_0 \varrho^{N+1} \frac{1-\varrho}{2-\varrho} \frac{1-(\varrho/2)^{k-n}}{1-\varrho^{N-n+1}}, \quad n+1 \leqslant k \leqslant N+1,$$

(iv) $$P_k = Q_0 \frac{\varrho^k}{2^{k-N-1}} \frac{1-\varrho}{2-\varrho} \frac{1-(\varrho/2)^{N-n+1}}{1-\varrho^{N-n+1}}, \quad k \geqslant N+2,$$

(v) $$Q_0 = \left[\frac{1}{1-\varrho} - \frac{(N-n+1)\varrho^{N+1}}{(2-\varrho)(1-\varrho^{N-n+1})}\right]^{-1}$$

Let us denote by $x_0, x_1, x_2, \dots$ the sequence of the moments of switching on and by $y_1, y_2, y_3, \dots$ the sequence of the moments of switching off. Define two sequences of random variables:

$$Y_i = y_i - x_{i-1},$$
$$X_i = x_i - y_i, \quad i = 1, 2, \dots$$

X_i and Y_i are independent random variables with distribution functions $\Phi(t)$ and $G(t)$, respectively.

From Lemma 1, assuming the initial condition $\bar{p}_i = \delta_{in}$, follows that $\Phi(t)$ is of the form

$$\Phi(t) = 1 - \sum_{k=1}^{N+1} A_k e^{-a_k t},$$

where $a_k > 0$ and $\sum_{k=1}^{N+1} A_k = 1$.

Similarly to (6), we can compute the expected value of X_i as

$$\mathrm{E}X_i = \frac{1-\varrho^{N-n+1}}{\lambda\varrho^N(1-\varrho)^2} - \frac{(N-n+1)\varrho}{\lambda(1-\varrho)}.$$

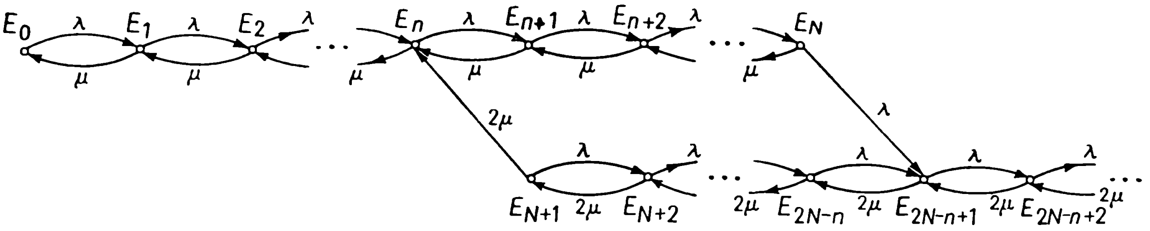

Fig. 2

To find the distribution function $G(t)$, we notice that the simultaneous work of both channels may be described in terms of a homogeneous random walk in continuous time $M^*(t)$ on the states $E_n, E_{N+1}, E_{N+2}, \dots$ (see Fig. 2). E_{2N-n+1} is the initial state and E_n is the absorbing state. The non-zero transition densities are λ for the transitions from E_{i-1} to E_i and 2μ for the transition from E_i to E_{i-1} ($i \geqslant N+2$) and 2μ from E_{N+1} to E_n. The length of the simultaneous work of the both channels is equivalent to the time in the random walk until the absorbing in the state E_n. Therefrom and from [2], XIV, § 6, it follows that

$$G(t) = P\{M^*(t) = n \mid M^*(0) = 2N-n+1\} = (L^{-1})^{*(N-n+1)}(t),$$

where $L^{-1}(t)$ is the distribution function of the first transition from the state E_i to E_{i-1}. The Laplace transform of the $L^{-1}(t)$ is

$$\int_0^\infty e^{-st} dL^{-1}(t) = \left[\frac{\lambda+2\mu+s+\sqrt{(\lambda+2\mu+s)^2-8\lambda\mu}}{4\mu}\right]^{-1}$$

(see Feller [2], XIV, § 6). Hence the Laplace transform of the distribution function $G(t)$ is of the form

$$G^*(s) = \left[\frac{\lambda+2\mu+s+\sqrt{(\lambda+2\mu+s)^2-8\lambda\mu}}{4\mu}\right]^{-(N-n+1)}$$

Knowing $G^*(s)$, we can compute moments of the random variable for example

$$\mathrm{E}Y_i = -\left.\frac{dG^*(s)}{ds}\right|_{s=0} = \frac{(N-n+1)\varrho}{(2-\varrho)\lambda}$$

and

$$\operatorname{Var} Y_i = \frac{(N-n+1)(6-\varrho)}{\mu^2(2-\varrho)^3}.$$

5. Acknowledgement. We wish to express our sincere thanks to Professor J. Łukaszewicz under whose guidance this work was carried out.

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SYSTEMY KOLEJKOWE Z REZERWOWYM KANAŁEM OBSŁUGI

STRESZCZENIE

W pracy autorzy rozwijają teorię systemów $M/M/2$ z dodatkowymi założeniami co do dostępności kanałów obsługi. Oba kanały są dostępne tylko wtedy, gdy w systemie znajduje się dużo jednostek. Przy małej liczbie jednostek w systemie jeden kanał zostaje wyłączony. Rozróżniane są dwa przypadki: przypadek asymetrii kanałów i przypadek symetrii kanałów. W przypadku asymetrii jeden wyróżniony kanał (podstawowy) jest zawsze dostępny, podczas gdy drugi kanał (rezerwowy) zostanie włączony w momencie, gdy liczba jednostek w systemie przekracza N, a wyłączony wtedy gdy w chwili zakończenia obsługi w tym kanale liczba jednostek w systemie okaże się nie większa niż $n (1 \leqslant n \leqslant N)$. W przypadku symetrii żaden z kanałów nie jest wyróżniony i jeśli oba pracują, to wyłączony będzie ten, który pierwszy zakończy obsługę w momencie, gdy stan systemu spadnie do liczby n jednostek.

Autorzy podają wzory rekurencyjne na rozkład prawdopodobieństwa stanu systemu (liczby jednostek w systemie) w warunkach stacjonarnych dla przypadku asymetrycznego oraz jawne wzory na analogiczny rozkład prawdopodobieństwa w przypadku symetrycznym. Ponadto autorzy podają dla obu przypadków metody uzyskania rozkładów długości okresów włączenia i wyłączenia kanału rezerwowego.

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СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ С РЕЗЕРВНЫМ КАНАЛОМ

РЕЗЮМЕ

В работе авторы предлагают теорию систем $M/M/2$ с дополнительным предположением относительно доступности каналов обслуживания. Оба канала доступны только тогда, когда в системе находится большое количество ожидающих требований. Когда число ожидающих требований не большое, один из каналов выключается, переходя в резервное состояние. Различаются два частных случая: случай асимметрии каналов и случай симметрии каналов. В случае асимметрии, определенный канал (основной) всегда доступен, так как второй канал (резервный) включается, когда число требований в системе превышает N и выключается если число требований в системе не более n $(1 \leqslant n \leqslant N)$. В случае симметрии оба канала работают на одинаковых условиях: если оба включены, то с течением времени выключается тот канал, который закончит обслуживание в момент времени, когда в системе число требований уменьшилось до n.

Авторы выводят рекуррентные формулы для распределения вероятности состояний системы (числиа требований в системе) в условиях стационарности для случая асимметрии и явные формулы для аналогичного распределения вероятности в случае симметрии. Кроме того, для обоих случаев указаны методы получения распределений длины периодов включения и выключения резервного канала.
