ZASTOSOWANIA MATEMATYKI
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I. KOPOCIŃSKA (Wrocław)

BUSY PERIOD PROBLEMS IN THE $GI/G/\infty$ QUEUE

1. Introduction. In the paper we investigate the busy and idle periods in a $GI/G/\infty$ queueing system. At the same time the system is considered as a counter in which every item blockades the counter. In view of this interpretation we consider some counter problems, namely we state the probability of an item to be counted and the waiting time for a counted item. These problems have been considered by Takács [2]. Here some methods of dam theory [1] are used.

2. Basic data. Let us introduce the main notation and assumptions for the basic data of the queueing system considered. Let $n = 0, 1, \ldots$ denote the numbers of successively arriving items, $t_0 = 0 < t_1 < t_2 < \ldots$ the input moments, $\xi_n = t_{n+1} - t_n$ the interarrival times, and χ_n the service times. We assume that the random variables ξ_n and χ_n $(n = 0, 1, \ldots)$ are positive, independent, and distributed as follows:

$$F(x) = \Pr(\xi_n < x), \quad H(y) = \Pr(\chi_n < y), \quad x \geqslant 0,\ y \geqslant 0.$$

In some situations we use, for simplicity, the random variables ξ and χ distributed according to F and G, respectively, assuming that they are mutually independent and also independent of the basic data.

3. Blockade time of the system. Denote by η_n the maximum of the service times of items being in the system immediately before the input moment of the n-th item. The sequence of random variables η_n $(n = 0, 1, \ldots)$ forms a Markov chain for which

$$\eta_{n+1} = \max\big(0, \max(\eta_n, \chi_n) - \xi_n\big), \quad n = 0, 1, \ldots, \tag{1}$$

where η_0 is the initial value of the chain.

Denote by ζ_n the maximum of the service times of the items being in the system at the arrival moment of the n-th item (including the service time of the arriving item). This quantity will be called the *blockade time* of the system. It is obvious that

$$\zeta_n = \max(\eta_n, \chi_n), \quad n = 0, 1, \ldots$$

For successive blockade times we have a Markovian dependence

(2) $$\zeta_{n+1} = \max(\zeta_n - \xi_{n+1}, \chi_{n+1}), \quad n = 0, 1, \ldots,$$

where ζ_0 is the initial value of the chain.

The distribution functions of the considered random variables are denoted by

$$Y_n(z) = \Pr(\eta_n < z) \quad \text{and} \quad Z_n(z) = \Pr(\zeta_n < z), \quad z \geqslant 0, \ n = 0, 1, \ldots$$

Note that $Y_n(0+)$ is the probability of the system being idle at the arrival moment of the n-th item.

THEOREM 1. *In the queueing system $GI/G/\infty$, independently of the initial value η_0 the limit*

$$\lim_{n\to\infty} Y_n(z) = Y(z), \quad z \geqslant 0,$$

exists and satisfies the integral equation

(3) $$Y(z) = \int_0^\infty Y(z+x)H(z+x)\,dF(x), \quad z \geqslant 0,$$

with the boundary condition $Y(\infty) = 1$.

Equation (3) *has the solution*

(4) $$Y(z) = \Pr(\chi_1 - \xi_1 < z, \chi_2 - \xi_1 - \xi_2 < z, \ldots), \quad z \geqslant 0.$$

COROLLARY 1. *Independently of the initial value ζ_0 the limit*

$$Z(z) = \lim_{n\to\infty} Z_n(z), \quad z \geqslant 0,$$

exists and is equal to $Z(z) = H(z)Y(z)$, $z \geqslant 0$.

Proof of Theorem 1. The recurrence formula (1) for random variables gives the recurrence formula for the distribution function:

(5) $$Y_{n+1}(z) = \int_0^\infty Y_n(z+x)H(z+x)\,dF(x), \quad n = 0, 1, \ldots,$$

where Y_0 is the distribution function of the initial value η_0. Now we prove that the limit $Y(z)$ exists. We show by induction that

(6) $$Y_n(z) = \Pr(\eta_0 - \xi_0 - \ldots - \xi_{n-1} < z,$$
$$\chi_{n-1} - \xi_{n-1} < z, \chi_{n-2} - \xi_{n-2} - \xi_{n-1} < z, \ldots, \chi_0 - \xi_0 - \ldots - \xi_{n-1} < z),$$
$$n = 1, 2, \ldots, \ z \geqslant 0.$$

We have

$$Y_1(z) = \Pr\big(\max(\eta_0, \chi_0) - \xi_0 < z\big) = \Pr(\eta_0 - \xi_0 < z, \ \chi_0 - \xi_0 < z).$$

By the assumption of induction we get

$$Y_{n+1}(z) = \Pr\big(\max(\eta_n, \chi_n) - \xi_n < z\big) = \Pr(\eta_n - \xi_n < z, \ \chi_n - \xi_n < z)$$
$$= \Pr(\eta_0 - \xi_0 - \ldots - \xi_n < z, \ \chi_n - \xi_n < z, \ldots, \chi_0 - \xi_0 - \ldots - \xi_n < z),$$

which proves (6). Using the independence assumption for the basic data, we can write (6) in the form

$$Y_n(z) = \Pr(A_n B_n), \quad n = 1, 2, \ldots,$$

where

$$A_n = \{\eta_0 - \xi_0 - \ldots - \xi_{n-1} < z\},$$
$$B_n = \{\chi_0 - \xi_0 < z,\ \chi_1 - \xi_1 - \xi_2 < z, \ldots, \chi_{n-1} - \xi_0 - \ldots - \xi_{n-1} < z\}.$$

The sequence of events $\{A_n\}$ is increasing, $\lim_{n\to\infty} \Pr(A_n) = 1$, the sequence of events $\{B_n\}$ is decreasing, $\lim_{n\to\infty} \Pr(B_n) = \Pr(\chi_0 - \xi_0 < z, \chi_1 - \xi_0 - \xi_1 < z, \ldots)$. In consequence, the limit

$$\lim_{n\to\infty} \Pr(A_n B_n) = Y(z)$$

exists and does not depend upon η_0. Taking the limit in (5), we obtain equation (3). Taking the limit in (6), we obtain the solution (4).

Example 1. If we have $F(x) = p\,1_{[0,\infty)}(x) + q\,1_{[a,\infty)}(x)$ for $0 \leqslant p \leqslant 1$, $q = 1-p$, $a > 0$, then

$$Y(z) = Y(z)H(z)p + Y(z+a)H(z+a)q.$$

Hence

$$Y(z) = \frac{H(z+a)q}{1-H(z)p}\, Y(z+a) = \frac{H(z+a)q}{1-H(z)p}\, \frac{H(z+2a)q}{1-H(z+a)p}\, Y(z+2a) = \ldots$$

Finally,

$$Y(z) = \prod_{i=1}^{\infty} \frac{H(z+ia)q}{1-H\big(z+(i-1)a\big)p}, \quad z \geqslant 0.$$

4. Distribution function of the time between counted arrivals. As mentioned earlier, the item is counted if there is no blockade at the input moment. Denote by $\omega(z)$, $z \geqslant 0$, the time from the arrival moment of a non-counted item to the arrival of the first counted item under the condition that the initial blockade time (including the service time of the arriving item) is z. The counted item starts the blockade time which is equal to the service time. Hence the intercount time is a compound random variable $\omega(\chi)$, where χ is a random variable distributed according to H and independent of the basic data.

The stochastic process $\omega(z)$, $z > 0$, satisfies the recursive equality

(7) $$\omega(z) = \xi + \omega_1(\zeta_1) 1_{z > \xi},$$

where $\zeta_1 = \max(z-\xi, \chi)$, and $\omega_1(z)$, $z>0$, is the probabilistic copy of the process $\omega(z)$, $z>0$, which is independent of ξ and χ.

The probability distribution of the value of the process $\omega(z)$ is defined by $W(z,w) = \Pr\{\omega(z)<w\}$, $z>0$, $w\geqslant 0$.

Formula (7) may be expressed in the extended form

$$\omega(z) = \begin{cases} \xi & \text{if } z \leqslant \xi, \\ \xi+\omega_1(\zeta_1) & \text{if } z>\xi. \end{cases} \tag{8}$$

For $w<z$ we have $W(z,w)=0$, since $\omega(z)\geqslant z$. For $w\geqslant z$ we get

$$\begin{aligned} W(z,w) &= \Pr(\xi<w, z\leqslant \xi)+\Pr\{\xi+\omega(\zeta_1)<w,\ z>\xi\} \\ &= \Pr(z\leqslant\xi<w)+\int_0^z \Pr\{\omega(\zeta_1)<w-x\}\,dF(x) \\ &= F(w)-F(z)+\int_0^z\int_0^{w-x} W\{\max(z-x,y), w-x\}\,dH(y)\,dF(x). \end{aligned}$$

Finally, we obtain the integral equation

$$W(z,w) = \begin{cases} 0 & \text{if } w<z, \\ F(w)-F(z)+\Theta(W)(z,w) & \text{if } w\geqslant z, \end{cases} \tag{9}$$

where the operator $\Theta(W)$ is defined as follows:

$$\Theta(W)(z,w) = \int_0^z\int_0^{w-x} W\{\max(z-x,y), w-x\}\,dH(y)\,dF(x). \tag{10}$$

The solution of equation (9) is of the form

$$W(z,w) = \sum_{n=0}^{\infty} \Theta^{*n}(f)(z,w), \tag{11}$$

where Θ^{*n} is the n-fold convolution of the operator (10) and

$$f(z,w) = \{F(w)-F(z)\}1_{w\geqslant z}, \quad z\geqslant 0,\ w\geqslant 0. \tag{12}$$

The convergence of the sequence (11) is proved in the following manner. Consider the Markov chain (2) with initial value $\zeta_0 = z$. Then from (8) we get recursively

$$\begin{aligned} \omega_0(\zeta_0) &= \xi_0 1_{\zeta_0\leqslant\xi_0}+\{\xi_0+\omega_1(\zeta_1)\}1_{\zeta_0>\xi_0} \\ &= \xi_0 1_{\zeta_0\leqslant\xi_0}+(\xi_0+\xi_1)1_{\zeta_0>\xi_0,\zeta_1\leqslant\xi_1}+\ldots+ \\ &\quad +(\xi_0+\ldots+\xi_{n+1})1_{\zeta_0>\xi_0,\ldots,\zeta_n>\xi_n,\zeta_{n+1}\leqslant\xi_{n+1}}+R_{n+1}(z), \end{aligned} \tag{13}$$

where

$$R_{n+1}(z) = \{\xi_0+\ldots+\xi_{n+1}+\omega_{n+1}(\zeta_{n+1})\}1_{\zeta_0>\xi_1,\ldots,\zeta_n>\xi_n}.$$

Consequently,

$$\Pr\big(\omega(z) < w\big) = \sum_{i=0}^{n} \Theta^{*i}(f)(z,t) + \Pr\big(R_{n+1}(z) < w\big),$$

where f is defined by (12).

The inequality $\Pr\big(R_{n+1}(z) < w\big) \leqslant \Pr(\xi_0 + \ldots + \xi_n < w) \leqslant F^n(w)$ implies that the distribution function of the random variable $R_n(z)$ tends to zero uniformly for z. This proves the uniqueness of the solution of equation (9). From (13) we obtain the following

COROLLARY 2. *The probability distribution function of the time length between the arrivals of counted items is equal to the distribution of the random variable:*

$$\omega(\chi) = \xi_0 + \xi_1 1_{\zeta_0 > \xi_0} + \xi_2 1_{\zeta_0 > \xi_0, \zeta_1 > \xi_1} + \ldots$$

5. Busy period of the system. Denote by $\tau(z)$, $z \geqslant 0$, the busy period of the system which starts at the arrival moment, given the initial blockade time z. The unconditional busy period starts at the arrival moment of an item which finds the system idle and is a compound random variable $\tau(\chi)$.

The stochastic process $\tau(z)$, $z \geqslant 0$, satisfies the relation

$$\tau(z) = z\, 1_{z \leqslant \xi} + \big(\xi + \tau_1(\zeta_1)\big) 1_{z > \xi}, \tag{14}$$

where $\tau_1(z)$, $z > 0$, is a probabilistic copy of the process $\tau(z)$, $z > 0$, which is independent of ξ and ζ_1.

The distribution function of the value of the process $\tau(z)$ is denoted by $T(z,t)$, $z > 0$, $t \geqslant 0$. Formula (14) may be expressed in the extended form

$$\tau(z) = \begin{cases} z & \text{if } z \leqslant \xi, \\ \xi + \tau_1(\zeta_1) & \text{if } z > \xi. \end{cases}$$

For $t < z$ we have $T(z,t) = 0$ since $\tau(z) \geqslant z$. For $t \geqslant z$ we get

$$\begin{aligned} T(z,t) &= \Pr(z < t, z \leqslant \xi) + \Pr\big(\xi + \tau_1(\zeta_1) < t, z > \xi\big) \\ &= 1 - F(z) + \int_0^z \int_0^{t-x} T\big(\max(z-x, y), t-x\big)\, dH(y)\, dF(x). \end{aligned}$$

Finally, we obtain the integral equation

$$T(z,t) = \begin{cases} 0 & \text{if } t < z, \\ 1 - F(z) + \Theta(T)(z,t) & \text{if } t \geqslant z, \end{cases} \tag{15}$$

where Θ is defined by (10). This equation is analogous to (9) and has the solution

$$T(z,t) = \sum_{n=0}^{\infty} \Theta^{*n}(f)(z,t),$$

where $f(z,t) = (1-F(z))1_{t\geqslant z}$, $z > 0$, $t \geqslant 0$.

6. Idle period of the system. Denote by $\lambda(z)$, $z>0$, the idle period of the system, assuming that the busy period of the system starts from the blockade time z. The unconditional idle time of the system is a compound random variable $\lambda(\chi)$.

The following equality holds:

$$\lambda(z) = (\xi - z)1_{z\leqslant\xi} + \lambda_1(\zeta_1)1_{z>\xi}, \tag{16}$$

where $\lambda_1(z)$, $z>0$, is a probabilistic copy of the process $\lambda(z)$, $z>0$, which is independent of ξ and ζ_1.

The distribution function of the value $\lambda(z)$, $z>0$, denoted by

$$L(z,l) = \Pr(\lambda(z) < l), \quad z>0,\ l\geqslant 0,$$

satisfies the equation

$$L(z,l) = \begin{cases} 0 & \text{if } l < z, \\ F(z+l) - F(z) + \Theta(L)(z,l) & \text{if } l \geqslant z, \end{cases} \tag{17}$$

where Θ is defined by (10). Equation (17) may be solved analogously as (9) and (15).

7. Expected values. Consider the expected values of the earlier-defined stochastic processes: $w(z) = \mathrm{E}\omega(z)$, $t(z) = \mathrm{E}\tau(z)$, $l(z) = \mathrm{E}\lambda(z)$, $z>0$. Using (7), (14), and (16) we obtain the integral equations

$$w(z) = \mu_1 + \theta(w)(z), \tag{18}$$

$$t(z) = z(1-F(z)) + \int_0^z x\,dF(x) + \theta(t)(z), \quad l(z) = w(z) - t(z), \quad z>0,$$

where the operator $\theta(w)$ is defined by

$$\theta(w)(z) = \int_0^z \int_0^\infty w(\max(z-x, y))\,dH(y)\,dF(x), \quad z>0,$$

and μ_1 is the expected value of the distribution function F.

The solutions of the above equations may be obtained in a standard manner, analogously as in the case of (9).

Now, we establish some conditions for the existence of the solutions. The operator θ is increasing and

$$\mu_1 \geqslant z(1-F(z)) + \int_0^z x dF(x), \quad z \geqslant 0,$$

so we restrict our considerations to some remarks on equation (18). Using (7) we get

$$\mathrm{E}w(\zeta) = \mathrm{E}\xi + \mathrm{E}w(\zeta_1)1_{\zeta>\xi}, \quad \text{where } \zeta_1 = \max(\zeta-\xi, \chi).$$

Hence

$$\mathrm{E}\omega(\zeta_0) = \mathrm{E}\xi_1 + \mathrm{E}\xi_2 1_{\zeta_0>\xi_1} + \ldots + \mathrm{E}\xi_{n+1} 1_{\zeta_0>\xi_1,\ldots,\zeta_{n-1}>\xi_n} + \\ + \mathrm{E}\omega(\zeta_{n+1}) 1_{\zeta_0>\xi_1,\ldots,\zeta_n>\xi_{n+1}}, \quad n = 1, 2, \ldots$$

The random variable ξ_{i+1} is independent of the random variables $1_{\zeta_0>\xi_1,\ldots,\zeta_{i-1}>\xi_i}$ for $i = 0, 1, \ldots, n$. Hence

$$\mathrm{E}\omega(\zeta_0) = \mu_1 \mathrm{E}v(\zeta_0) + r(\zeta_0),$$

where

$$\mathrm{E}v(\zeta_0) = 1 + \Pr(\zeta_0 > \xi_1) + \ldots + \Pr(\zeta_0 > \xi_1, \ldots, \zeta_{n-1} > \xi_n)$$

is the expected value of the time ζ_n over zero and

$$r(\zeta_0) = \lim_{n\to\infty} \mathrm{E}\left(\omega(\zeta_{n+1}) 1_{\zeta_0>\xi_1,\ldots,\zeta_n>\xi_{n+1}}\right).$$

Simple conditions for $r(\zeta_0) = 0$ are not known.

References

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

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