ZASTOSOWANIA MATEMATYKI
V (1961)

S. TRYBUŁA (Wrocław)

ON THE PARADOX OF THREE RANDOM VARIABLES

This paper is concerned with three random variables X, Y, Z and the numbers ξ, η, ζ where $\xi = P(X < Y)$, $\eta = P(Y < Z)$, $\zeta = P(Z < X)$. Most of the theorems contained in this paper were published (without proofs) in our preliminary note [1], and are the result of my collaboration with Professor H. Steinhaus, who has suggested the problems solved here, and has foreseen many of the solutions.

Problems solved here have arisen in practice. Suppose that in a testing laboratory iron bars of the same size and shape are compared with regard to resistance. Two bars are compared as follows: the edges of the bars are held by a frame in such a manner that we can apply the same force by turning a screw. The screw is turned until one of the bars breaks. Let x and y denote the breaking strengths of the first and the second bar, respectively. The experiment enables us to compare x and y.

Denote by X and Y the resistances of bars belonging to factories I and II, respectively; X and Y are random variables and we can speak about the probability $P(X < Y)$. The bars from factory I are worse than the bars from II if $P(X < Y) > \frac{1}{2}$. If we have three factories I, II, III it can happen that the bars of I are worse than the bars of II, the bars of II are worse than those of III, and the bars of III are worse than those of I when compared pairwise statistically by random sampling.

The title of the present paper refers to the phenomena described above. It may be formulated in terms of probability theory as follows: If we have three independent random variables X, Y, Z then $P(X < Y) > \frac{1}{2}$ and $P(Y < Z) > \frac{1}{2}$ does not imply $P(X < Z) > \frac{1}{2}$. That this is true can be seen from the following example: $P(X = 1) = p$, $P(X = 4) = 1-p$, $P(Y = 2) = 1$, $P(Z = 0) = 1-p$, $P(Z = 3) = p$. If X, Y, Z are independent we have $\xi = p$, $\eta = p$, $\zeta = 1-p^2$. If we put $p = \frac{1}{2}(\sqrt{5}-1)$, then $\xi = \eta = \zeta = \frac{1}{2}(\sqrt{5}-1) = 0\cdot618\ldots$, and we have a paradox. But, for example, we cannot have $\xi = \eta = \zeta = 1$ for any triplet X, Y, Z. Thus we have the following problem: Give the necessary and sufficient condition in order that for a triplet of numbers ξ, η, ζ there exist three independent random variables X, Y, Z such that $P(X < Y) = \xi$, $P(Y < Z) = \eta$, $P(Z < X) = \zeta$.

We shall now assume that the random variables X, Y and Z are discrete and can assume a finite number of values. Later in the paper this assumption will be dropped. In order to avoid trivial complications, we also suppose that $P(X = Y) = P(Y = Z) = P(Z = X) = 0$. We are going to define the class Σ of triplets (X, Y, Z) in the following manner: A triplet $(X, Y, Z)\in\Sigma$ if there exist 5 numbers $x_1 < y_1 < z < x_2 < y_2$ such that

$$P(X = x_1)+P(X = x_2) = 1,$$

$$P(Y = y_1)+P(Y = y_2) = 1,$$

$$P(Z = z) = 1.$$

LEMMA 1. *For each triplet (X, Y, Z) of independent random variables satisfying the above assumptions there exists a triplet (X_1, Y_1, Z_1) such that at least one of the triplets (X_1, Y_1, Z_1), (Y_1, Z_1, X_1), (Z_1, X_1, Y_1) belongs to class Σ* and $P(X_1 < Y_1) \geqslant P(X < Y)$, $P(Y_1 < Z_1) \geqslant P(Y < Z)$, $P(Z_1 < X_1) \geqslant P(Z < X)$.

Proof. Let $x_1 < \ldots < x_{n_1}$, $y_1 < \ldots < y_{n_2}$, $z_1 < \ldots < z_{n_3}$ be values of the random variables X, Y, Z respectively. If we admit that some of the numbers $P(X = x_i)$, $P(Y = y_j)$, $P(Z = z_k)$ can be equal to 0, then, without loss of generality, we can suppose that $n_1 = n_2 = n_3 = n$ and

$$x_1 < y_1 < z_1 < x_2 < y_2 < z_2 < \ldots < x_n < y_n < z_n. \tag{1}$$

Let t_i $(i = 1, \ldots, 3n)$ denote the i-th element of sequence (1). Put $t_0 < t_1$. Let t_k denote the maximal element of the sequence $\{t_j\}$ $(j = 0, 1, \ldots, 3n)$ such that at least one of the probabilities

$$P(X \leqslant t_j), \quad P(Y \leqslant t_j), \quad P(Z \leqslant t_j) \tag{2}$$

is equal to 0. Define three new independent random variables (X', Y', Z') in the following manner: $P(X' = t_i) = P(X = t_i)$, $P(Y' = t_i) = P(Y = t_i)$, $P(Z' = t_i) = P(Z = t_i)$ for $i > k$. Furthermore, if $k > 0$ and $P(X \leqslant t_k) = 0$, put $P(Y' = t_0) = P(Y \leqslant t_k)$, $P(Z' = t_k) = P(Z \leqslant t_k)$. When $k > 0$ and $P(X \leqslant t_k) \neq 0$ but $P(Y \leqslant t_k) = 0$, we put $P(Z' = t_0) = P(Z \leqslant t_k)$, $P(X' = t_k) = P(X \leqslant t_k)$. In the last case put $P(X' = t_0) = P(X \leqslant t_k)$, $P(Y' = t_k) = P(Y \leqslant t_k)$. It is easy to verify that in all cases

$$P(X' < Y') \geqslant P(X < Y), \quad P(Y' < Z') \geqslant P(Y < Z), \tag{3}$$

$$P(Z' < X') \geqslant P(Z < X);$$

and, furthermore, the random variables X, Y, Z have the following property: There exists a sequence of numbers

$$
\begin{aligned}
& z_1' < x_1' < y_1' < z_2' < x_2' < y_2' < \ldots < z_m' < x_m' < y_m' \\
(4) \qquad & \text{or } y_1' < z_1' < x_1' < y_2' < z_2' < x_2' < \ldots < y_m' < z_m' < x_m' \\
& \text{or } x_1' < y_1' < z_1' < x_2' < y_2' < z_2' < \ldots < x_m' < y_m' < z_m'
\end{aligned}
$$

such that

$$
(5) \qquad \sum_{i=1}^{m} P(X' = x_i') = \sum_{i=1}^{m} P(Y' = y_i') = \sum_{i=1}^{m} P(Z' = z_i') = 1
$$

and

$$
(6) \qquad P(X' = x_1') > 0, \quad P(Y' = y_1') > 0, \quad P(Z' = z_1') > 0.
$$

Without loss of generality we can assume that the first of the three cases formulated in (4) occurs. If $m = 1$, the lemma is proved. If $m > 1$, let

$$
(7) \qquad P(X' = x_i') = p_i, \quad P(Y' = y_i') = q_i, \quad P(Z' = z_i') = r_i
$$

and

$$
(8) \qquad
\begin{gathered}
P = \frac{p_1 q_1 + (p_1 + p_2) q_2}{(p_1 + p_2)(q_1 + q_2)}, \qquad Q = \frac{q_1 r_1 + (q_1 + q_2) r_2}{(q_1 + q_2)(r_1 + r_2)}, \\
R = \frac{r_1 p_2}{(r_1 + r_2)(p_1 + p_2)}.
\end{gathered}
$$

Formulae (6), (7), (8) imply

$$
(9) \qquad 0 < P \leqslant 1, \quad 0 < Q \leqslant 1, \quad 0 \leqslant R < 1.
$$

We shall prove that at least one of the following two cases must occur

$$
(10) \qquad P \leqslant 1 - QR, \quad Q \leqslant 1 - PR.
$$

Put $p = p_1/(p_1 + p_2)$, $q = q_1/(q_1 + q_2)$, $r = r_1/(r_1 + r_2)$. Inequalities (10) become

$$
(11) \quad (p-1)q \leqslant r(p-1)(1-r+qr), \quad (q-1)r \leqslant r(p-1)(1-q+pq).
$$

If $r = 0$ or $p = 1$ alternative (10) holds trivially. Suppose then that $r \neq 0$, $p \neq 1$. Inequalities (11) then become

$$
q \geqslant \frac{r}{1+r}, \quad q \leqslant \frac{1}{2-p}.
$$

Since $r/(1+r) \leqslant \frac{1}{2}$, $1/(2-p) \geqslant \frac{1}{2}$ one of the inequalities in (10) must occur.

Suppose that the first inequality occurs. Define the random variables X'', Y'', Z'' as follows: X'', Y'', Z'' are independent with

$$\begin{aligned} P(X'' = x_i') &= P(X' = x_i'), \\ P(Y'' = y_i') &= P(Y' = y_i'), \qquad \text{for} \quad i > 2 \\ P(Z'' = z_i') &= P(Z' = z_i'), \end{aligned}$$

and, furthermore,

$$\begin{aligned} P(X'' = x_1') &= (p_1+p_2)(1-R), & P(Y'' = y_2') &= (q_1+q_2)(1-Q), \\ P(X'' = x_2') &= (p_1+p_2)R, & P(Z'' = z_1') &= r_1+r_2. \\ P(Y'' = y_1') &= (q_1+q_2)Q, \end{aligned}$$

From inequalities (6) and (9) we have

$$P(X'' = x_1') > 0, \quad P(Y'' = y_1') > 0, \quad P(Z'' = z_1') > 0. \tag{12}$$

Furthermore, from the definition of the variables X'', Y'', Z'' and the inequality $P \leqslant 1-QR$ we obtain

$$\begin{aligned} P(X'' < Y'')-P(X' < Y') &= (p_1+p_2)(q_1+q_2)(1-QR-P) \geqslant 0, \\ P(Y'' < Z'')-P(Y' < Z') &= 0, \\ P(Z'' < X'')-P(Z' < X') &= 0. \end{aligned} \tag{13}$$

Hence if $m = 2$, the lemma is proved. If $m > 2$, we put $P(X''' = x_2') = P(X'' = x_2')+P(X'' = x_3')$, $P(Y''' = y_3') = P(Y'' = y_2')+P(Y'' = y_3')$, $P(X''' = x_i) = P(X'' = x_i')$, $P(Y''' = y_i') = P(Y'' = y_i')$ for $i \neq 2, 3$; $P(Z''' = z_i') = P(Z'' = z_i')$ for $i = 1, 2, \ldots, m$. For the random variables X''', Y''', Z''' the condition defined by formulae (4), (5), (6) holds, and

$$\begin{aligned} P(X''' < Y''') &\geqslant P(X'' < Y'') \geqslant P(X < Y), \\ P(Y''' < Z''') &\geqslant P(Y'' < Z'') \geqslant P(Y < Z), \\ P(Z''' < X''') &\geqslant P(Z'' < X'') \geqslant P(Z < X). \end{aligned} \tag{14}$$

Suppose now that $P > 1-QR$. Then, by (10), $Q \leqslant 1-PR$ and $R > 0$. If in this case we put

$$\begin{aligned} P(X''' = x_2') &= p_1+p_2, & P(Z''' = z_1') &= (r_1+r_2)R, \\ P(Y''' = y_1') &= (q_1+q_2)(1-P), & P(Z''' = z_2') &= (r_1+r_2)(1-R), \\ P(Y''' = y_2') &= (q_1+q_2)P, \end{aligned} \tag{15}$$

then from the definition of the variables X''', Y''', Z''', and the assumption $Q \leqslant 1-PR$ we obtain

$$(16)\qquad \begin{gathered} P(X''' < Y''') \geqslant P(X < Y), \quad P(Y''' < Z''') \geqslant P(Y < Z), \\ P(Z''' < X''') \geqslant P(Z < X). \end{gathered}$$

Construction (15) allows us to eliminate the value x_1. Furthermore, assuming $R > 0$ and from formulae (6) and (9) we have

$$P(Z''' = z_1') > 0, \quad P(X''' = x_2') > 0, \quad P(Y''' = y_2') > 0.$$

It is also in this case that the random variables X, Y, Z satisfy the condition defined by (4), (5), (6).

Then in both cases of (10) the variables X''', Y''', Z''' satisfy the restrictions imposed on X', Y', Z', and the method described above permits us to put at least $P(X''' = x_1') = 0$. For the variables X''', Y''', Z''' one may apply an analogous method as for X', Y', Z', repeating this procedure until we obtain a triplet (X_1, Y_1, Z_1) such that one of its permutations belongs to Σ. From inequalities (14) and (16) we see that the triplet (X_1, Y_1, Z_1) will satisfy the inequalities formulated in the lemma.

We are going to formulate lemma 2 which permits us to generalize lemma 1 for the triplets (X, Y, Z) of independent random variables with non-arbitrary distributions.

LEMMA 2. *For each triplet (X, Y, Z) of the independent random variables and for each $\varepsilon > 0$ there exist three independent discrete random variables $X_\varepsilon, Y_\varepsilon, Z_\varepsilon$ assuming a finite number of values and such that*

$$\begin{gathered} |P(X < Y) - P(X_\varepsilon < Y_\varepsilon)| < \varepsilon, \quad |P(Y < Z) - P(Y_\varepsilon < Z_\varepsilon)| < \varepsilon, \\ |P(Z < X) - P(Z_\varepsilon < X_\varepsilon)| < \varepsilon. \end{gathered}$$

The proof of this elementary lemma is omitted.

COROLLARY. *For each triplet (X, Y, Z) of independent random variables there exists a triplet (X_1, Y_1, Z_1) of independent variables such that at least one of the triplets (X_1, Y_1, Z_1), (Y_1, Z_1, X_1), (Z_1, X_1, Y_1) belongs to Σ and*

$$(17)\qquad \begin{gathered} P(X_1 < Y_1) \geqslant P(X < Y), \quad P(Y_1 < Z_1) \geqslant P(Y < Z), \\ P(Z_1 < X_1) \geqslant P(Z < X). \end{gathered}$$

The proof of this corollary is a simple consequence of lemmata 1 and 2 and the fact that for each triplet (X, Y, Z) of independent random variables there exists a triplet $(\bar{X}, \bar{Y}, \bar{Z})$ of independent variables satisfying the condition $P(\bar{X} = \bar{Y}) = P(\bar{Y} = \bar{Z}) = P(\bar{Z} = \bar{X}) = 0$, and such that $P(\bar{X} < \bar{Y}) \geqslant P(X < Y)$, $P(\bar{Y} < \bar{Z}) \geqslant P(Y < Z)$, $P(\bar{Z} < \bar{X}) \geqslant P(Z < X)$.

Denote by $\alpha(\xi, \eta)$ the function defined in the following manner:

$$\alpha(\xi,\eta) = \begin{cases} \max\left(\frac{1-\xi}{\eta}, \frac{1-\eta}{\xi}, 1-\xi\eta\right) & \text{for} \quad \xi+\eta > 1, \\ 1 & \text{for} \quad \xi+\eta \leqslant 1 \end{cases} \tag{18}$$
$$(0 \leqslant \xi \leqslant 1,\ 0 \leqslant \eta \leqslant 1).$$

THEOREM 1. *In order that for every triplet* (ξ, η, ζ) $(0 \leqslant \xi, \eta, \zeta \leqslant 1)$ *there exist three independent random variables* X, Y, Z *such that*

$$P(X = Y) = P(Y = Z) = P(Z = X) = 0, \tag{19}$$

it is necessary and sufficient that

$$1-\alpha(1-\xi, 1-\eta) \leqslant \zeta \leqslant \alpha(\xi, \eta). \tag{20}$$

Proof. (a) We shall first prove that condition (20) is necessary. We are going to demonstrate that if a triplet (X, Y, Z) or some cyclic permutation (Y, Z, X), (Z, X, Y) belongs to the class Σ then the inequality $\zeta \leqslant \alpha(\xi, \eta)$ holds.

If $(X, Y, Z) \epsilon \Sigma$, then the values of these variables are ordered as follows: $x_1 < y_1 < z < x_2 < y_2$. We then have

$$\xi = 1-P(X = x_2)P(Y = y_1), \quad \eta = P(Y = y_1), \quad \zeta = P(X = x_2).$$

Hence $\xi = 1-\eta\zeta$. In other cases we have $\eta = 1-\xi\zeta$, $\zeta = 1-\xi\eta$. Then the inequality $\zeta \leqslant \alpha(\xi, \eta)$ holds.

The function $\alpha(\xi, \eta)$ has the following property: If $\xi_1 > \xi_0$ and $\eta_1 > \eta_0$ then $\alpha(\xi_1, \eta_1) \leqslant \alpha(\xi_0, \eta_0)$. For $\xi_0+\eta_0 > 1$ it is a consequence of the fact that the functions $(1-\xi)/\eta$, $(1-\eta)/\xi$, $1-\xi\eta$ have this property, and that $\alpha(\xi, \eta)$ for $\xi+\eta > 1$ is identical with their maximum. Since for $\xi+\eta \leqslant 1$ the inequality $\zeta \leqslant \alpha(\xi, \eta)$ is trivial, suppose, on the contrary, that there exists a triplet (X_0, Y_0, Z_0) of independent random variables such that $P(X_0 < Y_0) = \xi_0$, $P(Y_0 < Z_0) = \eta_0$, $P(Z_0 < X_0) = \zeta_0$, $\xi_0+\eta_0 > 1$ and $\zeta_0 > \alpha(\xi_0, \eta_0)$. From the corollary to lemma 2 we know that there exists a triplet (X_1, Y_1, Z_1) such that some of its permutations belong to Σ, and $\xi_1 \geqslant \xi_0$, $\eta_1 \geqslant \eta_0$, $\zeta_1 \geqslant \zeta_0$. But for such triplets we have proved $\zeta_1 \leqslant \alpha(\xi_1, \eta_1)$. We then have $\zeta_0 \leqslant \zeta_1 \leqslant \alpha(\xi_1, \eta_1) \leqslant \alpha(\xi_0, \eta_0)$, which contradicts the assumption $\zeta_0 > \alpha(\xi_0, \eta_0)$.

We shall prove the left-hand side of formula (20) as follows: Put $X^{(c)} = -X$, $Y^{(c)} = -Y$, $Z^{(c)} = -Z$ and use assumption (19). Then $\xi^{(c)} = 1-\xi$, $\eta^{(c)} = 1-\eta$, $\zeta^{(c)} = 1-\zeta$. From the result concerning the right-hand side of (20) it follows that

$$1-\zeta = \zeta^{(c)} \leqslant \alpha(\xi^{(c)}, \eta^{(c)}) = \alpha(1-\xi, 1-\eta).$$

Proving the inequality $\zeta \leqslant a(\xi, \eta)$ we did not use assumption (19). This circumstance will be used in the sequel.

(b) Sufficiency. We are going to prove that for every triplet of numbers (ξ, η, ζ) satisfying condition (20) there exists a triplet (X, Y, Z) such that $P(X < Y) = \xi$, $P(Y < Z) = \eta$, $P(Z < X) = \zeta$ and $P(X = Y) = P(Y = Z) = P(Z = X) = 0$.

Denote by S a surface defined as follows: S is composed of the surface given by equation $\zeta = a(\xi, \eta)$ and of the planes $\xi = 0$ for $\eta + \zeta \leqslant 1$, $\eta \geqslant 0$, $\zeta \geqslant 0$ and $\eta = 0$ for $\xi + \zeta \leqslant 1$, $\xi \geqslant 0$, $\zeta \geqslant 0$. Denote by D the space defined by (20). The surface S has the following property. If there is a point $(\xi, \eta, \zeta) \in D$, then there exists a point $(u, v, w) \in S$ and a constant $0 \leqslant \lambda \leqslant 1$ such that

$$\begin{gathered} \xi = \lambda u + (1-\lambda)(1-u), \qquad \eta = \lambda v + (1-\lambda)(1-v), \\ \zeta = \lambda w + (1-\lambda)(1-w). \end{gathered} \tag{21}$$

We shall prove first that if the point $(\xi, \eta, \zeta) \in S$ then there exists a corresponding triplet (X, Y, Z) of independent random variables. Put $P(X = 1) = 1$, $P(Z = -1) = 1$, $P(Y = -2) = \eta$, $P(Y = 0) = 1 - \xi - \eta$, $P(Y = 2) = \xi$. Then $P(X < Y) = \xi$, $P(Y < Z) = \eta$, $P(Z < X) = 1$. Then, if one of the numbers ξ, η, ζ is equal to 1, and the sum of the remaining two numbers is not greater than 1, then the desired triplet (X, Y, Z) can be found. Suppose then that $(\xi, \eta, \zeta) \in S$ and that the case considered above does not occur. Then the surface S may be reduced to the surface given by the equation

$$\zeta = \max\left(\frac{1-\xi}{\eta}, \frac{1-\eta}{\xi}, 1 - \xi\eta\right) \quad (\xi + \eta > 1).$$

But in this case a maximum is attained for a triplet (X, Y, Z) such that some of its permutations belong to the class Σ, because in part (a) of the proof we showed that for such random variables there are $\zeta = (1-\xi)/\eta$, $\zeta = (1-\eta)/\xi$, $\zeta = 1 - \xi\eta$ respectively; and as it is easy to prove that for each pair ξ, η such that $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$, $\xi + \eta > 1$ we can find random variables which are members of class Σ. In both cases considered X, Y, Z satisfy condition (19) and are bounded.

Now suppose that numbers ξ, η, ζ satisfy (20). They may be represented in the form (21) where $(u, v, w) \in S$. Consider a triplet of bounded independent random variables (U, V, W) such that $P(U < V) = u$, $P(V < W) = v$, $P(W < U) = w$ and $P(U = V) = P(V = W) = P(W = U) = 0$. We proved above that such a triplet exists. Let μ_1, μ_2, μ_3 denote, respectively, the distributions of U, V, W. Let $\mu_1^{(c)}, \mu_2^{(c)}, \mu_3^{(c)}$ denote, respectively, the distributions of $2C - U$, $2C - V$, $2C - W$ where a constant C is so large that $P(U < C) = P(V < C) = P(W < C) = 1$. Suppose that X, Y, Z are independent random variables with distributions $\lambda\mu_1 + (1-\lambda)\mu_1^{(c)}$, $\lambda\mu_2 + (1-\lambda)\mu_2^{(c)}$, $\lambda\mu_3 + (1-\lambda)\mu_3^{(c)}$ where λ is the same as in (21). Then

$$\begin{aligned} P(X < Y) &= \lambda^2 P(U < V) + \lambda(1-\lambda)P(U < 2C - V) + \\ &\quad + \lambda(1-\lambda)P(2C - U < V) + (1-\lambda)^2 P(2C - U < 2C - V) \\ &= \lambda^2 u + \lambda(1-\lambda) + (1-\lambda)^2(1-u) \\ &= \lambda u + (1-\lambda)(1-u) = \xi, \\ P(Y < Z) &= \lambda v + (1-\lambda)(1-v) = \eta, \\ P(Z < X) &= \lambda w + (1-\lambda)(1-w) = \zeta, \end{aligned}$$

and because $P(U = V) = P(V = W) = P(W = U) = 0$, we get

$$P(X = Y) = P(Y = Z) = P(Z = X) = 0.$$

This completes the proof.

COROLLARY 1.1. *If the random variables X, Y, Z are independent and $\xi = P(X < Y)$, $\eta = P(Y < Z)$, $\zeta = P(Z < X)$, then*

$$\zeta \leqslant \alpha(\xi, \eta).$$

Corollary is a consequence of the remark made at the end of part (a) of the proof of theorem 1.

If we omit assumption (19), then the left-hand side of (20) no longer holds, as can easily be seen for random variables defined as follows: $P(X = 0) = P(Y = 0) = P(Z = 0) = 1$.

COROLLARY 1.2. *If the random variables X, Y, Z are independent, and $P(X < Y) = P(Y < Z) = P(Z < X)$, then $\max\limits_{(X,Y,Z)} P(X < Y) = \frac{1}{2}(\sqrt{5}-1)$.*

Proof. Since $\alpha(\xi, \eta)$ does not increase when ξ and η increase, $\max\limits_{(X,Y,Z)} \xi$ is the largest root of the equation $\alpha(\xi, \xi) = \xi$, i. e. of $\xi^2 + \xi - 1 = 0$. Thus we have $\max\limits_{(X,Y,Z)} \xi = \frac{1}{2}(\sqrt{5}-1)$.

COROLLARY 1.3. *For each triplet of independent random variables (X, Y, Z) we have*

$$P(X < Y) + P(Y < Z) + P(Z < X) \leqslant 2. \tag{22}$$

Proof. It is sufficient to prove that $\xi + \eta + \alpha(\xi, \eta) \leqslant 2$. If $\xi + \eta \leqslant 1$, it is trivial. If $\xi + \eta > 1$ the inequality $\xi + \eta + \alpha(\xi, \eta) \leqslant 2$ follows from the inequalities $\xi + \eta + (1-\xi)/\eta \leqslant 2$, $\xi + \eta + (1-\eta)/\xi \leqslant 2$, $\xi + \eta + 1 - \xi\eta \leqslant 2$

Later we shall show that the assumption of independence is not necessary for the above result to be obtained.

COROLLARY 1.4. *For each triplet of independent random variables* (X, Y, Z) *we have*

$$P(X < Y)P(Y < Z)P(Z < X) \leqslant \tfrac{1}{4}. \tag{23}$$

The proof is analogous, to the proof of corollary 1.3.

As can be seen the independence of X, Y, Z does not imply that if $P(X < Y) > \frac{1}{2}$ and $P(Y < Z) > \frac{1}{2}$ then $P(X < Z) > \frac{1}{2}$. The situation does not change if we suppose, in addition, that X, Y, Z have the same means and variances. Let $P(X = 1) = P(Y = -1) = t$, $P(X = -a) = P(X = a) = 1-t$, $P(Z = -ab) = P(Z = ab) = \varepsilon$, $P(Z = 0) = 1-2\varepsilon$. If we suppose $0 < \varepsilon < \frac{1}{2}(\sqrt{2}-1)$ and put

$$\frac{\sqrt{5}-1}{2} < t = \frac{2\varepsilon-1+\sqrt{4\varepsilon^2-8\varepsilon+5}}{2} < \frac{\sqrt{2}}{2},$$

$$a = \frac{t}{1-t} > 1, \quad b = \frac{1}{\sqrt{2a\varepsilon}} > 1,$$

then we have

$$E(X) = E(Y) = E(Z), \quad D^2(X) = D^2(Y) = D^2(Z),$$

$$P(X < Y) = P(Y < Z) = P(Z < X) = 1-t^2 > \tfrac{1}{2}.$$

By $E(X)$ and $D^2(X)$ we have denoted the mean and the variance of X.) When $\varepsilon \to 0$, $P(X < Y) \to \frac{1}{2}(\sqrt{5}-1)$.

Suppose that X, Y, Z are independent and have, up to an additive constant, the same distributions. We now prove

THEOREM 2. *If the random variables* X, Y, Z *are independent, and* $P(X < t) = F(t)$, $P(Y < t) = F(t-c)$, $P(Z < t) = F(t-d)$, *then from* $P(X < Y) > \frac{1}{2}$, $P(Y < Z) > \frac{1}{2}$ *we have* $P(X \leqslant Z) > \frac{1}{2}$.

Proof. X, $Y-c$, $Z-d$ have the same distribution. $X-Y+c$, $Y-Z+d-c$, $Z-X-d$ then have the same symmetrical distribution about 0. Write $P(X-Y+c < t) = G(t)$. We have $P(X < Y) = G(c)$, $P(Y < Z) = G(d-c)$, $P(Z < X) = G(-d)$. If $G(c) > \frac{1}{2}$ and $G(d-c) > \frac{1}{2}$, then $d > c > 0$. We then obtain $P(Z < X) = G(-d) \leqslant 1-G(d) \leqslant 1-$ $-G(c) < \frac{1}{2}$, which completes the proof.

In the case when X, Y, Z are not independent the situation is simpler. Suppose that $X = (X_1, \ldots, X_n)$ is an n-dimensional random variable defined on Borel subsets of Euclidean space E_n. Denote the distribution of X by μ and let $\xi_i = P(X_i < X_{i+1})$, for $i = 1, \ldots, n-1$, $\xi_n = P(X_n < X_1)$.

THEOREM 3. *Each random variable* $X = (X_1, \ldots, X_n)$ *satisfies the condition*

$$\sum_{i=1}^{n} \xi_i \leqslant n-1. \tag{24}$$

Proof. Denote by $\delta(x_1, \ldots, x_n)$ a function in E_n defined as follows: $\delta(x_1, \ldots, x_n) = k$ if exactly k of the following n inequalities are satisfied:

$$x_1 < x_2,\ x_2 < x_3,\ \ldots,\ x_{n-1} < x_n,\ x_n < x_1.$$

It is obvious that $\delta \leqslant n-1$ and $\int_{E_n} \delta \, d\mu = \sum_{i=1}^{n} \xi_i$. Since $\int_{E_n} d\mu = 1$, we have the inequality (24).

We state without proof the following corollaries:

COROLLARY 3.1. *For each random variable* $X = (X_1, \ldots, X_n)$ *such that* $P(X_1 = X_2) = P(X_2 = X_3) = \ldots = P(X_{n-1} = X_n) = P(X_n = X_1) = 0$ we have

$$\sum_{i=1}^{n} \xi_i \geqslant 1.$$

COROLLARY 3.2. *Each random variable* $X = (X_1, \ldots, X_n)$ *satisfies the condition*

$$\xi_1 \xi_2 \cdots \xi_n \leqslant \left(\frac{n-1}{n}\right)^n.$$

By the result of this paper we can explain the paradox of three players known especially among chess-players. We shall say that player I is better than player II if he wins more than matches he loses against II. In practice we often observe the phenomenon that player I is better than II, player II is better than III and player III is better than I. It may be explained in the following manner: Suppose that the strength of the play of each player is a random variable and that a player A wins against his opponent B if the strength of the play of A is higher than that of B. Then the problem reduces to the problem of the existence of three independent random variables X, Y, Z such that $P(X > Y) > \frac{1}{2}$, $P(Y > Z) > \frac{1}{2}$, $P(Z > X) > \frac{1}{2}$. By the results of the paper there are many triplets (X, Y, Z) satisfying the above conditions.

References

H. Steinhaus and S. Trybuła, *On a paradox in applied probabilities*, Bull. de l'Acad. Polon. des Sci. VII (1959), p. 67.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Praca wpłynęła 21. 4. 1959

S. TRYBUŁA (Wrocław)

O PARADOKSIE TRZECH ZMIENNYCH LOSOWYCH

STRESZCZENIE

W pracy podane są dowody twierdzeń sformułowanych w pracy [1] oraz pewne uogólnienia zawartych tam rezultatów. Głównym wynikiem pracy jest odpowiedź na pytanie postawione przez H. Steinhausa: Dla jakich trójek liczb rzeczywistych ξ, η, ζ istnieją trzy niezależne zmienne losowe X, Y, Z takie, że

$$P(X < Y) = \xi, \quad P(Y < Z) = \eta, \quad P(Z < X) = \zeta. \tag{1}$$

Odpowiedź ta brzmi następująco: Oznaczmy

$$a(\xi, \eta) = \begin{cases} \max\left(1-\xi\eta, \dfrac{1-\xi}{\eta}, \dfrac{1-\eta}{\xi}\right), & \text{gdy} \quad \xi+\eta > 1, \\ 1 \qquad , & \text{gdy} \quad \xi+\eta \leqslant 1 \end{cases}$$

($0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$). Trójka X, Y, Z niezależnych zmiennych losowych spełniających warunek (1) istnieje wtedy i tylko wtedy, gdy

$$1-a(1-\xi, 1-\eta) \leqslant \zeta \leqslant a(\xi, \eta).$$

W szczególności, jeżeli $\xi = \eta = \zeta$, to

$$\frac{3-\sqrt{5}}{2} \leqslant \zeta \leqslant \frac{\sqrt{5}-1}{2} = 0{,}618\ldots$$

Z twierdzenia tego wynika, że dla niezależnych zmiennych losowych X, Y, Z

$$P(X < Y)P(Y < Z)P(Z < X) \leqslant \tfrac{1}{4}.$$

Nierówność

$$P(X < Y)+P(Y < Z)+P(Z < X) \leqslant 2$$

jest prawdziwa również dla zmiennych losowych zależnych.

Powyższe rezultaty rozciągają się na przypadek, gdy X, Y, Z mają ponadto jednakowe pierwsze dwa momenty. Niektóre wyniki uogólnia się także na przypadek n zmiennych losowych ($n \geqslant 3$).

Końcowa część pracy dotyczy różnych zastosowań praktycznych otrzymanych rezultatów. Oto jedno z takich zastosowań: Będziemy mówili, że gracz G_1 bije gracza G_2 jeżeli częściej z nim wygrywa, niż przegrywa. Wśród graczy, np. szachistów, znane jest następujące zjawisko: Gracz A bije gracza B, gracz B bije C i gracz C bije A. Zjawisko to można wytłumaczyć w sposób następujący. Załóżmy, że forma każdego gracza jest zmienną losową i że wygrywa ten gracz, który w danej chwili ma wyższą formę. Oznaczmy przez X, Y, Z odpowiednio formę gracza A, B, C. Z rezultatów pracy wynika, że mogą istnieć trzy zmienne losowe X, Y, Z takie, że $P(X > Y) > \frac{1}{2}$, $P(Y > Z) > \frac{1}{2}$, $P(Z > X) > \frac{1}{2}$.

С. ТРЫБУЛА (Вроцлав)

О ПАРАДОКСЕ ТРЁХ СЛУЧАЙНЫХ ВЕЛИЧИН

РЕЗЮМЕ

В работе даются доказательства утверждений, сформулированных в [1], а также приводятся некоторые их обобщения. Главным результатом работы является следующая теорема:

Обозначим:

$$a(\xi,\eta)=\begin{cases}\max\left(1-\xi\eta,\ \dfrac{1-\xi}{\eta},\ \dfrac{1-\eta}{\xi}\right), & \text{при}\quad \xi+\eta>1,\\ 1\quad , & \text{при}\quad \xi+\eta\leqslant 1\end{cases}\qquad (0\leqslant\xi\leqslant 1,\ 0\leqslant\eta\leqslant 1).$$

Для того, чтобы для трёх действительных чисел ξ,η,ζ существовали три независимые случайные величины X, Y, Z такие, что

$$P(X<Y)=\xi,\quad P(Y<Z)=\eta,\quad P(Z<X)=\zeta,$$

необходимо и достаточно, чтобы

$$1-a(1-\xi,1-\eta)\leqslant\zeta\leqslant a(\xi,\eta).$$

Заключительная часть работы касается практического приложения полученных результатов.