ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIV, 3 (1974)

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SOME FOURTH-ORDER FORMULAE OF A CERTAIN METHOD OF ZURMÜHL

1. Let us consider the initial-value problem

$$\frac{dx}{dt} = f(t, x), \tag{1.1}$$

$$x(t_0) = x_0. \tag{1.2}$$

In this paper we describe two one-step fourth-order methods analogous to the Runge-Kutta method.

We assume that it is possible to evaluate the function

$$g(t, x) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} f. \tag{1.3}$$

Zurmühl gave in paper [3] a method of numerical integration of problem (1.1)-(1.2) using the function $g(t, x)$. It is a particular case of the first of the methods described in Section 2.

The approximative solution of problem (1.1)-(1.2) at the point $t_1 = t_0 + h$ will be expressed by

$$x_1 = x_0 + h\varphi(t_0, x_0, h), \qquad x(t_1) = x_1 + O(h^5), \tag{1.4}$$

where φ is a linear combination of values of the functions f and g at some selected points of the set $\{t_0 \leqslant t \leqslant t_1, -\infty < x < \infty\}$.

2. The first method is of the form

$$x_1 = x_0 + a_0 k_0 + a_1 k_1 + b_0 g_0 + b_1 g_1, \tag{2.1}$$

where

$$\begin{cases} g_0 = 0.5h^2 g(t_0, x_0), \qquad k_0 = hf(t_0, x_0), \\ g_1 = 0.5h^2 g(t_0 + M_1 h, x_0 + L_{10} k_0 + P_{10} g_0), \\ k_1 = hf(t_0 + M_1 h, x_0 + R_{10} k_0 + E_{10} g_0 + E_{11} g_1). \end{cases} \tag{2.2}$$

The second method is of the form

$$x_1 = x_0 + a_0 k_0 + a_1 k_1 + a_2 k_2, \tag{2.3}$$

where

$$\begin{aligned} k_0 &= hf(t_0, x_0), \quad k_1 = hf(t_0 + M_1 h, x_0 + R_{10} k_0 + E_{11} g_1), \\ k_2 &= hf(t_0 + M_2 h, x_0 + R_{20} k_0 + R_{21} k_1 + E_{22} g_2) \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} g_1 &= 0.5h^2 g(t_0 + M_1 h, x_0 + L_{10} k_0), \\ g_2 &= 0.5h^2 g(t_0 + M_2 h, x_0 + L_{20} k_0 + L_{21} k_1). \end{aligned}$$

Let us introduce the operator D,

$$D = \frac{\partial}{\partial t} + f_0 \frac{\partial}{\partial x}, \tag{2.5}$$

where $f_0 = f(t_0, x_0)$, and $Df_0 = Df|_{(t_0, x_0)}$.

We expand the exact solution $x(t)$ at the point t_1 into the Taylor series in the neighborhood of the point t_0. Thus we obtain

$$\begin{aligned} x(t_1) &= x(t_0) + hf_0 + \frac{1}{2} h^2 Df_0 + \frac{1}{3!} h^3 \big(D^2 f_0 + (f_x)_0 Df_0\big) + \\ &+ \frac{1}{4!} h^2 \big(D^3 f_0 + (f_x)_0 D^2 f_0 + (f_x^2)_0 Df_0 + 3Df_0 (Df_x)_0\big) + \\ &+ \frac{1}{5!} h^5 \big(D^4 f_0 + 6Df_0 (D^2 f_x)_0 + 4D^2 f_0 (Df_x)_0 + (f_x^2)_0 D^2 f_0 + \\ &+ (f_x^3)_0 Df_0 + 3(Df_0)^2 (f_{xx})_0 + (f_x)_0 D^3 f_0 + \\ &+ 7(f_x)_0 Df_0 (Df_x)_0\big) + O(h^6). \end{aligned} \tag{2.6}$$

Now we expand also the right-hand parts of (2.1) and (2.3) into power series. From (2.3) we have, evidently, the expansions for g_1 and k_1,

(2.7′)

$$\begin{aligned} g_1 &= \frac{h^2}{2} Df_0 + \frac{h^3}{2} \big(D_{11} Df_0 + (f_x)_0 D_{11} f_0\big) + h^4 \Big[\frac{1}{4} P_{10} \big(Df_0 (Df_x)_0 + (f_x^2)_0 Df_0\big) + \\ &+ \frac{1}{4} \big(D_{11}^2 Df_0 + 2(D_{11} f_x)_0 D_{11} f_0 + (f_x)_0 D_{11}^2 f_0\big)\Big] + \\ &+ h^5 \Big[\frac{1}{4} P_{10} \big(Df_0 (D_{11} Df_x)_0 + 2(f_x)_0 Df_0 (D_{11} f_x)_0 + \\ &+ (f_{xx})_0 Df_0 D_{11} f_0\big) + \frac{1}{12} \big(D_{11}^3 Df_0 + 3D_{11}^2 f_0 (D_{11} f_x)_0 + \\ &+ 3D_{11} f_0 (D_{11}^2 f_x)_0 + (f_x)_0 D_{11}^3 f_0\big)\Big] + O(h^6), \end{aligned}$$

$$(2.7'')\quad k_1 = hf_0 + h^2 D_{12}f_0 + h^3\left[\frac{1}{2}(E_{10}+E_{11})(f_x)_0 Df_0 + \frac{1}{2}D_{12}^2 f_0\right] +$$
$$+ h^4\left[\frac{E_{11}}{2}\big((f_x)_0 D_{11}Df_0 + (f_x^2)_0 D_{11}f_0\big) + \frac{1}{2}(E_{10}+E_{11})Df_0(D_{12}f_x)_0 + \right.$$
$$\left. + \frac{1}{3!}D_{12}^3 f_0\right] + h^5\left[\frac{P_{10}E_{11}}{4}\big((f_x)_0 Df_0(Df_x)_0 + (f_x^3)_0 Df_0\big) + \right.$$
$$+ \frac{E_{11}}{4}\big((f_x)_0^2 D_{11}^2 Df_0 + 2(f_x)_0(D_{11}f_x)_0 D_{11}f_0 + (f_x^2)_0 D_{11}^2 f_0\big) +$$
$$+ \frac{E_{11}}{2}\big(D_{11}Df_0(D_{12}f_x)_0 + (f_x)_0(D_{12}f_x)_0 D_{11}f_0\big) +$$
$$+ \frac{1}{4!}D_{12}^4 f_0 + \frac{1}{8}(E_{10}+E_{11})^2(Df_0)^2(f_{xx})_0 +$$
$$\left. + \frac{1}{4}(E_{10}+E_{11})Df_0(D_{12}^2 f_x)_0\right] + O(h^6),$$

where

$$(2.8)\qquad D_{11} = M_1\frac{\partial}{\partial t} + L_{10}f_0\frac{\partial}{\partial x},\qquad D_{12} = M_1\frac{\partial}{\partial t} + R_{10}f_0\frac{\partial}{\partial x}.$$

The explicit expansion of x_1 in (2.1) can immediately be found from formulae (2.7).

Analogously, for g_1, k_1, g_2 and k_2 from (2.4) we have

$$g_1 = \frac{h^2}{2}Df_0 + \frac{h^3}{2}\big(D_{11}Df_0 + (f_x)_0 D_{11}f_0\big) +$$
$$+ \frac{h^4}{4}\big(D_{11}^2 Df_0 + 2(D_{11}f_x)_0 D_{11}f_0 + (f_x)_0 D_{11}^2 f_0\big) +$$
$$+ \frac{h^5}{12}\big(D_{11}^3 Df_0 + 3D_{11}^2 f_0(D_{11}f_x)_0 + 3D_{11}f_0(D_{11}^2 f_x)_0 + (f_x)_0 D_{11}^3 f_0\big) +$$
$$+ O(h^6),$$

(2.9′)

$$k_1 = hf_0 + h^2 D_{12}f_0 + h^3\left(\frac{1}{2}E_{11}(f_x)_0 Df_0 + \frac{1}{2}D_{12}^2 f_0\right) +$$
$$+ h^4\left[\frac{E_{11}}{2}\big((f_x)_0 D_{11}Df_0 + (f_x^2)_0 D_{11}f_0 + Df_0(D_{12}f_x)_0\big) + \frac{1}{3!}D_{12}^3 f_0\right] +$$
$$+ h^5\left[\frac{E_{11}}{4}\big((f_x)_0 D_{11}^2 Df_0 + 2(f_x)_0(D_{11}f_x)_0 D_{11}f_0 + \right.$$

$$+(f_x^2)_0 D_{11}^2 f_0 + 2D_{11} Df_0 (D_{12} f_x)_0 + 2(f_x)_0 D_{11} f_0 (D_{12} f_x)_0 +$$

$$+ Df_0 (D_{12}^2 f_x)_0\Big) + \frac{1}{8} E_{11}^2 (Df_0)(f_{xx})_0 + \frac{1}{4!} D_{12}^4 f_0\Big] + O(h^6),$$

$$g_2 = \frac{h^2}{2} Df_0 + \frac{h^3}{2}\big(D_{21} Df_0 + (f_x)_0 D_{21} f_0\big) + h^4\Big[\frac{1}{2} L_{21} D_{12} f_0\big((f_x^2)_0 + (Df_x)_0\big) +$$

$$+ \frac{1}{4}\big(D_{21}^2 Df_0 + 2(D_{21} f_x)_0 D_{21} f_0 + (f_x)_0 D_{21}^2 f_0\big)\Big] + O(h^6),$$

(2.9″)

$$k_2 = hf_0 + h^2 D_{22} f_0 + h^3\left(R_{21}(f_x)_0 D_{12} f_0 + \frac{E_{22}}{2}(f_x)_0 Df_0 + \frac{1}{2} D_{22}^2 f_0\right) +$$

$$+ h^4\Big[\frac{1}{2} R_{21}\big(E_{11}(f_x^2)_0 Df_0 + (f_x)_0 D_{12}^2 f_0\big) +$$

$$+ \frac{E_{22}}{2}\big((f_x)_0 D_{21} Df_0 + (f_x^2)_0 D_{21} f_0\big) + R_{21} D_{12} f_0 (D_{22} f_x)_0 +$$

$$+ \frac{E_{22}}{2} Df_0 (D_{22} f_x)_0 + \frac{1}{3!} D_{22}^3 f_0\Big] + h^5\Big[\frac{R_{21} E_{11}}{2}\big((f_x^2)_0 D_{11} Df_0 +$$

$$+ (f_x^3)_0 D_{11} f_0 + (f_x)_0 Df_0 (D_{12} f_x)_0\big) + \frac{R_{21}}{3}(f_x)_0 D_{12}^3 f_0 +$$

$$+ \frac{E_{22} L_{21}}{2} D_{12} f_0\big((f_x)_0 (Df_x)_0 + (f_x^3)_0\big) + \frac{1}{4} E_{22}\big((f_x)_0 D_{21}^2 Df_0 +$$

$$+ 2(f_x)_0 (D_{21} f_x)_0 D_{21} f_0 + (f_x^2)_0 D_{21}^2 f_0\big) + R_{21}\left(\frac{1}{2} D_{12}^2 f_0 (D_{22} f_x)_0 +\right.$$

$$\left. + \frac{1}{2} E_{11} (f_x)_0 Df_0 (D_{22} f_x)_0\right) + \frac{E_{22}}{2}\big(D_{21} Df_0 (D_{22} f_x)_0 +$$

$$+ (f_x)_0 (D_{21} f_x)_0 (D_{22} f_x)_0\big) + \frac{1}{2}\left(R_{21} D_{12} f_0 + \frac{E_{22}}{2} Df_0\right)^2 (f_{xx})_0 +$$

$$+ \frac{1}{2}(D_{22}^2 f_x)_0 \left(R_{21} D_{12} f_0 + \frac{E_{22}}{2} Df_0\right) + \frac{1}{4!} D_{22}^4 f_0\Big] + O(h^6).$$

D_{11} and D_{12} are defined as in formula (2.8) and

(2.10)

$$D_{21} = M_2 \frac{\partial}{\partial t} + (L_{20} + L_{21}) f_0 \frac{\partial}{\partial x}, \qquad D_{22} = M_2 \frac{\partial}{\partial t} + (R_{20} + R_{21}) f_0 \frac{\partial}{\partial x}.$$

From expansions (2.9) we can obtain — as a linear combination — the expansion for x_1 in (2.3).

The constant parameters in formulae (2.1)-(2.2) and also in (2.3)-(2.4) can be evaluated similarly as in the Runge-Kutta process (see Ralston [2], Section 5.6.3), that is (for a fourth-order method) the coefficients at h^r, $r = 1, 2, 3, 4$, in the Taylor series of (2.1) and (2.3) must agree with the respective coefficients of the Taylor series for the exact solution (2.6).

If we assume that, for the first method,

$$L_{10} = M_1, \quad P_{10} = M_1^2, \quad R_{10} = M_1, \quad E_{10}+E_{11} = M_1^2, \tag{2.11}$$

and, for the second method,

$$\begin{gathered} R_{10} = L_{10} = M_1, \quad E_{11} = M_1^2, \quad L_{20}+L_{21} = M_2, \\ M_1 L_{21} = 0.5 M_2^2, \quad R_{20}+R_{21} = M_2, \quad R_{21} M_1 + 0.5 E_{22} = 0.5 M_2^2, \end{gathered} \tag{2.12}$$

then from (2.11) and (2.12) we have

$$D_{11} = D_{12} = M_1 D, \quad D_{21} = D_{22} = M_2 D. \tag{2.13}$$

We obtain the remaining coefficients in (2.1)-(2.2) from the following system of equations:

$$\left\{ \begin{aligned} a_0 + a_1 &= 1, \\ M_1 a_1 + \frac{1}{2} b_0 + b_1 &= \frac{1}{2}, \\ 0.5 M_1^2 a_1 + 0.5 M_1 b_1 &= \frac{1}{6}, \\ 0.5 E_{11} M_1 a_1 + 0.25 M_1^2 b_1 &= \frac{1}{24}, \\ 0.5 M_1^2 a_1 + 0.75 M_1^2 b_1 &= \frac{1}{8}. \end{aligned} \right. \tag{2.14}$$

Solving this system, we have

$$\begin{gathered} a_0 = \frac{2M_1^3 - 2M_1 + 1}{2M_1^3}, \quad a_1 = \frac{2M_1 - 1}{2M_1^3}, \\ b_0 = \frac{6M_1^2 - 8M_1 + 3}{6M_1^2}, \quad b_1 = \frac{3 - 4M_1}{6M_1^2}, \quad E_{11} = \frac{M_1^2}{3}, \end{gathered} \tag{2.15}$$

and from (2.11) we obtain

$$E_{10} = \frac{2M_1^2}{3}. \tag{2.16}$$

The coefficients in formulae (2.3) - (2.4) are evaluated from the following system of equations

$$
\left\{
\begin{aligned}
a_0+a_1+a_2 &= 1, \\
M_1a_1+M_2a_2 &= \frac{1}{2}, \\
0.5M_1^2a_1+0.5M_2^2a_2 &= \frac{1}{6}, \\
0.5M_1^3a_1+0.5R_{21}M_1^2+E_{22}M_2a_2 &= \frac{1}{24}, \\
0.25M_1^3a_1+0.5M_2^3a_2 &= \frac{1}{8}.
\end{aligned}
\right.
\tag{2.17}
$$

As the result from (2.17) and (2.12) we have

$$M_2 = \frac{3-4M_1}{2(2-3M_1)} \tag{2.18}$$

and

(2.19)

$$a_0 = \frac{6M_1M_2-3(M_1+M_2)+2}{6M_1M_2},$$

$$a_1 = \frac{3M_2-2}{6M_1(M_2-M_1)}, \quad a_2 = \frac{2-3M_1}{6M_2(M_2-M_1)},$$

$$L_{21} = 0.5\frac{M_2^2}{M_1}, \quad L_{20} = 0.5M_2\frac{2M_1-M_2}{M_1},$$

$$R_{20} = \frac{M_2(M_2-M_1+8M_1M_2-18M_1^2M_2+6M_1M_2^2+6M_1^3-4M_2^2)}{2M_1(2M_2-M_1)(2-3M_1)},$$

$$R_{21} = \frac{M_2(M_2-M_1)[4(M_2+M_1)-6M_1M_2-1]}{2M_1(2M_2-M_1)(2-3M_1)},$$

$$E_{22} = \frac{M_2(M_2-M_1-3M_1^2M_2+4M_1^2-2M_1M_2)}{(2M_2-M_1)(2-3M_1)}.$$

3. The free parameter M_1 in formulae (2.1) - (2.2) and (2.3) - (2.4) can be chosen so that the absolute value of the coefficient a_5 at h^5 in the power series for the difference $x_1-x(t_1)$ will have a minimal overestimation. The coefficient a_5 can be described by

$$a_5 = \sum_{i=1}^{8} d_i(M_1)e_i = \big(\boldsymbol{d}(M_1), \boldsymbol{e}\big). \tag{3.1}$$

For the first method,

$$(3.2)\qquad \begin{aligned}
d_1(M_1) &= \frac{5}{4} M_1^2 E_{11} a_1 + \frac{1}{2} M_1^3 b_1 - \frac{7}{120},\\
d_2(M_1) &= \frac{1}{4} M_1^2 E_{11} a_1 - \frac{1}{120},\\
d_3(M_1) &= \frac{1}{4} M_1^2 E_{11} a_1 + \frac{1}{12} M_1^3 b_1 - \frac{1}{120},\\
d_4(M_1) &= \frac{1}{4} M_1^2 E_{11} a_1 - \frac{1}{120},\\
d_5(M_1) &= \frac{1}{2} M_1^2 E_{11} a_1 + \frac{1}{4} M_1^3 b_1 - \frac{1}{30},\\
d_6(M_1) &= \frac{1}{8} M_1^4 a_1 + \frac{1}{4} M_1^3 b_1 - \frac{1}{40},\\
d_7(M_1) &= \frac{1}{4} M_1^4 a_1 + \frac{1}{2} M_1^3 b_1 - \frac{1}{20},\\
d_8(M_1) &= \frac{1}{24} M_1^4 a_1 + \frac{1}{12} M_1^3 b_1 - \frac{1}{120},
\end{aligned}$$

and, for the second method,

$$(3.3)\qquad \begin{aligned}
d_1(M_1) &= M_1^4 a_1 + \left(\frac{1}{2} R_{21} M_1^3 + \frac{5}{4} E_{22} M_2^2 + \frac{1}{2} R_{21} M_1^2 M_2\right) a_2 - \frac{7}{120},\\
d_2(M_1) &= \left(\frac{1}{2} R_{21} M_1^3 + \frac{1}{4} E_{22} M_2^2\right) a_2 - \frac{1}{120},\\
d_3(M_1) &= \frac{1}{4} M_1^4 a_1 + \left(\frac{1}{6} R_{21} M_1^3 + \frac{1}{4} E_{22} M_2^2\right) a_2 - \frac{1}{120},\\
d_4(M_1) &= \frac{1}{4} M_1^4 a_1 + \frac{1}{2} (R_{21} M_1^3 + E_{22} M_2^2) a_2 - \frac{1}{120},\\
d_5(M_1) &= \frac{1}{2} M_1^4 a_1 + \frac{1}{2} (M_1^2 M_2 R_{21} + E_{22} M_2^2) a_2 - \frac{1}{30},\\
d_6(M_1) &= \frac{1}{8} M_1^4 a_1 + \frac{1}{8} M_2^4 a_2 - \frac{1}{40},\\
d_7(M_1) &= \frac{1}{4} M_1^4 a_1 + \frac{1}{4} M_2^4 a_2 - \frac{1}{20},\\
d_8(M_1) &= \frac{1}{24} M_1^4 a_1 + \frac{1}{24} M_2^4 a_2 - \frac{1}{120}.
\end{aligned}$$

The components of the vector $\boldsymbol{e}$ are equal to

$$(3.4)\quad e_1 = (f_x)_0 Df_0 (Df_x)_0, \quad e_2 = (f_x^3)_0 Df_0, \quad e_3 = (f_x)_0 D^3 f_0, \quad e_4 = (f_x^2)_0 D^2 f_0,$$
$$e_5 = D^2 f_0 (Df_x)_0, \quad e_6 = (Df)_0^2 (f_{xx})_0, \quad e_7 = Df_0 (D^2 f_x)_0, \quad e_8 = D^4 f_0.$$

From (3.1) and in virtue of the Schwarz inequality, we get

$$(3.5)\qquad |a_5| \leqslant |\boldsymbol{d}(M_1)|\,|\boldsymbol{e}|.$$

$|a_5|$ will have a minimal overestimation in the class of functions $f(t, x) \in C^4$ for $\boldsymbol{d}$ having the minimal norm. Therefore, we must find the minimum of the function

$$(3.6)\qquad \varphi(M_1) = \sum_{i=1}^{8} d_i^2(M_1)$$

in the interval $(0, 1\rangle$.

For method (2.1)-(2.2), we get

$$(3.7)\qquad M_1 = 6.4037505_{10}-1$$

and

$$(3.8)\qquad |\boldsymbol{d}(M_1)| = 9.11728888_{10}-3.$$

For the second method (2.3)-(2.4), we have

$$(3.9)\qquad M_1 = 3.0446_{10}-1$$

and

$$(3.10)\qquad |\boldsymbol{d}(M_1)| = 7.01518036_{10}-3.$$

Formulae (2.1)-(2.2) for $M_1 = 0.5$ give, as a particular case, the method of Zurmühl (see [3])

$$(3.11)\qquad x_1 = x_0 + k_0 + \tfrac{1}{3} g_0 + \tfrac{2}{3} g_1,$$

where

$$(3.12)\qquad \begin{aligned} g_0 &= 0.5h^2 g(t_0, x_0), \qquad k_0 = hf(t_0, x_0), \\ g_1 &= 0.5h^2 g(t_0 + 0.5h, x_0 + 0.5k_0 + 0.25g_0). \end{aligned}$$

In this case

$$(3.13)\qquad |\boldsymbol{d}(M_1)| = 2.57600514_{10}-2.$$

Substituting M_1 from (3.7) into (2.15)-(2.16), we obtain

$$(3.14)\qquad x_1 = x_0 + 0.465451992k_0 + 0.534548008k_1 + \\ + 0.137160497g_0 + 0.178217088g_1,$$

where

$$(3.15)\quad \begin{cases} k_0 = hf(t_0, x_0), \quad g_0 = 0.5h^2 g(t_0, x_0), \\ g_1 = 0.5h^2 g(t_0 + 0.64037505h, x_0 + 0.64037505k_0 + \\ \qquad\qquad + 0.410080205g_0), \\ k_1 = hf(t_0 + 0.64037505h, x_0 + 0.64037505k_0 + \\ \qquad\qquad + 0.136693402g_0 + 0.273386803g_1). \end{cases}$$

Substituting M_1 from (3.9) into (2.18)-(2.19), we have

$$(3.16)\quad x_1 = x_0 + 0.0831141507k_0 + 0.4885493590k_1 + 0.42833649k_2,$$

where

$$(3.17)\quad \begin{cases} k_0 = hf(t_0, x_0), \quad g_1 = 0.5h^2 g(t_0 + 0.30446h, x_0 + 0.30446k_0), \\ k_1 = hf(t_0 + 0.30446h, x_0 + 0.30446k_0 + 0.0926958916g_1), \\ g_2 = 0.5h^2 g(t_0 + 0.820047487h, x_0 + 1.10437805k_1 - \\ \qquad\qquad - 0.28433056k_0), \\ k_2 = hf(t_0 + 0.820047487h, x_0 + 0.956853942k_1 - \\ \qquad\qquad - 0.136806456k_0 + 0.0898303778g_2). \end{cases}$$

Let us assume that the functions f and g satisfy the Lipschitz condition with respect to x. We can easily prove that the one-step methods described in this paper are convergent since they are consistent in the sense defined by Henrici (see [1], Section 2.2.1), namely, they satisfy

$$(3.18)\quad \varphi(t, x, 0) = f(t, x),$$

where, for the first method,

$$\varphi(t_0, x_0, h) = \frac{1}{h}(a_0k_0 + a_1k_1 + b_0g_0 + b_1g_1)$$

and, for the second one,

$$\varphi(t_0, x_0, h) = \frac{1}{h}(a_0k_0 + a_1k_1 + a_2k_2),$$

which is the sufficient and necessary condition of convergence.

4. Let us assume that

$$(4.1)\quad |f| < M, \quad \left|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}\right| < \frac{L^{i+1}}{M^{j-1}}, \quad i + j \leqslant 4.$$

Then we can estimate α_5 in the manner similar to that proposed by Ralston (see [2], Section 5.6.3). For method (2.1)-(2.2) we have the minimal estimation for $M_1 = 6.4209415\cdot 10^{-1}$, namely,

$$|\alpha_5| \leqslant 1.00111836\cdot 10^{-1} L^4 M,$$

whereas for M_1 defined by (3.7) we obtain

$$|\alpha_5| \leqslant 1.00749365\cdot 10^{-1} L^4 M.$$

For the second method (2.3)-(2.4) the "optimal" M_1 is equal to $3.42423592\cdot 10^{-1}$ and for this value we have

$$|\alpha_5| \leqslant 5.52066848\cdot 10^{-2} L^4 M,$$

whereas for the value defined by (3.9) we have

$$|\alpha_5| \leqslant 0.602897564 L^4 M.$$

The criterion to choose M_1 without assuming (4.1), proposed in Section 3, is, evidently, useful in practice.

5. These methods give in many cases somewhat better results than the popular Runge-Kutta process defined by

$$x_1 = x_0 + \tfrac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3), \tag{5.1}$$

where

$$\begin{aligned} k_0 &= hf(t_0, x_0), & k_1 &= hf(t_0 + 0.5h, x_0 + 0.5k_0), \\ k_2 &= hf(t_0 + 0.5h, x_0 + 0.5k_1), & k_3 &= hf(t_0 + h, x_0 + k_2). \end{aligned} \tag{5.2}$$

If we assume that inequalities (4.1) hold, then we have here (¹) — in the same manner as above — the estimation $|\alpha_5| \leqslant 0.1014 L^4 M$.

The above-proposed methods were tested on the Odra 1204 computer with 37-bit floating-point mantissa. The results were compared with respective results of method (5.1)-(5.2). Some results are shown in the tables enclosed below.

I. $x' = x + t + 1$, $x(0) = -1$, $h = 0.1$. Exact solution: $x(t) = \exp(t) - 2 - t$.

t_k	x_k from (3.14)-(3.15)	error	x_k from (3.16)-(3.17)	error
0.1	−0.994829092	$-0.99_{10}-8$	−0.994829043	$0.39_{10}-7$
0.5	−0.851278803	$-0.73_{10}-7$	−0.851278440	$0.29_{10}-6$
0.8	−0.574459230	$-0.16_{10}-6$	−0.574458447	$0.62_{10}-6$
1.0	−0.281718413	$-0.24_{10}-6$	−0.281717217	$0.95_{10}-6$

(¹) The estimation for the "optimal" variant of the Runge-Kutta method, given by Ralston, is equal to $5.4627\cdot 10^{-2} L^4 M$ and is wrong; it should be of the form $0.1038261 L^4 M$. This remark is due to T. Pokora.

t_k	x_k from (5.1)-(5.2)	error
0.1	-0.994829167	$-0.85_{10}-7$
0.5	-0.851279361	$-0.63_{10}-6$
0.8	-0.574460437	$-0.14_{10}-5$
1.0	-0.281720256	$-0.21_{10}-5$

II. $x' = -x\operatorname{cotan}\left(\frac{1}{t}\right)\frac{1}{t^2}$, $x(1) = 1$, $h = 0.1$. Exact solution

$$x(t) = \frac{1}{\sin(1)}\sin\left(\frac{1}{t}\right).$$

t_k	x_k from (3.14)-(3.15)	error	x_k from (3.16)-(3.17)	error
1.1	0.937578322	$-0.61_{10}-6$	0.937578983	$0.56_{10}-7$
1.5	0.734866728	$-0.92_{10}-6$	0.734867696	$0.48_{10}-7$
1.7	0.659432220	$-0.85_{10}-6$	0.659433100	$0.34_{10}-7$
2.0	0.569746230	$-0.73_{10}-6$	0.569746984	$0.21_{10}-7$

t_k	x_k from (5.1)-(5.2)	error
1.1	0.937579254	$0.33_{10}-6$
1.5	0.734868152	$0.50_{10}-6$
1.7	0.659433537	$0.47_{10}-6$
2.0	0.569747379	$0.42_{10}-6$

The time of calculation is, generally, longer for methods (3.14)-(3.15) and (3.16)-(3.17) than for (5.1)-(5.2), since the time of calculating the value of the function g is greater than that for the function f. For example, for problem II, the calculation time for methods (3.14)-(3.15) is twice longer than that for (5.1)-(5.2). For methods (3.16)-(3.17) this time is even greater. Sometimes, however, the function g is simpler than f. It is so, e.g., for

$$x' = \frac{t^3(4+3t)}{1+t} + \frac{x}{1+t},$$

where $g(t, x) = 12t^2$. For this equation the calculation time for methods (3.14)-(3.15) is the same as for methods (5.1)-(5.2).

References

[1] P. Henrici, *Discrete variable methods in ordinary differential equations*, J. Wiley, New York 1964.
[2] A. Ralston, *A first course in numerical analysis*, New York 1965 (see also Polish edition: *Wstęp do analizy numerycznej*, PWN, Warszawa 1971).
[3] R. Zurmühl, *Runge-Kutta-Verfahren unter Verwendung höherer Ableitungen*, Z. Angew. Math. Mech. 32 (1952), p. 153-154.

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Received on 10. 3. 1973;
revised version on 5. 1. 1974

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KILKA WZORÓW RZĘDU CZWARTEGO PEWNEJ METODY ZURMÜHLA

STRESZCZENIE

W pracy podane są dwie jednokrokowe metody rozwiązywania zadania początkowego dla zwyczajnego równania różniczkowego pierwszego rzędu, przy założeniu możliwości obliczania pierwszej pochodnej prawej strony równania. Podane wzory są czwartego rzędu; wyprowadzone zostały podobnie jak wzory w metodzie Rungego-Kutty. Zilustrowano je przykładami obliczeń numerycznych. Uzyskane wyniki były porównywane z wynikami klasycznej metody Rungego-Kutty czwartego rzędu.