ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
Hugo Steinhaus Jubilee Volume
X (1969)

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THE CORRELOID

The paper presents a simple theorem that has a bearing upon the distinctions between (i) unirelational vs multirelational models, and (ii) two types of multirelational models, namely causal chain systems vs interdependent systems. The notion of correloid is introduced for the sake of geometric interpretation of the points at issue under (i)-(ii).

1. Let

$$R_n = \begin{bmatrix} 1 & r_{12} & \ldots & r_{1n} \\ r_{21} & 1 & \ldots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & \ldots & 1 \end{bmatrix} \tag{1}$$

be a correlation matrix of order n; that is, a matrix with entries r_{ik} that are correlation coefficients of n random variates, say z_i,

$$r_{ik} = r(z_i, z_k); \quad i, k = 1, \ldots, n.$$

As is well known, R_n is real-valued and symmetric,

$$r_{ik} = r_{ki}; \quad i, k = 1, \ldots, n$$

and nonnegative definite; this last property is equivalent to

$$\det R_k \geqslant 0; \quad k = 1, \ldots, n.$$

We shall refer to R_n as a point in a Euclidean space $X_{\frac{1}{2}n(n-1)}$ with coordinates

$$x_{ik}; \quad i < k = 1, \ldots, n$$

letting r_{ik} for each pair i, k $(i < k)$ be measured along the x_{ik} axis.

2. Two definitions: the correloid and its points of infinity for a corresponding regression coefficient.

The correloid C_n is the solid in $X_{\frac{1}{2}n(n-1)}$, the surface of which is formed by points R_n such that

$$\det R_n = 0. \tag{2}$$

The correloids C_2 and C_3 are shown in Fig. 1a-b. For $n = 2$ we obtain C_2 as the linear segment

$$-1 \leqslant x_{12} \leqslant 1$$

with a "surface" formed by the two points $x_{12} = \pm 1$.

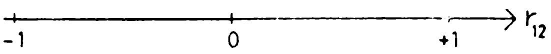

Fig. 1a. The correloid C_2

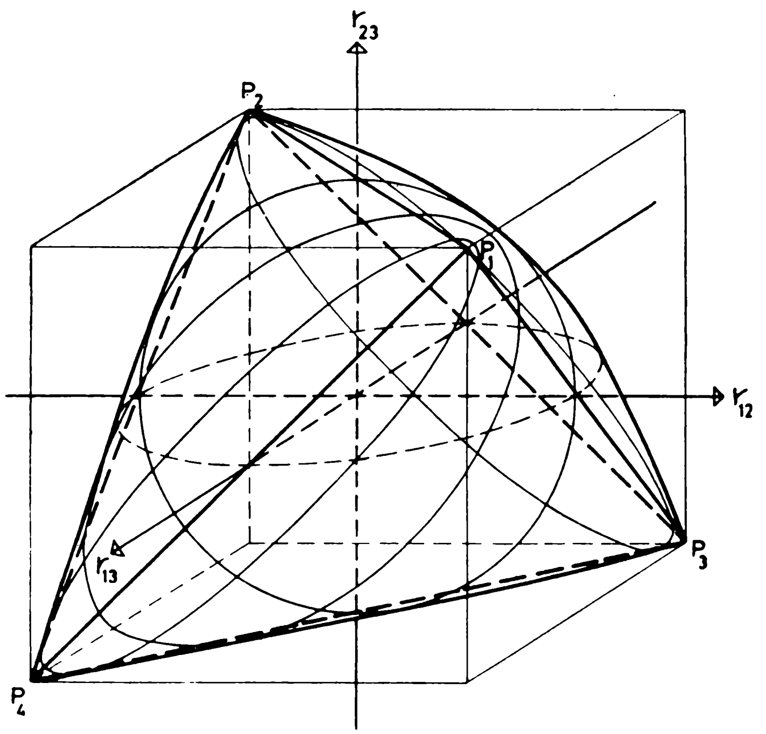

Fig. 1b. The correloid C_3

For $n = 3$ relation (2) gives

$$(3) \qquad 1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2 r_{12} r_{13} r_{23} = 0$$

as the equation for the surface of the correloid C_3. The surface (3) is of 3rd degree. We see that the surface has pointed peaks in the four points P_1 to P_4 with coordinates as follows,

	r_{12}	r_{13}	r_{23}
P_1	1	1	1
P_2	−1	−1	1
P_3	1	−1	−1
P_4	−1	1	−1

The six line segments between the points P_1 to P_4 form the edges of a tetrahedron. These six line segments lie in the surface of the correloid C_3. They make no edges of the correloid, however; except for the corners P_i its surface is curvilinear. The plane $r_{13} = 0$ cuts the surface (3) along the circle

$$r_{12}^2 + r_{23}^2 = 1$$

as shown in Fig. 1b. The planes

$$r_{13} = \pm\lambda \quad \text{with} \quad 0 < \lambda < 1$$

cut the surface (3) along ellipses. The three ellipses with $\lambda = 0.9$, $\lambda = 0.5$, $\lambda = -0.9$ are shown in Fig. 1b. As $\lambda \to 1$ the ellipses degenerate to the line segment

$$r_{12} = r_{23}$$

between the points P_1 and P_4; as $\lambda \to -1$ the ellipses degenerate to

$$r_{12} = -r_{23}$$

giving the segment between the points P_2 and P_3. The intersections between the surface (3) and the planes $r_{12} = \pm\lambda$ and $r_{23} = \pm\lambda$ give the same picture. The broken curve in Fig. 1b is the circle (in perspective seen as an ellipse) along which the plane $r_{23} = 0$ cuts the correloid.

We note two properties of the general correloid C_n that are immediate implications of its definition:

(i) The correloid C_n is convex.

(ii) If a point R_n belongs to C_n, the point with coordinates

$$x_{ik} = \lambda r_{ik}; \quad 0 \leqslant \lambda < 1 \quad (i < k;\ i, k = 1, \ldots, n)$$

will for any λ lie in the interior of C_n.

Points of infinity. Given n random variates $z_1, \ldots, z_n$ with correlation matrix (1), let

$$y = \beta_1 z_1 + \ldots + \beta_n z_n + \varepsilon \tag{4}$$

be the regression of a random variate y on $z_1, \ldots, z_n$. We shall consider a sequence of vector variates

$$V^{(1)}, V^{(2)}, \ldots$$

with

$$V^{(k)} = (y^{(k)}, z_1^{(k)}, \ldots, z_n^{(k)})$$

such that

$$R_n^{(k)} \to R_n \quad \text{as} \quad k \to \infty$$

where, in obvious notation,

$$R_n^{(1)}, R_n^{(2)}, \ldots$$

is the corresponding sequence of points (1). Forming the corresponding regressions of type (4), we write

$$\beta_1^{(k)}, \ldots, \beta_n^{(k)}; \quad k = 1, 2, \ldots$$

for the resulting sequence of regression coefficients. We shall then call R_n a *point of infinity* of β_i if

$$\beta_i^{(k)} \to \pm \infty \quad \text{as} \quad k \to \infty.$$

3. THEOREM. *For any regression* (4), *all points of infinity lie on the surface of the correloid* C_n.

Proof. The theory of regression gives

$$\beta_i = \frac{R^{0i}}{\det R_n}$$

where R^{0i} is a minor of the correlation matrix of the variates $y, z_1, \ldots, z_n$. Hence (2) is a necessary condition for $\beta_i = \pm \infty$. Since the points that satisfy (2) constitute the surface of C_n, the theorem is proved.

COMMENT.(1) The regression (4) is a typical unirelational model. For this type of model, according to the theorem, all points of infinity lie on the surface of the correloid C_n.

The theorem extends to multirelational models of the type known as *causal chain systems*, inasmuch as any relation of the structural form of a causal chain is specified as an ordinary regression (2). On the other hand, the theorem does not allow extension to *interdependent systems*. In fact, as can be shown by simple examples, the relations in the structural form of such a system will in general have points of infinity in the interior of the relevant correloid. To quote broadly from a forthcoming publication (Ref. [2]), large β-values make much more of a nuisance in interdependent systems than in ordinary regressions or causal chains.

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(1) This comment links up with the treatment of causal chains vs interdependent systems in recent works by the author; see e.g. [1].

References

[1] H. Wold, *Toward a verdict on macroeconomic simultaneous equations*, pages 115-166 in *La semaine d'étude sur le rôle de l'analyse économétrique dans la formulation de plans de développement*, ed. P. Salviucci, Vatican City, Pontifical Academy of Sciences, Scripta Varia, 28 (1965).

[2] E. Mosbaek and H. Wold, *Interdependent systems, Structure and estimation*, North-Holland Publ. Co., Amsterdam. (In press, 1969.)

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Received on 24. 5. 1967