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THE CALCULATION OF α-SETS OF REPRESENTATIVES

1. Introduction. A number of combinatorial problems can be regarded as the minimal representatives problem.

Given a set $Y = \{1, 2, \ldots, n\}$ and m subsets $Y_1, Y_2, \ldots, Y_m$ of Y, for an integer number $\alpha < n$, find a subset Y^* of Y such that

1° $|Y_i \cap Y^*| \geqslant \alpha$ for $i = 1, 2, \ldots, m$,

2° there is no subset of Y with fewer elements than Y^* which has this property.

The subset Y^* of Y which satisfies 1° is called an *α-set of representatives* and the one satisfying 1°-2° — (*absolutely*) *minimal α-set of representatives.*

The matrix formulation of this problem is as follows:

Let $A = (a_{ij})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ denote the incidence matrix of subsets and elements, that is, $a_{ij} = 1$ iff $j \in Y_i$, $a_{ij} = 0$ otherwise. Find an m by $\varepsilon(\alpha)$ submatrix A^* of A such that

1° every row sum of A^* is at least α,

2° there is no submatrix of A with fewer columns than A^* which has this property.

In practical applications (see, for instance, [4]) it is interesting to obtain (all) minimal α-sets of representatives for a fixed matrix A. It can be made by the use of an algorithm of integer linear programming but the known ILP-algorithms, even for small problems, are not effective. The present short paper contains a method of calculating all minimal α-sets of representatives for a fixed matrix A which was derived from Boolean considerations [3].

2. The method. The method of calculating all minimal α-sets of representatives is analogous to the method of calculating all minimal externally stable sets [3].

Definition 1. An α-set of representatives $P \subseteq Y$ is called *minimal* if each $P' \subseteq Y$ such that $P' \subset P$ ceases to be an α-set of representatives.

Definition 2. An a-set of representatives $P \subseteq Y$ is called ***absolutely minimal*** if there is no a-set of representatives P' having fewer elements than P.

Let to each subset P of Y be associated the characteristic vector $(x_1, \ldots, x_n)$, where $x_i = 1$ iff $i \in P$.

It is obvious that $P \subseteq Y$ is an a-set of representatives iff, for each row i of A, there exist a indices $j_1, j_2, \ldots, j_a \in Y$ which satisfy the condition

$$a_{ij_1} a_{ij_2} \ldots a_{ij_a} x_{j_1} x_{j_2} \ldots x_{j_a} = 1.$$

Hence we have

THEOREM 1. *A set $P \subseteq Y$ is an a-set of representatives for the matrix A iff its characteristic vector $(x_1, x_2, \ldots, x_n)$ satisfies the Boolean equation*

$$\bigcap_{i=1}^{m} \bigcup a_{ij_1} \ldots a_{ij_a} x_{j_1} \ldots x_{j_a} = 1, \tag{1}$$

where the disjunction is extended over $\binom{n}{a}$ combinations $j_1, \ldots, j_a$ with $j_1, \ldots, j_a \in Y$.

Performing the necessary multiplications and all possible absorptions, we obtain equation (1) in the form

$$\bigcup x_{k_1} x_{k_2} \ldots x_{k_{l(k)}} = 1. \tag{2}$$

THEOREM 2. *If $x_{k_1} x_{k_2} \ldots x_{k_{l(k)}}$ is one of the elementary conjunctions in the left-hand side of equation (2), then the vector $(x_1', x_2', \ldots, x_n')$, defined by*

$$x_i' = \begin{cases} 1 & \text{if } i = k_1, k_2, \ldots, k_{l(k)}, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

is the characteristic vector of a minimal a-set of representatives for the matrix A, and all minimal a-set of representatives can be obtained in this way.

Remark 1. An a-set of representatives for the matrix A exists iff an a is not greater than the minimal row sum of the matrix A.

Remark 2. The characteristic vector of a 1-set of representatives satisfies the Boolean equation

$$\bigcap_{i=1}^{m} \bigcup_{j=1}^{n} a_{ij} x_j = 1. \tag{4}$$

Example.

$$A = \begin{vmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

For this matrix A and $a = 1$, equation (4) is as follows:

$$(x_1 \cup x_2 \cup x_5)(x_3 \cup x_4 \cup x_6)(x_1 \cup x_2 \cup x_4)(x_2 \cup x_3 \cup x_4 \cup x_5 \cup x_6)(x_5 \cup x_6) = 1.$$

After performing the multiplications and absorptions, we get

$$x_4 x_5 \cup x_1 x_6 \cup x_2 x_6 \cup x_1 x_3 x_5 \cup x_2 x_3 x_5 = 1.$$

Hence, the minimal 1-sets of representatives for the matrix A are $\{4, 5\}$, $\{1, 6\}$, $\{2, 6\}$, $\{1, 3, 5\}$ and $\{2, 3, 5\}$, and the absolutely ones are $\{4, 5\}$, $\{1, 6\}$ and $\{2, 6\}$.

Remark 3. We can associate with each column a non-negative cost c_j. Let

$$c(P) = \sum_{j \in P} c_j, \quad \text{where } P \subseteq Y.$$

The covering problem [1] consists of finding $\min_P c(P)$, where P is a 1-set of representatives.

The minimum can be extended only over the minimal 1-set of representatives (not necessarily absolutely minimal, see example) because, for each 1-set of representatives Q, there exists a minimal 1-set of representatives $P \subseteq Q$ and $c(P) \leqslant c(Q)$.

Example (contd.). Let $c = (4, 4, 3, 10, 3, 10)$; then $c(\{1, 3, 5\}) = c(\{2, 3, 5\}) = 10$ and $\{1, 3, 5\}$, $\{2, 3, 5\}$ are the minimal cost coverings.

Added in proof. The Boolean methods of solution of (minimal, absolutely minimal and minimal cost) covering problems are presented in papers [5] and [6]. The latter also contains the procedure declaration (ALGOL-60) of the method of determination of all minimal coverings presented in this paper.

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WYZNACZANIE a-ZBIORÓW REPREZENTANTÓW

STRESZCZENIE

Wiele problemów kombinatorycznych można sprowadzić do zagadnienia (absolutnie) minimalnego a-zbioru reprezentantów.

Dany jest zbiór $Y = \{1, 2, \ldots, n\}$ i m podzbiorów $Y_1, Y_2, \ldots, Y_m$ zbioru Y; dla liczby naturalnej $a < n$ należy znaleźć podzbiór Y^* zbioru Y, taki że

1° $|Y_i \cap Y^*| \geqslant a$ dla $i = 1, 2, \ldots, m$,

2° Y^* jest minimalnym podzbiorem Y spełniającym 1°.

W tej krótkiej pracy podano równanie boolowskie, którego rozwiązaniami są wektory charakterystyczne wszystkich (absolutnie minimalnych lub tylko minimalnych) a-zbiorów reprezentantów.