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PIECEWISE MARKOV PROCESSES ON A GENERAL STATE SPACE

1. Introduction. In applications of stochastic processes the following class of processes often appears. The process starts from some point x of the state space \mathcal{X} and it evolves as some Markov process (the transition function of which depends on $x \in \mathcal{X}$); at the instant τ_1 the process jumps to another point of the state space \mathcal{X} according to a given distribution (which depends on the state just before the jump). Next the process evolves as some Markov process, etc. Such processes are called *piecewise Markov processes* (in the sequel abbreviated to P.M.P.).

A *semi-Markov process* (see, for example, [5]) is the simplest example of a P.M.P.

Let \mathcal{X} be a complete subset of the Euclidean n -dimensional space \mathcal{R}^n and let \mathfrak{F} be the σ -algebra of Borel subsets of \mathcal{X} . The measurable space $(\mathcal{X}, \mathfrak{F})$ is called the *state space*.

On $(\mathcal{X}, \mathfrak{F})$, there are given:

(i) a class of measurable, Markov transition functions $\{P^x(t, y, A), x \in \mathcal{X}\}$, i.e. $P^x(t, y, \cdot)$ is a probability measure on $(\mathcal{X}, \mathfrak{F})$ for every $x, y \in \mathcal{X}$, $t > 0$, $P^x(\cdot, \cdot, A)$ is a measurable function with respect to the product σ -algebra $\mathfrak{B}_+ \times \mathfrak{F}$ for every $x \in \mathcal{X}$, $A \in \mathfrak{F}$, and \mathfrak{B}_+ denotes the σ -algebra of Borel subsets of $\mathcal{R}_+ = (0, \infty)$;

(ii) a Markov kernel $Q(x, A)$, i.e. $Q(x, \cdot)$ is a probability measure for every $x \in \mathcal{X}$, and $Q(\cdot, A)$ is an \mathfrak{F} -measurable function for every $A \in \mathfrak{F}$;

(iii) a class $\{\mu^x, x \in \mathcal{X}\}$ of probability measures on $(\mathcal{R}_+, \mathfrak{B}_+)$.

Definition 1. A stochastic process $X(t)$, $t > 0$, on the probability space $(\Omega, \mathfrak{S}, \mathbf{P})$ with values in the state space $(\mathcal{X}, \mathfrak{F})$, the trajectories of which are right-side continuous and have left-hand limits, is said to be a *piecewise Markov process* (P.M.P.) if there exist random points $0 = \tau_1 < \tau_2 < \dots$ (called *regenerative points*) such that, for $n = 1, 2, \dots$,

$$\begin{aligned} \text{(a)} \quad \mathbf{P}(\tau_n - \tau_{n-1} \in B | \mathfrak{S}_0^s, s = \tau_{n-1}) &= \mathbf{P}(\tau_n - \tau_{n-1} \in B | \mathfrak{S}_s^s, s = \tau_{n-1}) \\ &= \mu^{X(\tau_{n-1})}(B), \quad B \in \mathfrak{B}_+, \end{aligned}$$

and $\mathfrak{S}_s^t [\mathfrak{S}_s^{t-}]$, $0 \leq s \leq t$, is the σ -algebra generated by the family of sets of the form

$$\{\omega: (X(t_1), \dots, X(t_k)) \in B_1 \times \dots \times B_k, s \leq t_1 \leq \dots \leq t_k \leq t$$

$$[s \leq t_1 \leq \dots \leq t_k < t]\}, \quad B_k \in \mathfrak{F}, \quad k = 1, 2, \dots,$$

(b) $\mathbf{P}(X(\tau_n) \in A | \mathfrak{S}_0^{s-}, s = \tau_n) = Q(X(\tau_n -), A),$

(c) $\mathbf{P}(X(t) \in A | \tau_{n-1} \leq s < t < \tau_n, \mathfrak{S}_0^s) = \mathbf{P}(X(t) \in A | \mathfrak{S}_s^s \vee \mathfrak{F}_{X(\tau_{n-1})})$
 $= X(\tau_{n-1})\mathbf{P}(X(t-s), X(s), A),$

where $\mathfrak{F}_{X(\tau_{n-1})}$ is the σ -algebra generated by the random variable $X(\tau_{n-1})$, and $\mathfrak{S}_s^s \vee \mathfrak{F}_{X(\tau_{n-1})}$ is the σ -algebra generated by the union $\mathfrak{S}_s^s \cup \mathfrak{F}_{X(\tau_{n-1})}$.

All equalities in (a), (b), and (c) are given \mathbf{P} -almost everywhere. Let

$$\mathbf{P}(\lim_{n \rightarrow \infty} \tau_n = \infty) = 1.$$

The aim of this paper is to derive relations between the stationary probability distribution of a P.M.P. provided it exists and the stationary probability distributions of the imbedded chains $X(\tau_n -)$ and $X(\tau_n)$ for $n = 1, 2, \dots$. These relations are obtained by the use of the contraction semigroup theory.

A P.M.P., in the case where P^x and μ^x do not depend on $x \in \mathcal{X}$, has been analyzed by Baklan in [2]. He has shown that

$$\limsup_{t \rightarrow \infty} \sup_{x \in \mathcal{X}} \left| \mathbf{P}(X(t) \in A | X(0) = x) - \right.$$

$$\left. - \frac{1}{\int_{\mathcal{X}} t \mu(dt)} \int_{\mathcal{X}_+} \int_{\mathcal{X}} N^+(dx) \mu((t, \infty)) P(t, x, A) dt \right| = 0,$$

where N^+ is the stationary probability distribution of the Markov chain $X(\tau_n)$ for $n = 1, 2, \dots$. In the Corollary to Theorem 5 we obtain a similar result for a general P.M.P., we prove, however, point convergence only and our assumptions are different from those of Baklan.

The case where \mathcal{X} is a discrete space was analyzed by Kuczura [9] by the use of renewal theory. In [6], different proofs of Kuczura's theorems by the application of the extended Markov process and prospective Kolmogorov equations are given. This method was used earlier in the theory of queues by Kopocińska and Kopociński (see [7] and [8]).

2. Notions from the contraction semigroup theory. Definitions and theorems, except Definition 4 and Theorem 3, are quoted from [4]. Assume that $(\mathcal{X}, \mathfrak{F})$ is a state space, $P(t, x, A)$ is a measurable, Markov transition function, $\mathcal{M} = \mathcal{M}(\mathcal{X}, \mathfrak{F})$ denotes the Banach space of signed measures on $(\mathcal{X}, \mathfrak{F})$ having finite variation, and $\mathcal{B} = \mathcal{B}(\mathcal{X}, \mathfrak{F})$ is the Banach space of real, bounded functions.

Markov transition functions induce the following two semigroups of contraction operators:

$$T_t: \mathcal{B} \ni f(\cdot) \rightarrow \int_x f(x)P(t, \cdot, dx) \in \mathcal{B}, \quad t > 0,$$

and

$$U_t: \mathcal{M} \ni M(\cdot) \rightarrow \int_x M(dx)P(t, x, \cdot) \in \mathcal{M}, \quad t > 0.$$

We denote by T the semigroup T_t , and by U the semigroup U_t , $t > 0$.

Let \mathcal{C} be a Banach subspace of the space \mathcal{B} such that $\mathcal{C}^* \supset \mathcal{M}$, where \mathcal{C}^* denotes the Banach space conjugate to \mathcal{C} .

Definition 2. The sequence of measures $M_n \in \mathcal{M}$, $n = 1, 2, \dots$, is said to be *weakly convergent* to the measure M ($w\text{-}\lim_{n \rightarrow \infty} M_n = M$) if, for every $f \in \mathcal{C}$,

$$\lim_{n \rightarrow \infty} \int_x f(x)M_n(dx) = \int_x f(x)M(dx).$$

Let us write

$$(1) \quad \mathcal{L} = \{M \in \mathcal{M} : w\text{-}\lim_{t \searrow 0} U_t M = M\},$$

$$(2) \quad \mathcal{D} = \left\{ M \in \mathcal{M} : w\text{-}\lim_{t \searrow 0} \frac{U_t M - M}{t} \text{ exists} \right\}.$$

Definition 3. (i) The operator A is said to be an *infinitesimal generator* of the semigroup U if, for $M \in \mathcal{D}$,

$$AM = w\text{-}\lim_{t \searrow 0} \frac{U_t M - M}{t}.$$

(ii) The operator R_λ is said to be a *resolvent* of the semigroup U if

$$R_\lambda M = \int_0^\infty e^{-\lambda t} U_t M dt.$$

(iii) The operator R is said to be a *potential* of the semigroup U if

$$RM = \int_0^\infty U_t M dt$$

provided the right-hand side exists.

THEOREM 1. If $M_1 = RM_2$ and if $M_2 \in \mathcal{L}$, then $M_1 \in \mathcal{D}$ and $-AM_1 = M_2$.

THEOREM 2. The operator $\lambda I - A$, $\lambda > 0$, is a one-to-one mapping of \mathcal{D} onto \mathcal{L} and $R_\lambda = (\lambda I - A)^{-1}$, where I denotes the identical operator.

Definition 4. The non-negative measure $N \in \mathcal{M}$, $N \neq 0$, is said to be *invariant* with respect to the semigroup U if $U_t N = N$ for every $t > 0$.

In the sequel we assume that, for invariant measures, $N(\mathcal{X}) = 1$. The following theorem shows how an invariant measure can be found.

THEOREM 3. *The non-negative measure N ($N(\mathcal{X}) = 1$) is the only invariant measure with respect to the semigroup U if and only if the measure N belongs to \mathcal{D} and N is the unique solution of the equation $AN = 0$.*

Proof. We prove only the sufficient condition; the necessary condition can be proved in a similar way. From Theorem 2, putting $\lambda = 1$, we know that there exists a measure $N_1 \in \mathcal{L}$ such that $N - AN = N_1$ and $N = R_1 N_1$. Since $AN = 0$, we have $R_1 N = N$ which implies (see [1]) that N is invariant with respect to the semigroup U . Now, let us assume, to the contrary, that there exists another invariant measure $N_2 \neq N$. We have $AN_2 = 0$ which is contradictory, since N is the unique solution of the equation $AN = 0$.

3. The transition function for an extended process. In the sequel the following notation is used: $\mathcal{R} = (-\infty, \infty)$, and \mathfrak{B} is the σ -algebra of subsets of \mathcal{R} ; \mathcal{X} is a complete subset of \mathcal{R}^n ; \mathfrak{F} denotes the σ -algebra of subsets of \mathcal{X} ;

$$(\mathcal{X} \times \mathcal{X} \times \mathcal{R}_+, \mathfrak{F} \times \mathfrak{F} \times \mathfrak{B}_+) = (\bar{\mathcal{X}}, \bar{\mathfrak{F}}); \quad (\tilde{\mathcal{X}} \times \tilde{\mathcal{X}}, \tilde{\mathfrak{F}} \times \tilde{\mathfrak{F}}) = (\tilde{\bar{\mathcal{X}}}, \tilde{\bar{\mathfrak{F}}});$$

the bar (wave) mark is used for the notation connected with the space $(\bar{\mathcal{X}}, \bar{\mathfrak{F}})$ ($(\tilde{\bar{\mathcal{X}}}, \tilde{\bar{\mathfrak{F}}})$); Greek letters are reserved for probability measures on $(\mathcal{R}_+, \mathfrak{B}_+)$, and capital letters for probability measures on different spaces. The indicator function of a set B is denoted by $\chi_B(\cdot)$.

Let $X(t)$, $t \geq 0$, be a P.M.P. with state space $(\mathcal{X}, \mathfrak{F})$, let

$$\begin{aligned} Y(t) &= X(\tau_{n-1}) \\ Z(t) &= \tau_n - t \end{aligned} \quad \text{for } \tau_{n-1} \leq t < \tau_n, \quad n = 1, 2, \dots,$$

and let, for $\bar{A} \in \bar{\mathfrak{F}}$ and $\bar{x} \in \bar{\mathcal{X}}$,

$$P((X(t), Y(t), Z(t)) \in \bar{A} | (X(0), Y(0), Z(0)) = \bar{x}) = \bar{P}(t, \bar{x}, \bar{A}).$$

Let $V_k(\bar{x}, \cdot)$ for $k = 1, 2, \dots$ be a measure on $(\mathcal{R}_+^k \times \mathcal{X}, \mathfrak{B}_+^k \times \mathfrak{F})$ such that, for $B_1 \times \dots \times B_k \times A \in \mathfrak{B}_+^k \times \mathfrak{F}$ and for $\bar{x} = (x, y, z) \in \bar{\mathcal{X}}$,

$$\begin{aligned} &V_k(\bar{x}, B_1 \times \dots \times B_k \times A) \\ &= \int_{\mathcal{X}} P^y(z, x, ds_1) \int_{\mathcal{X}} Q(s_1, dx_1) \int_{B_1} \mu^{x_1}(dz_1) \times \dots \times \\ &\quad \times \int_{\mathcal{X}} P^{x_{k-1}}(z_{k-1}, x_{k-1}, ds_k) \int_A Q(s_k, dx_k) \mu^{x_k}(B_k). \end{aligned}$$

THEOREM 4. *The process $(X(t), Y(t), Z(t))$, $t > 0$, is Markov and its transition function satisfies either*

$$(3) \quad \bar{P}(t, (x, y, z), \cdot) = \bar{\Pi}(t, (x, y, z), \cdot) + \chi_{(0,t]}(z) \int_{\mathcal{X}} P^\nu(z, x, dx_1) \int_{\mathcal{X}} Q(x_1, dx_2) \int_{\mathcal{R}_+} \mu^{x_2}(dz_1) \bar{P}(t-z, (x_2, x_2, z_1), \cdot),$$

where $x, y \in \mathcal{X}$, $z \in \mathcal{R}_+$, and $\bar{\Pi}$ is a transition function such that, for $x, y \in \mathcal{X}$, $z \in \mathcal{R}_+$,

$$\bar{\Pi}(t, (x, y, z), A \times B \times C) = P^\nu(t, x, A) \chi_B(y) \chi_{C+t}(z),$$

or

$$(4) \quad \bar{P}(t, (x, y, z), \cdot) = \bar{\Pi}(t, (x, y, z), \cdot) + \sum_{k=1}^{\infty} \int_{\mathcal{R}_+^k \times \mathcal{X}} V_k((x, y, z), dz_1 \times \dots \times dz_k \times dx_1) \times \bar{\Pi}(t-z-z_1-\dots-z_{k-1}, (x_1, x_1, z_k), \cdot) \chi_{(0,t]}(z+z_1+\dots+z_{k-1}).$$

Proof. The behaviour of the process $(X(t), Y(t), Z(t))$, $t > 0$, after the moment t depends only on the state of this process at the moment t , thus it is a Markov process.

Let $\xi(t)$ be the number of regenerative points in the interval $(0, t]$, i.e.

$$\xi(t) = \max_{k \geq 1} \{k: \tau_k \leq t\} + 1.$$

Then, for $A \times B \times C \in \bar{\mathcal{F}}$ and $\bar{x} = (x, y, z) \in \bar{\mathcal{X}}$ we have

$$\begin{aligned} & \mathbf{P}(\bar{X}(t) \in A \times B \times C | \bar{X}(0) = \bar{x}) \\ &= \sum_{k=0}^{\infty} \mathbf{P}(\bar{X}(t) \in A \times B \times C, \xi(t) = k | \bar{X}(0) = \bar{x}) \\ &= P^\nu(t, x, A) \chi_B(y) \chi_{C+t}(z) + \sum_{k=1}^{\infty} \int_{\mathcal{R}_+^k \times \mathcal{X}} V_k(\bar{x}, dz_1 \times \dots \times dz_k \times dx_1) \times \\ & \times \bar{\Pi}(t-z-z_1-\dots-z_{k-1}, (x_1, x_1, z_k), A \times B \times C) \chi_{(0,t]}(z+z_1+\dots+z_{k-1}), \end{aligned}$$

and hence we obtain (4). In a similar way we can show that

$$\begin{aligned} & \mathbf{P}(\bar{X}(t) \in A \times B \times C | \bar{X}(0) = \bar{x}) \\ &= \mathbf{P}(\bar{X}(t) \in A \times B \times C, \xi(t) = 0 | \bar{X}(0) = \bar{x}) + \\ & \quad + \mathbf{P}(\bar{X}(t) \in A \times B \times C, \xi(t) \geq 1 | \bar{X}(0) = \bar{x}) \\ &= P^\nu(t, x, A) \chi_B(y) \chi_{C+t}(z) + \chi_{(0,t]}(z) \int_{\mathcal{X}} P^\nu(z, x, dx_1) \int_{\mathcal{X}} Q(x_1, ds_1) \times \\ & \quad \times \int_{\mathcal{R}_+} \mu^{s_1}(dz_1) \bar{P}(t-z, (s_1, s_1, z_1), A \times B \times C), \end{aligned}$$

which completes the proof.

We have two transition functions \bar{P} and $\bar{\Pi}$ on $(\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. They induce two semigroups of U contraction operators: \bar{P}_t and $\bar{\Pi}_t$, respectively. Let $\mathcal{D}(\bar{P})$, $\mathcal{D}(\bar{\Pi})$, and $\mathcal{L}(\bar{P})$, $\mathcal{L}(\bar{\Pi})$ denote sets \mathcal{D} and \mathcal{L} defined by (2) and (1), respectively. $A(\bar{P})$ and $A(\bar{\Pi})$ stand for the corresponding infinitesimal generators, $R_\lambda(\bar{P})$ and $R_\lambda(\bar{\Pi})$ for the corresponding resolvents, $R(\bar{P})$ and $R(\bar{\Pi})$ for the corresponding potentials. We write

$$\bar{P}_\lambda(\bar{x}, \cdot) = \int_{\mathcal{R}_+} e^{-\lambda t} \bar{P}(t, \bar{x}, \cdot) dt \quad \text{and} \quad \bar{\Pi}_\lambda(\bar{x}, \cdot) = \int_{\mathcal{R}_+} e^{-\lambda t} \bar{\Pi}(t, \bar{x}, \cdot) dt.$$

COROLLARY. *We have*

$$(5) \quad \bar{P}_\lambda(\bar{x}, \cdot) = \bar{\Pi}_\lambda(\bar{x}, \cdot) + \int_{\bar{\mathcal{X}}} \bar{V}(\bar{x}, d\bar{y}) \bar{\Pi}_\lambda(\bar{y}, \cdot) \quad \text{for } \bar{x} \in \bar{\mathcal{X}}.$$

Proof. Multiplying (4) by $e^{-\lambda t}$ and integrating from 0 to ∞ we get the assertion of the Corollary.

4. Main theorems. Let $\mathcal{C}_0(\mathcal{X})$ denote the class of real-valued, bounded, continuous functions vanishing at ∞ and let $\bar{\mathcal{C}}_0(\bar{\mathcal{X}})$ be the subspace of real-valued, bounded, continuous functions generated by functions of the form $f(x, y)g(z)$, where $f \in \mathcal{C}_0(\tilde{\mathcal{X}})$ and g is bounded and continuous. In the sequel we assume the following propositions:

$$(A.1) \quad \mathcal{C} = \bar{\mathcal{C}}_0(\bar{\mathcal{X}}).$$

$$(A.2) \quad \text{For every } f \in \bar{\mathcal{C}}_0(\bar{\mathcal{X}}),$$

$$\lim_{t \searrow 0} \int_{\bar{\mathcal{X}}} \bar{\Pi}(t, \bar{x}, d\bar{v}) f(\bar{v}) = f(\bar{x}).$$

$$(A.3) \quad \text{For every } f \in \bar{\mathcal{C}}_0(\bar{\mathcal{X}}),$$

$$\limsup_{t \searrow 0} \left| \int_{\mathcal{R}_+} \mu^x(dz) \int_{\bar{\mathcal{X}}} \bar{\Pi}(t, (x, x, z), d\bar{v}) f(\bar{v}) - \int_{\mathcal{R}_+} \mu^x(dz) f(x, x, z) \right| = 0.$$

$$(A.4) \quad \text{For every } f \in \bar{\mathcal{C}}_0(\bar{\mathcal{X}}),$$

$$\int_{\bar{\mathcal{X}}} Q(x, dx_1) \int_{\mathcal{R}_+} \mu^{x_1}(dz_1) f(x_1, x_1, z_1) = g(x) \in \mathcal{C}_0(\mathcal{X}).$$

(A.5) There exists a continuous function $\varepsilon: \mathcal{R}_+ \rightarrow \mathcal{R}_+$, $\lim_{t \searrow 0} \varepsilon(t) = 0$, such that

$$\sup_{s \in \mathcal{X}} \int_{\mathcal{X}} Q(s, dx) \mu^x((0, t]) \leq \varepsilon(t), \quad t > 0.$$

$$(A.6) \quad \sup_{x \in \mathcal{X}} \int_{\bar{\mathcal{X}}} \mu^x((t, \infty)) dt \leq M < \infty.$$

(A.7) There exists a stationary probability distribution \bar{N} , i.e.

$$w\text{-}\lim_{t \rightarrow \infty} \bar{P}(t, \bar{x}, \cdot) = \bar{N}(\cdot),$$

such that, for some version of the conditional probability

$\tilde{N}(A \times B|z)$, for every $f \in \mathcal{C}_0(\tilde{\mathcal{X}})$ we have

$$\lim_{z \rightarrow 0} \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy|z) \int_{\mathcal{X}} P^y(z, x, dx_1) f(x_1, y) = \int_{\tilde{\mathcal{X}}} \tilde{N}^-(dx \times dx) f(x, y),$$

where \tilde{N}^- is a probability measure on $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$. This stationary probability distribution \bar{N} is the unique invariant measure with respect to the semigroup \bar{P}_t .

Remark. Let us note that \tilde{N}^- is the stationary distribution of the Markov chain $(X(\tau_n^-), Y(\tau_n^-))$ for $n = 1, 2, \dots$

In the sequel, the invariant measure with respect to the semigroup \bar{P}_t will be denoted by \bar{N} . According to our convention, we write

$$\begin{aligned} \tilde{N}(A \times B) &= \bar{N}(A \times B \times \mathcal{R}_+), & N(A) &= \bar{N}(A \times \mathcal{X} \times \mathcal{R}_+), \\ \nu(C) &= \bar{N}(\mathcal{X} \times \mathcal{X} \times C), & N^-(A) &= \tilde{N}^-(A \times \mathcal{X}), \quad A, B \in \mathcal{Y}, C \in \mathcal{B}_+. \end{aligned}$$

LEMMA 1. $\bar{N} \in \mathcal{L}(\bar{\Pi})$.

The proof of Lemma 1 is omitted.

LEMMA 2. $\bar{N} \in \mathcal{D}(\bar{\Pi})$.

Proof. Since \bar{N} is invariant, we have $\mathbf{R}_1(\bar{P})\bar{N} = \bar{N}$ (see [1]). Using (5) we obtain

$$\begin{aligned} \bar{N}(\cdot) &= \mathbf{R}_1(\bar{P})\bar{N}(\cdot) = \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \bar{P}_1(\bar{x}, \cdot) = \mathbf{R}_1(\bar{\Pi})\bar{N}(\cdot) + \\ &+ \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \int_{\bar{\mathcal{Y}}} \bar{V}(\bar{x}, d\bar{y}) \bar{\Pi}_1(\bar{y}, \cdot) = \mathbf{R}_1(\bar{\Pi})\bar{N}(\cdot) + \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \int_{\bar{\mathcal{Y}}} \bar{V}(\bar{x}, d\bar{y}) \bar{\Pi}_1(\bar{y}, \cdot). \end{aligned}$$

Thus the operator $\mathbf{R}_1(\bar{\Pi})$ maps one-to-one $\mathcal{L}(\bar{\Pi})$ onto $\mathcal{D}(\bar{\Pi})$ (see Theorem 2). Hence it suffices to show that

$$\int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \bar{V}(\bar{x}, \cdot) \in \mathcal{L}.$$

Let $f \in \mathcal{C}_0(\bar{\mathcal{X}})$. Then

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \int_{\bar{\mathcal{X}}} \bar{V}(\bar{x}, d\bar{x}_1) \int_{\bar{\mathcal{X}}} \bar{\Pi}(t, \bar{x}_1, d\bar{x}_2) f(\bar{x}_2) \\ &= \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \int_{\bar{\mathcal{X}}} \bar{V}(\bar{x}, d\bar{x}_1) \left(\lim_{t \rightarrow 0} \int_{\bar{\mathcal{X}}} \bar{\Pi}(t, \bar{x}_1, d\bar{x}_2) f(\bar{x}_2) \right) && \text{(by the bounded convergence theorem)} \\ &= \int_{\bar{\mathcal{X}}} \bar{N}(d\bar{x}) \int_{\bar{\mathcal{X}}} \bar{V}(\bar{x}, d\bar{x}_1) f(\bar{x}_1) && \text{(by (A.2))} \end{aligned}$$

and the proof is completed.

LEMMA 3. There exists a $c > 0$ such that $\nu((0, \tau]) \leq c\tau$, $\tau > 0$.

Proof. Since $\bar{N} \in \mathcal{D}(\bar{\Pi})$, there exists a signed measure φ with finite variation such that

$$\nu(C) = \bar{N}(\mathcal{X} \times \mathcal{X} \times C) = \int_{\mathcal{X}_+} \left(\int_{\mathcal{X}_+} e^{-\lambda t} \chi_{C+t}(z) dt \right) \varphi(dz).$$

Let φ^+ be the positive part of φ . Then

$$\begin{aligned} \nu((0, \tau]) &\leq \int_{\mathcal{X}_+} \left(\int_{\mathcal{X}_+} e^{-\lambda t} \chi_{(t, t+\tau]}(z) dt \right) \varphi^+(dz) \\ &\leq \int_{\mathcal{X}_+} \left(\int_{\mathcal{X}_+} \chi_{(t, t+\tau]}(z) dt \right) \varphi^+(dz) \leq \varphi^+(\mathcal{X}_+) \tau. \end{aligned}$$

THEOREM 5. *The invariant measure \bar{N} satisfies the equation*

$$A(\bar{\Pi}) \bar{N} + m \bar{N}^+ = 0,$$

where

$$\bar{N}^+(A \times B \times C) = \int_{\mathcal{X}} N^-(ds) \int_A Q(s, dx) \chi_B(x) \mu^x(C)$$

and

$$m^{-1} = \int_{\mathcal{X}_+} \int_{\mathcal{X}} N^+(dx) \mu^x((t, \infty)) dt.$$

Remark. Let us note that \bar{N}^+ is the stationary distribution of the Markov chain $(X(\tau_n), Y(\tau_n), \tau_{n+1} - \tau_n)$ for $n = 1, 2, \dots$

Proof. From Theorem 3 we infer that \bar{N} is the unique solution of the equation $A(\bar{P})\bar{N} = 0$. By Theorem 4 we have

$$\begin{aligned} A(\bar{P})\bar{N}(\cdot) &= \text{w-lim}_{t \searrow 0} \frac{1}{t} (\bar{\Pi}_t \bar{N}(\cdot) - \bar{N}(\cdot)) = \text{w-lim}_{t \searrow 0} \frac{1}{t} \int_{\mathcal{X}} \bar{N}(dx \times dy \times dz) \chi_{(0, t]}(z) \times \\ &\times \int_{\mathcal{X}} P^\nu(z, x, ds) \int_{\mathcal{X}} Q(s, dx_1) \int_{\mathcal{X}_+} \mu^{x_1}(dz_1) \bar{P}(t - z, (x_1, x_1, z_1), \cdot) = 0. \end{aligned}$$

Since, by Lemma 2, $\bar{N} \in \mathcal{D}(\bar{\Pi})$, we infer that

$$\text{w-lim}_{t \searrow 0} \frac{1}{t} (\bar{\Pi}_t \bar{N} - \bar{N}) = A(\bar{\Pi})\bar{N}$$

exists. Hence for $f \in \bar{\mathcal{C}}_0(\bar{\mathcal{X}})$ the limit

$$\begin{aligned} (6) \quad \lim_{t \searrow 0} \frac{1}{t} \int_{\mathcal{X}} \bar{N}(dx \times dy \times dz) \chi_{(0, t]}(z) \int_{\mathcal{X}} P^\nu(z, x, ds_1) \int_{\mathcal{X}} Q(s_1, dx_1) \times \\ \times \int_{\mathcal{X}_+} \mu^{x_1}(dz_1) \int_{\mathcal{X}} \bar{P}(t - z, (x_1, x_1, z_1), d\bar{v}) f(\bar{v}) \end{aligned}$$

exists which, as we will show, is equal to

$$\lim_{t \searrow 0} \frac{1}{t} \int_{\bar{x}} \bar{N}(dx \times dy \times dz) \chi_{(0,t]}(z) \int_{\bar{x}} P^\nu(z, x, ds_1) \int_{\bar{x}} Q(s_1, dx_1) \times \\ \times \int_{\mathfrak{A}_+} \mu^{x_1}(dz_1) \int_{\bar{x}} \bar{\Pi}(t-z, (x_1, x_1, z_1), d\bar{v}) f(\bar{v}).$$

To prove it let us note that

$$\begin{aligned} & \frac{1}{t} \int_{\bar{x}} \bar{N}(dx \times dy \times dz) \chi_{(0,t]}(z) \int_{\bar{x}} P^\nu(z, x, ds_1) \int_{\bar{x}} Q(s_1, dx_1) \times \\ & \quad \times \int_{\mathfrak{A}_+} \mu^{x_1}(dz_1) \int_{\bar{x}} (\bar{P}(t-z, (x_1, x_1, z_1), d\bar{v}) - \bar{\Pi}(t-z, (x_1, x_1, z_1), d\bar{v})) f(\bar{v}) \\ & = \frac{1}{t} \int_{\bar{x}} \bar{N}(dx \times dy \times dz) \chi_{(0,t]}(z) \int_{\bar{x}} P^\nu(z, x, ds_1) \int_{\bar{x}} Q(s_1, dx_1) \int_{\mathfrak{A}_+} \mu^{x_1}(dz_1) \times \\ & \quad \times \chi_{(0,t-z]}(z_1) \int_{\bar{x}} P^{x_1}(z_1, x_1, ds_2) \int_{\bar{x}} Q(s_2, dx_2) \int_{\mathfrak{A}_+} \mu^{x_2}(dz_2) \times \\ & \quad \times \int_{\bar{x}} \bar{P}(t-z-z_1, (x_2, x_2, z_2), d\bar{v}) f(\bar{v}) \quad (\text{by (3)}) \\ & \leq \frac{1}{t} \int_0^t \nu(dz) \int_{\bar{x}} \bar{N}(dx \times dy | z) \int_{\bar{x}} P^\nu(z, x, ds_1) \int_{\bar{x}} Q(s_1, dx_1) \times \\ & \quad \times \mu^{x_1}((0, t-z]) \sup_{\bar{v} \in \bar{x}} |f(\bar{v})| \\ & \leq \frac{1}{t} \varepsilon(\theta_t t) \nu((0, t]) \sup_{\bar{v} \in \bar{x}} |f(\bar{v})| \rightarrow 0 \quad \text{as } t \searrow 0, \quad 0 < \theta_t \leq 1 \end{aligned}$$

(by Lemma 3 and (A.5)).

Now, assume that

$$(7) \quad \nu(C) = \int_C h(z) dz, \quad C \in \mathfrak{B}_+ \text{ and } \lim_{z \searrow 0} h(z) = m.$$

Then for $f \in \overline{\mathcal{C}}_0(\overline{\mathcal{X}})$ we have

$$\begin{aligned} & \lim_{t \searrow 0} \frac{1}{t} \int_0^t h(z) dz \int_{\overline{\mathcal{X}}} \tilde{N}(dx \times dy | z) \int_{\overline{\mathcal{X}}} P^y(z, x, ds_1) \int_{\overline{\mathcal{X}}} Q(s_1, dx_1) \times \\ & \qquad \qquad \qquad \times \int_{\mathcal{X}_+} \mu^{x_1}(dz_1) \int_{\overline{\mathcal{X}}} \overline{\Pi}(t-z, (x_1, x_1, z_1), d\bar{v}) f(\bar{v}) \\ & = \lim_{t \searrow 0} \frac{1}{t} \int_0^t h(z) dz \int_{\overline{\mathcal{X}}} \tilde{N}(dx \times dy | z) \int_{\overline{\mathcal{X}}} P^y(z, x, ds_1) \int_{\overline{\mathcal{X}}} Q(s_1, dx_1) \times \\ & \qquad \qquad \qquad \times \int_{\mathcal{X}_+} \mu^{x_1}(dz_1) f(x_1, x_1, z_1) \quad (\text{by (A.3)}) \\ & = m \int_{\overline{\mathcal{X}}} N^-(dx) \int_{\overline{\mathcal{X}}} Q(x, dx_1) \int_{\mathcal{X}_+} \mu^{x_1}(dz_1) f(x_1, x_1, z_1) \\ & \qquad \qquad \qquad (\text{by (A.4) and (A.7)}). \end{aligned}$$

Hence we obtain

$$(8) \qquad A(\overline{\Pi})\overline{N} + m\overline{N}^+ = 0.$$

To eliminate assumption (7) we show that the solution of equation (8) (which is unique) satisfies this assumption. Using Theorem 1 we have

$$\overline{N}(\cdot) = m \int_{\overline{\mathcal{X}}} \left(\int_{\mathcal{X}_+} \overline{\Pi}(t, \bar{x}, \cdot) dt \right) \overline{N}^+(d\bar{x}),$$

and hence

$$v((z, \infty)) = m \int_{\mathcal{X}_+} \int_{\overline{\mathcal{X}}} N^+(dx) \mu^x((z+t, \infty)) dt$$

(the right-hand side has the sense by (A.6)). Thus

$$h(z) = m \int_{\overline{\mathcal{X}}} N^+(dx) \mu^x((z, \infty)) \quad \text{and} \quad \lim_{z \searrow 0} h(z) = m.$$

This completes the proof.

COROLLARY. (i) For $A, B \in \mathfrak{F}$ and $C \in \mathfrak{B}_+$,

$$\overline{N}(A \times B \times C) = m \int_{\mathcal{X}_+} \int_{\overline{\mathcal{X}}} N^+(dx) \int_{C+t} \mu^x(dz) P^x(t, x, A) \chi_B(x) dt.$$

(ii) For $A \in \mathfrak{F}$,

$$\lim_{t \rightarrow \infty} P(X(t) \in A | X(0) = x) = m \int_{\mathcal{X}_+} N^+(dx) \int_{\overline{\mathcal{X}}} P^x(t, x, A) \mu^x((t, \infty)) dt.$$

In order to give a formula which relates \tilde{N} , \tilde{N}^+ and \tilde{N}^- , we need some stronger assumptions. Instead of (A.3) and (A.5) we assume:

(A.3') For every $f \in \overline{\mathcal{C}_0(\mathfrak{X})}$,

$$\limsup_{t \searrow 0} \sup_{x, y \in \mathfrak{X}} \left| \int_{\mathfrak{X}_+} \mu^x(dz) \int_{\overline{\mathfrak{X}}} \overline{\Pi}(t, (x, y, z), d\bar{v}) f(\bar{v}) - \int_{\mathfrak{X}_+} \mu^x(dz) f(x, y, z) \right| = 0.$$

(A.5') $\limsup_{t \searrow 0} \sup_{x \in \mathfrak{X}} \mu^x((0, t]) = 0.$

We need the following lemmas.

LEMMA 4. For every $g \in \mathcal{C}_0(\tilde{\mathfrak{X}})$,

$$\limsup_{t \searrow 0} \sup_{x, y \in \mathfrak{X}} \left| \int_{\mathfrak{X}} P^y(t, x, dx_1) g(x_1, y) - g(x, y) \right| = 0.$$

Proof. Let $\varepsilon > 0$. From (A.3'), putting $f(x, y, z) = g(x, y)$, it follows that there exists a T_1 such that, for $0 < t < T_1$,

$$\sup_{x, y \in \mathfrak{X}} \left| \mu^x((t, \infty)) \int_{\mathfrak{X}} P^y(t, x, dx_1) g(x_1, y) - g(x, y) \right| < \frac{\varepsilon}{2},$$

and from (A.5') it follows that there exists a T_2 such that, for $0 < t < T_2$,

$$\sup_{x \in \mathfrak{X}} (1 - \mu^x((t, \infty))) \leq \frac{\varepsilon}{2} \sup_{x, y \in \mathfrak{X}} |g(x, y)|.$$

Hence, for $t < \min(T_1, T_2)$,

$$\begin{aligned} & \sup_{x, y \in \mathfrak{X}} \left| \int_{\mathfrak{X}} P^y(t, x, dx_1) f(x_1, y) - f(x, y) \right| \\ & \leq \sup_{x, y \in \mathfrak{X}} \left| \mu^x((t, \infty)) \int_{\mathfrak{X}} P^y(t, x, dx_1) f(x_1, y) - f(x, y) \right| + \\ & \quad + \sup_{x, y \in \mathfrak{X}} \left| \int_{\mathfrak{X}} P^y(t, x, dx_1) f(x_1, y) \right| \sup_{x \in \mathfrak{X}} (1 - \mu^x((t, \infty))) \leq \varepsilon. \end{aligned}$$

LEMMA 5. For every $f \in \mathcal{C}_0(\tilde{\mathfrak{X}})$, we have

$$\lim_{z \searrow \theta} \int_{\tilde{\mathfrak{X}}} \tilde{N}(dx \times dy | z) f(x, y) = \int_{\tilde{\mathfrak{X}}} \tilde{N}^-(dx \times dy) f(x, y).$$

The lemma follows from Lemma 4 and (A.7).

THEOREM 6. We have

$$A(\tilde{I})\tilde{N} = m(\tilde{N}^- - \tilde{N}^+),$$

where, for $A \times B \in \mathfrak{F} \times \mathfrak{F}$ and $x, y \in \mathfrak{X}$,

$$\tilde{I}(t, (x, y), A \times B) = P^y(t, x, A) \chi_B(y).$$

Proof. Let $f(x, y, z) = g(x, y) \in \overline{\mathcal{C}}_0(\overline{\mathcal{X}})$. Then, using the identities

$$\begin{aligned} & \frac{1}{t} \left(\int_{\tilde{\mathcal{X}}} \overline{N}(dx \times dy \times dz) \int_{\tilde{\mathcal{X}}} \overline{\Pi}(t, (x, y, z), d\xi_1 \times d\xi_2 \times d\eta) f(\xi_1, \xi_2, \eta) - \right. \\ & \qquad \qquad \qquad \left. - \int_{\tilde{\mathcal{X}}} \overline{N}(d\bar{x}) f(\bar{x}) \right) \\ &= \frac{1}{t} \left(\int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy \times (t, \infty)) \int_{\tilde{\mathcal{X}}} \tilde{\Pi}(t, (x, y), d\xi_1 \times d\xi_2) g(\xi_1, \xi_2) - \right. \\ & \qquad \qquad \qquad \left. - \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy) g(x, y) \right) \\ &= \frac{1}{t} \left(\int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy) \int_{\tilde{\mathcal{X}}} \tilde{\Pi}(t, (x, y), d\xi_1 \times d\xi_2) g(\xi_1, \xi_2) - \right. \\ & \qquad \qquad \qquad \left. - \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy) g(x, y) \right) - \\ & \qquad \qquad \qquad - \frac{1}{t} \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy \times (0, t]) \int_{\tilde{\mathcal{X}}} \tilde{\Pi}(t, (x, y), d\xi_1 \times d\xi_2) g(\xi_1, \xi_2), \end{aligned}$$

we obtain

$$\begin{aligned} (9) \quad & \frac{1}{t} \left(\int_{\tilde{\mathcal{X}}} \overline{N}(dx \times dy \times dz) \int_{\tilde{\mathcal{X}}} \overline{\Pi}(t, (x, y, z), d\xi_1 \times d\xi_2 \times d\eta) f(\xi_1, \xi_2, \eta) - \right. \\ & \qquad \qquad \qquad \left. - \int_{\tilde{\mathcal{X}}} \overline{N}(dx \times dy \times dz) f(x, y, z) \right) + \\ & \qquad \qquad \qquad + \frac{1}{t} \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy \times (0, t]) \int_{\tilde{\mathcal{X}}} P^\nu(t, x, d\xi_1) g(\xi_1, y) \\ &= \frac{1}{t} \left(\int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy) \int_{\tilde{\mathcal{X}}} \tilde{\Pi}(t, x, d\xi_1 \times d\xi_2) g(\xi_1, \xi_2) - \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy) g(x, y) \right). \end{aligned}$$

Hence we have

$$\begin{aligned} & \lim_{t \searrow 0} \frac{1}{t} \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy \times (0, t]) \int_{\tilde{\mathcal{X}}} P^\nu(t, x, d\xi_1) g(\xi_1, y) \\ &= \lim_{t \searrow 0} \frac{1}{t} \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy \times (0, t]) g(x, y) \quad (\text{by Lemma 4}) \\ &= \lim_{t \searrow 0} \frac{1}{t} \int_0^t \nu(dz) \int_{\tilde{\mathcal{X}}} \tilde{N}(dx \times dy | z) g(x, y) \\ &= m \int_{\tilde{\mathcal{X}}} \tilde{N}^-(dx \times dy) g(x, y) \quad (\text{by Lemma 5}). \end{aligned}$$

Since $\bar{N} \in \mathcal{D}(\bar{\Pi})$, the limits (for $t \searrow 0$) on the left-hand side of (9) exist, and hence the limit on the right-hand side of (9) exists and is equal to $A(\bar{\Pi})\bar{N}$. Thus we get

$$\begin{aligned} \lim_{t \searrow 0} \frac{1}{t} & \left(\int_{\bar{\mathcal{X}}} \bar{N}(dx \times dy \times dz) \int_{\bar{\mathcal{X}}} \bar{\Pi}(t, (x, y, z), d\xi_1 \times d\xi_2 \times d\eta) g(\xi_1, \xi_2) - \right. \\ & \left. - \int_{\bar{\mathcal{X}}} \bar{N}(dx \times dy \times dz) g(x, y) \right) \\ & = \int_{\bar{\mathcal{X}}} (A(\bar{\Pi})\bar{N} - m\bar{N}^-)(dx \times dy) g(x, y) \end{aligned}$$

which, by Theorem 5, completes the proof.

Remark. In the case where $P^x = P$ and $\mu^x = \mu$, $x \in \mathcal{X}$, it suffices to use assumptions (A.3) and (A.5), and the assertion of Theorem 6 reduces to

$$A(P)N = m(N^- - N^+), \quad \text{where } m^{-1} = \int_{\mathcal{X}_+} x \mu(dx).$$

5. Applications. Consider the queueing system $GI^X + M^Y/G/1$ as an example of a P.M.P. Such a system is described as follows: groups of customers of the first kind arrive in the queueing system at the instants $0 = \tau_0 < \tau_1 < \dots$, where $\tau_{i+1} - \tau_i$ ($i = 1, 2, \dots$) are independent, identically distributed random variables (abbreviated to i.i.d.r.v.) with the distribution function (abbreviated to d.f.) equal to F . Groups of customers of the second kind arrive in the system according to a Poisson process with rate λ_2 . The size of groups in the general input stream (Poisson stream) are i.i.d.r.v. with common d.f. equal to H_1 (H_2). The service times of customers of the first kind (the second kind) are i.i.d.r.v. with continuous d.f. equal to G_1 (G_2). The virtual waiting time $W(t)$ (see [3] for the definition) is the P.M.P. ($\mathcal{X} = (0, \infty)$) for which the instants τ_n , $n = 1, 2, \dots$, are the regenerative points. Let

$$K_2(y) = \int_{\mathcal{X}_+} G_2^{*z}(y) H_2(dz)$$

(G^{*n} denotes the n -fold convolution of G , $G^{*0}(x) = \chi_{[0, \infty)}(x)$) be the d.f. of the total sum of service times in one group. The transition function of the process between consecutive regenerative points is equal to

$$P(t, x, [0, y]) = \exp[-\lambda_2 t] \sum_{k=0}^{\infty} \frac{(\lambda_2 t)^k}{k!} K_2^{*k}(y - x + t) + o(1), \quad t, x, y \geq 0$$

(see [10], (2.55), p. 79), and the Markov kernel is

$$Q(x, A) = \int_{A-x} d \int_{\mathcal{X}_+} G_1^{*z}(y) H_1(dz), \quad x \in [0, \infty), A \in \mathfrak{F}.$$

We assume that the system is stable, i.e. assumption (A.7) is satisfied.

Now let us examine the remaining assumptions. Let us note that, for $f \in \mathcal{C}_0(\mathcal{X})$,

$$\int_{[0, \infty)} f(y)P(t, x, dy) = \exp[-\lambda_2 t]f((x-t)_+) + o(1),$$

where $(a)_+$ denotes $\max(a, 0)$, tends to $f(x)$ uniformly on $[0, \infty)$.

Since, for $f(x)g(z) \in \mathcal{C}_0(\mathcal{X})$,

$$\int_{\bar{x}} \bar{\Pi}(t, (x, z), d\xi \times d\zeta)f(\xi, \zeta) = \int_{[0, \infty)} P(t, x, d\xi)f(\xi)\chi_{a\tau+t}(z)g(\zeta),$$

one can show that (A.1), (A.2), and (A.3) are fulfilled.

Assumptions (A.4), (A.5), and (A.6) are satisfied, since the family of transient functions and distributions of intervals between successive regenerative points do not depend on the state $x \in \mathcal{X}$.

Let W^- (W^+) denote the stationary distribution of the Markov chain $W(\tau_n^-)$ ($W(\tau_n)$) for $n = 1, 2, \dots$. Then using the Corollary to Theorem 5 we get

$$(10) \quad \lim_{t \rightarrow \infty} \mathbf{P}(W(t) \leq y | W(0) = x) = W([0, y]) \\ = \lambda_1 \int_{\mathcal{X}_+} (1 - F(t)) \int_{[0, \infty)} W^+(dx)P(t, x, [0, y]) dt, \quad y \geq 0,$$

where

$$\lambda_1^{-1} = \int_{\mathcal{X}_+} (1 - F(x)) dx,$$

and $P(t, x, [0, y])$ is given in [10], p. 79.

Now we apply Theorem 6. We need to know $A(P)W$. First we calculate $A(P')W((y, \infty))$ for $y > 0$, where $P'(t, x, A) = \chi_A(x-t)_+$. We have

$$A(P')W((y, \infty)) = \mathbf{w}\text{-}\lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{[0, \infty)} W(dx)P'(t, x, (y, \infty)) - W((y, \infty)) \right) \\ = \mathbf{w}\text{-}\lim_{t \rightarrow 0} \frac{1}{t} (W((y+t, \infty)) - W((y, \infty))) = \frac{d}{dy} W((y, \infty))$$

(here $\mathbf{w}\text{-}\lim_{t \rightarrow \infty} M_n((y, \infty)) = M((y, \infty))$ is taken at points of continuity of $M((y, \infty))$). Hence

$$A(P)W((y, \infty)) = \mathbf{w}\text{-}\lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{[0, \infty)} W(dx)P(t, x, (y, \infty)) - W((y, \infty)) \right) \\ = \mathbf{w}\text{-}\lim_{t \rightarrow 0} \left(\frac{1}{t} \left(\int_{[0, \infty)} W(dx)P'(t, x, (y, \infty)) \exp[-\lambda_2 t] - W((y, \infty)) \right) + \right. \\ \left. + \lambda_2 \exp[-\lambda_2 t] \int_{[0, \infty)} W(dx)K_2((y-x+t, \infty)) + \exp[-\lambda_2 t] \frac{o(t)}{t} \right) \\ = \frac{d}{dy} W((y, \infty)) - \lambda_2 W((y, \infty)) + \lambda_2 W * K_2((y, \infty)).$$

Now, using Theorem 6, we obtain

$$(11) \quad -\frac{d}{dy} W((y, \infty)) + \lambda_2 W * K_2((y, \infty)) - \lambda_2 W((y, \infty)) \\ = \lambda_1 (W^-((y, \infty)) - W^+((y, \infty))).$$

Let us note that if $\lambda_2 = 0$, we obtain the queueing system GI/G/1. Then formula (10) reduces to

$$(12) \quad \lim_{t \rightarrow \infty} P(W(t) \leq y | W(0) = x) = \lambda_1 \int_{\mathcal{R}_+} (1 - F(t)) W^+([0, y + t]) dt$$

and equation (11) takes the form

$$(13) \quad \frac{d}{dy} W((y, \infty)) = \lambda_1 (W^-((y, \infty)) - W^+((y, \infty))), \quad y > 0.$$

The solution of (13) is

$$(14) \quad W((y, \infty)) = \varrho W^- * \hat{K}_2((y, \infty)), \quad y > 0,$$

where

$$\hat{K}_2([0, x]) = \mu \int_0^x K_2((y, \infty)) dy, \quad \mu^{-1} = \int_0^{\infty} K_2((y, \infty)) dy \quad \text{and} \quad \varrho = \frac{\lambda_1}{\mu}.$$

The results (12) and (14) were obtained earlier by Takács in [11] and [12], respectively.

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PROCESY PRZEDZIAŁAMI MARKOWA Z OGÓLNA PRZESTRZENIĄ FAZOWĄ

STRESZCZENIE

Kuczura [9] wprowadził pojęcie procesu przedziałami Markowa z dyskretną przestrzenią fazową. W niniejszej pracy rozpatruje się procesy przedziałami Markowa, przyjmujące wartości z pewnego podzbioru n -wymiarowej przestrzeni euklidesowej \mathcal{R}^n . Badane są związki między stacjonarnymi rozkładami pewnych włożonych łańcuchów Markowa a rozkładem stacjonarnym procesu. Rezultaty pracy zastosowano do analizy systemu kolejkowego $GI^X + M^Y/G/1$.
