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STEADY-STATE DISTRIBUTIONS OF QUEUE LENGTH IN THE $GI/G/s$ QUEUE

Consider the $GI/G/s$ queue. We characterize the state of this ueue-q ing system by the extended Markov process

$$(1)\qquad \big(n(t);\, X_{m(t)}(t);\, Y(t)\big), \qquad t \geqslant 0,$$

where $n(t)$ is the queue length at the moment t,

$$X_{m(t)}(t) = \big(X_1(t), X_2(t), \ldots, X_{\min(n(t),s)}(t)\big)$$

are the residual service times of items being served at the moment t, $Y(t)$ is the nearest arrival time after the moment t, and $m(t) = \min\big(n(t), s\big)$. We assume that at the beginning of service the service lines are renumbered in such a manner that

$$X_1(t) \geqslant X_2(t) \geqslant \ldots \geqslant X_{m(t)}(t), \qquad t \geqslant 0,$$

and that the process (1) is continuous from the left with respect to each component. We denote the arrival moments of items to the system by $t^{**} = t_1^{**}, t_2^{**}, \ldots$, and the departure moments by $t^* = t_1^*, t_2^*, \ldots$

Consider three marginal processes for (1), namely, $n(t)$, $\big(n(t); X_{m(t)}(t)\big)$, $\big(n(t); Y(t)\big)$, and the following five embedded stochastic chains: $n(t^{**})$, $n(t^*)$, $\big(n(t^{**}); X_{m(t^{**})}(t^{**})\big)$, $\big(n(t^*); Y(t^*)\big)$, $\big(n(t^*); X_{m(t^*)-1}(t^*)\big)$.

In this note we assume the stationarity of (1) and we find — under this assumption — the relations between the probability distributions of the above-mentioned processes and stochastic chains. This is an extension of well-known relations for the $M/G/1$ and $GI/M/1$ queues (see, e.g., [1]) to the case of a $GI/G/s$ queue.

Let us introduce the following notation for the probability distributions being of interest to us:

$$p_n = \Pr\big(n(t) = n\big), \quad p_n^{**} = \Pr\big(n(t^{**}) = n\big), \quad p_{n+1}^* = \Pr\big(n(t^*) = n+1\big), \qquad n \geqslant 0,$$

$$p_n(x) = \Pr(n(t) = n; X_m(t) \leqslant x), \quad p_n^{**}(x) = \Pr(n(t^{**}) = n; X_m(t^{**}) \leqslant x), \quad n \geqslant 1,$$

$$p_n^*(x) = \Pr(n(t^*) = n; X_{m-1}(t^*) \leqslant x), \quad n \geqslant 1,$$

$$P_n(y) = \Pr(n(t) = n; Y(t) \leqslant y), \quad P_{n+1}^*(y) = \Pr(n(t^*) = n+1; Y(t^*) \leqslant y), \quad n \geqslant 0,$$

where $m = \min(n, s)$.

Denote by A and B the probability distribution functions of inter-arrival and service times, respectively. We assume that these distributions are non-lattice, continuous at zero and have finite expected values which are denoted by $1/a$ and $1/b$, respectively. These assumptions do not guarantee the existence and necessary regularity of the considered probability distributions. These problems are not dealt with in this note.

THEOREM. (a) *In the $GI/G/s$ queue the steady-state distributions of queue length at the arrival and departure moments are related by the equalities*

$$p_n^{**} = p_{n+1}^*, \quad n \geqslant 0 \tag{2}$$

(Krakowski [2]).

(b) *In the $GI/G/s$ queue the steady-state distributions of the process $(n(t); Y(t))$ and the embedded stochastic chain $(n(t^*); Y(t^*))$ are related by the equalities*

$$\begin{aligned} (1/a)P_0'(y) &= p_1^* - P_1^*(y), \\ (1/a)P_n'(y) &= P_n^*(y) - p_n^* A(y) + p_{n+1}^* - P_{n+1}^*(y), \quad n \geqslant 1. \end{aligned} \tag{3}$$

(c) *In the $GI/G/1$ queue the steady-state distributions of the process $(n(t); X_1(t))$ and the embedded stochastic chain $(n(t^{**}); X_1(t^{**}))$ are related by the equalities*

$$\begin{aligned} (1/a)p_1'(x) &= p_0^{**} - p_0^{**}B(x) + p_1^{**}(x) - p_1^{**}B(x), \\ (1/a)p_n'(x) &= p_{n-1}^{**} - p_{n-1}^{**}(x) + p_n^{**}(x) - p_n^{**}B(x), \quad n > 1. \end{aligned} \tag{4}$$

(d) *In the $GI/G/s$ queue, for $s > 1$, the steady-state distributions of the process $(n(t); X_{m(t)}(t))$ and the embedded stochastic chains $(n(t^*); X_{m(t^*)-1}(t^*))$ and $(n(t^{**}); X_{m(t^{**})}(t^{**}))$ are related by the equalities*

$$\begin{aligned} (1/a)p_1'(x) &= p_0^{**} - p_0^{**}B(x) + p_1^{**}(x) - p_2^*(x), \\ (1/a)p_n'(x) &= p_{n-1}^{**} - p_{n-1}^{**}(x) + p_n^{**}(x) - p_{n+1}^*(x) - \end{aligned} \tag{5}$$

$$- \begin{cases} p_{n-1}^{**}B(x) - p_{n-1}^{**}(x)B(x) & \text{for } 1 < n < s, \\ p_s^{**}B(x) - p_{s+1}^*(x)B(x) + p_{s-1}^{**}B(x) - p_{s-1}^{**}(x)B(x) & \text{for } n = s, \\ p_n^{**}B(x) - p_{n+1}^*(x)B(x) & \text{for } n > s. \end{cases}$$

Integrating (3) over y from 0 to ∞, we obtain

COROLLARY 1. *In the GI/G/s queue the steady-state distributions of queue length depend upon the steady-state distribution of queue length at the departure moments as follows:*

$$p_0 = (a/a_1)p_1^*, \quad p_n = \big(1-(a/a_n)\big)p_n^* + (a/a_{n+1})p_{n+1}^*, \quad n \geqslant 1,$$

where

$$(1/a_n) = \int_0^\infty \Pr\big(Y(t^*) > y \mid n(t^*) = n\big)dy, \quad n \geqslant 1.$$

Integrating (4) over x from 0 to ∞, we obtain

COROLLARY 2. *In the GI/G/1 queue the steady-state distributions of queue length depend upon the steady-state distribution of queue length at the arrival moments as follows:*

$$p_n = (a/b_{n-1})p_{n-1}^{**} + \big((a/b)-(a/b_n)\big)p_n^{**}, \quad n \geqslant 1,$$

where

$$b_0 = b, \quad (1/b_n) = \int_0^\infty \Pr\big(X_m(t^{**}) > x \mid n(t^{**}) = n\big)\,dx, \quad n \geqslant 1.$$

Integrating (5) over x from 0 to ∞, we obtain

COROLLARY 3. *In the GI/G/s queue, for $s > 1$, the steady-state distributions of queue length depend upon the steady-state distributions of queue length at both the arrival and departure moments as follows:*

$$p_1 = (a/b)p_0^{**} + \big((a/c_2)-(a/b_1)\big)p_1^{**},$$

$$p_n = (a/b_{n-1})p_{n-1}^{**} + \big((a/c_{n+1})-(a/b_n)\big)p_n^{**} - \\ - \begin{cases} (a/b_{n-1}^*)p_{n-1}^{**} & \text{for } 1 < n < s, \\ (a/c_{s+1}^*)p_s^{**} + (a/b_{s-1}^*)p_{s-1}^{**} & \text{for } n = s, \\ (a/c_{n+1}^*)p_n^{**} & \text{for } n > s, \end{cases}$$

where

$$(1/c_n) = \int_0^\infty \Pr\big(X_{m-1}(t^*) > x \mid n(t^*) = n\big)dx, \quad n \geqslant 1,$$

$$(1/b_n^*) = \int_0^\infty \Pr\big(X_m(t^{**}) > x \mid n(t^{**}) = n\big)B(x)dx, \quad n \geqslant 1,$$

$$(1/c_n^*) = \int_0^\infty \Pr\big(X_{s-1}(t^*) > x \mid n(t^*) = n\big)B(x)dx, \quad n > s.$$

In the $M/G/s$ queue the random variable $Y(t^*)$ does not depend upon $n(t^*)$ and has the expected value $1/a$. From Corollary 1 we obtain

COROLLARY 4. *In the $M/G/s$ queue the steady-state distributions of queue length and those of queue length at the departure moments are related by the equality*

$$p_n = p_{n+1}^*, \quad n \geqslant 0$$

(the correct version of Corollary 3 from [1]).

In the $GI/M/s$ queue, for $n(t^*) = n$, the random variables $X_1(t^*)$, $X_2(t^*), \ldots, X_{m-1}(t^*)$ form order statistics in the sequence of independent and exponentially distributed (with parameter b) random variables. Thus $X_{m-1}(t^*)$ is exponentially distributed with parameter $(m-1)b$. We find

$$\frac{1}{c_n} = \frac{1}{(m-1)b}, \; n > 1, \quad \frac{1}{c_n^*} = \frac{1}{(s-1)b} - \frac{1}{sb}, \; n > s.$$

For $n(t^{**}) = n$, the random variables $X_1(t^{**}), X_2(t^{**}), \ldots, X_m(t^{**})$ form order statistics in the sequence of independent and exponentially distributed (with parameter b) random variables. Therefore, $X_m(t^{**})$ has the exponential distribution with parameter mb. We find

$$\frac{1}{b_n} = \frac{1}{mb}, \quad \frac{1}{b_n^*} = \frac{1}{mb} - \frac{1}{(m+1)b}, \quad n \geqslant 1.$$

From Corollaries 2 and 3 we obtain

COROLLARY 5. *In the $GI/M/s$ queue the steady-state distributions of queue length and those of queue length at the arrival moments satisfy the equality*

$$p_n = (a/mb)p_{n-1}^{**}, \quad n \geqslant 1$$

(Kopocińska and Kopociński [1], Takács [3]).

Proof of the Theorem. Consider the Markov process (1) and two embedded Markov chains defined at the arrival moments t^{**} and at the departure moments t^*. Symbolically,

$$(6) \quad \begin{aligned} &\big(n(t^{**}); \boldsymbol{X}_{m(t^{**})}(t^{**})\big) = \big(n(t); \boldsymbol{X}_{m(t)}(t) \mid Y(t) = 0\big), \\ &\big(n(t^*); \boldsymbol{X}_{m(t^*)-1}(t^*); Y(t^*)\big) = \big(n(t); \boldsymbol{X}_{m(t)-1}(t); Y(t) \mid n(t) > 0, X_{m(t)}(t) = 0\big), \end{aligned}$$

where we write, for convenience, $\boldsymbol{x}_i = (x_1, x_2, \ldots, x_i) = (\boldsymbol{x}_{i-1}, x_i)$.

Introduce the following notation for the steady-state distributions of the process (1) and for the chains (6):

$$
(7)\quad
\begin{aligned}
p_0(\cdot\,;y) &= P_0(y) = \Pr\big(n(t)=0;\ Y(t)\leqslant y\big),\\
p_n(\boldsymbol{x}_m;y) &= \Pr\big(n(t)=n;\ \boldsymbol{X}_m(t)\leqslant \boldsymbol{x}_m;\ Y(t)\leqslant y\big), \qquad n\geqslant 1,\\
p_n^{**}(\boldsymbol{x}_m) &= \Pr\big(n(t^{**})=n;\ \boldsymbol{X}_m(t^{**})\leqslant \boldsymbol{x}_m\big), \qquad n\geqslant 1,\\
P_n^{*}(\boldsymbol{x}_{m-1};y) &= \Pr\big(n(t^{*})=n;\ \boldsymbol{X}_{m-1}(t^{*})\leqslant \boldsymbol{x}_{m-1};\ Y(t^{*})\leqslant y\big), \qquad n\geqslant 0.
\end{aligned}
$$

Let $\boldsymbol{I}_j = (1,1,\ldots,1)$ be a j-component vector and let $\boldsymbol{G}(\boldsymbol{x}_j)$ be a function which arranges the components of $\boldsymbol{x}_j$ in non-increasing order. Since we use the function $\boldsymbol{G}$ only at the beginning of service, we may limit ourselves to the consideration of (7) for $\boldsymbol{x}_m = (x_1, x_2, \ldots, x_m)$ such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_m$.

Let V denote a random variable distributed according to B and independent of $\big(n(t);\ \boldsymbol{X}_{m(t)}(t);\ Y(t)\big)$. Analyzing the state of the process (1) at the moments t and $t+h$, $h>0$, assuming stationarity of the process we find that

$$
(8)\quad
\begin{aligned}
p_0(\cdot\,;y)+o(h) &= p_0(\cdot\,;y+h)-p_0(\cdot\,;h)+p_1(h;y),\\
p_n(\boldsymbol{x}_m;y)+o(h) &= p_n(\boldsymbol{x}_m+h\boldsymbol{I}_m;y+h)-p_n(\boldsymbol{x}_{m-1},h;y)-p_n(\boldsymbol{x}_m;h)+
\end{aligned}
$$

$$
+\begin{cases}
p_{n+1}(\boldsymbol{x}_n,h;y)+\Pr\big(n(t)=n-1;\ \boldsymbol{X}_{n-1}(t)\leqslant \boldsymbol{x}_{n-1},\ \boldsymbol{G}(\boldsymbol{X}_{n-1}(t),V)\leqslant \boldsymbol{x}_n;\\
\qquad\qquad Y(t)\leqslant h\big)A(y) \quad \text{for } 0<n<s,\\
\Pr\big(n(t)=n+1;\ \boldsymbol{X}_{s-1}(t)\leqslant \boldsymbol{x}_{s-1},\ X_s(t)\leqslant h,\ \boldsymbol{G}(\boldsymbol{X}_{s-1}(t),V)\leqslant \boldsymbol{x}_s;\\
\qquad\qquad Y(t)\leqslant y\big)+
\end{cases}
$$

$$
+\begin{cases}
\Pr\big(n(t)=s-1;\ \boldsymbol{X}_{s-1}(t)\leqslant \boldsymbol{x}_{s-1},\ \boldsymbol{G}(\boldsymbol{X}_{s-1}(t),V)\leqslant \boldsymbol{x}_s;\ Y(t)\leqslant h\big)A(y)\\
\qquad\qquad \text{for } n=s,\\
p_{n-1}(\boldsymbol{x}_s;h)A(y) \qquad \text{for } n>s.
\end{cases}
$$

Let $(\boldsymbol{x}_m;y)$ be a continuity point of the probability distributions (7), where $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_m$. We have

$$
(9)\quad \frac{\partial}{\partial x_m} p_n(\boldsymbol{x}_{m-1},0;y) = \lim_{h\to 0}\frac{1}{h}\Pr\big(n(t)>0;\ X_m(t)\leqslant h\big)\Pr\big(n(t)=n;
$$

$$
\boldsymbol{X}_{m-1}(t)\leqslant \boldsymbol{x}_{m-1};\ Y(t)\leqslant y \mid n(t)>0,\ X_m(t)\leqslant h\big) = a^{*}P_n^{*}(\boldsymbol{x}_{m-1};y), \qquad n\geqslant 1,
$$

where

$$
a^{*} = \lim_{h\to 0}\frac{1}{h}\Pr\big(n(t)>0;\ X_m(t)\leqslant h\big),
$$

$$
(10)\quad \frac{\partial}{\partial y} p_n(\boldsymbol{x}_m;0) = \lim_{h\to 0}\frac{1}{h}\Pr\big(Y(t)\leqslant h\big)\Pr\big(n(t)=n;
$$

$$
\boldsymbol{X}_m(t)\leqslant \boldsymbol{x}_m \mid Y(t)\leqslant h\big) = a p_n^{**}(\boldsymbol{x}_m), \qquad n\geqslant 0,
$$

$$(11)\quad \lim_{h\to 0}\frac{1}{h}\Pr\big(n(t) = n+1;\, \boldsymbol{X}_{s-1}(t)\leqslant \boldsymbol{x}_{s-1},\, X_s(t)\leqslant h,\, G\big(\boldsymbol{X}_{s-1}(t),\, V\big)\leqslant \boldsymbol{x}_s;\; Y(t)\leqslant y\big)$$
$$= a^*\Pr\big(n(t^*) = n+1;\, \boldsymbol{X}_{s-1}(t^*)\leqslant \boldsymbol{x}_{s-1},\, G\big(\boldsymbol{X}_{s-1}(t^*),\, V\big)\leqslant \boldsymbol{x}_s;\, Y(t^*)\leqslant y\big)\quad \text{for } n\geqslant s,$$

$$(12)\quad \lim_{h\to 0}\frac{1}{h}\Pr\big(n(t) = n-1;\, \boldsymbol{X}_{n-1}(t)\leqslant \boldsymbol{x}_{n-1},\, G\big(\boldsymbol{X}_{n-1}(t),\, V\big)\leqslant \boldsymbol{x}_n;\; Y(t)\leqslant h\big)$$
$$= a\Pr\big(n(t^{**}) = n-1;\, \boldsymbol{X}_{n-1}(t^{**})\leqslant \boldsymbol{x}_{n-1},\, G\big(\boldsymbol{X}_{n-1}(t^{**}),\, V\big)\leqslant \boldsymbol{x}_n\big)\quad \text{for } n\leqslant s.$$

Forming difference quotients in (8), taking the limit as $h\to 0$ and using (9)-(12) we get

$$(13)\quad 0 = \frac{\partial}{\partial y}p_0(\cdot;y) - ap_0^{**} + a^*P_1^*(\cdot;y),$$

$$(14)\quad 0 = \sum_{j=1}^{m}\frac{\partial}{\partial x_j}p_n(\boldsymbol{x}_m;y) + \frac{\partial}{\partial y}p_n(\boldsymbol{x}_m;y) - a^*P_n^*(\boldsymbol{x}_{m-1};y) - ap_n^{**}(\boldsymbol{x}_m) +$$
$$+\begin{cases} a^*P_{n+1}^*(\boldsymbol{x}_n;y) + a\Pr\big(n(t^{**}) = n-1;\\ \qquad \boldsymbol{X}_{n-1}(t^{**})\leqslant \boldsymbol{x}_{n-1},\, G\big(\boldsymbol{X}_{n-1}(t^{**}),\, V\big)\leqslant \boldsymbol{x}_n\big)A(y) & \text{for } 0<n<s,\\ a^*\Pr\big(n(t^*) = n+1;\, \boldsymbol{X}_{s-1}(t^*)\leqslant \boldsymbol{x}_{s-1},\, G\big(\boldsymbol{X}_{s-1}(t^*),\, V\big)\leqslant \boldsymbol{x}_s;\, Y(t^*)\leqslant y\big) + \end{cases}$$
$$+\begin{cases} a\Pr\big(n(t^{**}) = s-1;\, \boldsymbol{X}_{s-1}(t^{**})\leqslant \boldsymbol{x}_{s-1},\, G\big(\boldsymbol{X}_{s-1}(t^{**}),\, V\big)\leqslant \boldsymbol{x}_s\big)A(y) & \\ & \text{for } n = s,\\ ap_{n-1}^{**}(\boldsymbol{x}_s)A(y) & \text{for } n>s. \end{cases}$$

Taking the limit of (14) as $\boldsymbol{x}_m\to\boldsymbol{\infty}_m$, we obtain

$$(15)\quad \frac{\partial}{\partial x_j}p_n(\boldsymbol{x}_m;y)\to 0\quad \text{for } 1\leqslant j\leqslant m$$

and, therefore,

$$(16)\quad \begin{aligned} 0 &= P_0'(y) - ap_0^{**} + a^*P_1^*(y),\\ 0 &= P_n'(y) - a^*P_n^*(y) - ap_n^{**} + a^*P_{n+1}^*(y) + ap_{n-1}^{**}A(y),\qquad n\geqslant 1. \end{aligned}$$

Taking the limit as $y\to\infty$, and because of $P_n'(y)\to 0$, we obtain $ap_{n-1}^{**} = a^*p_n^*$, $n\geqslant 1$. Hence we have $a^* = a$ and (2), and after substitution into (16) we get (3).

Taking the limit of (14) as $y \to \infty$, $\boldsymbol{x}_{m-1} \to \boldsymbol{\infty}_{m-1}$ and using the substitution $x_m = x$ we have

$$\frac{\partial}{\partial y} p_n(\boldsymbol{x}_m; y) \to 0$$

and (15) holds for $1 \leqslant j \leqslant m-1$. Using the fact that the last component of $G(X_j, V)$ is equal to $\min(X_j, V)$ we get

$$(17) \quad 0 = p_n'(x) - ap_n^* - ap_n^{**}(x) +$$
$$+\begin{cases} ap_{n+1}^*(x) + a\Pr\big(n(t^{**}) = n-1;\ \min\big(X_{n-1}(t^{**}), V\big) \leqslant x\big) & \text{for } 0 < n < s, \\ a\Pr\big(n(t^*) = n+1;\ \min\big(X_{s-1}(t^*), V\big) \leqslant x\big) + \end{cases}$$
$$+\begin{cases} a\Pr\big(n(t^{**}) = s-1;\ \min\big(X_{s-1}(t^{**}), V\big) \leqslant x\big) & \text{for } n = s, \\ ap_{n-1}^{**}(x) & \text{for } n > s. \end{cases}$$

It is easy to obtain

$$\Pr\big(n(t^{**}) = n-1;\ \min\big(X_{n-1}(t^{**}), V\big) \leqslant x\big)$$
$$= \begin{cases} p_{n-1}^{**}B(x) & \text{for } n = 1, \\ p_{n-1}^{**}(x) + p_{n-1}^{**}B(x) - p_{n-1}^{**}(x)B(x) & \text{for } n > 1, \end{cases}$$
$$\Pr\big(n(t^*) = n+1;\ \min\big(X_{s-1}(t^*), V\big) \leqslant x\big)$$
$$= \begin{cases} p_{n+1}^{*}B(x) & \text{for } s = 1, \\ p_{n+1}^{*}(x) + p_{n+1}^{*}B(x) - p_{n+1}^{*}(x)B(x) & \text{for } s > 1. \end{cases}$$

From this and from (17) and (2) we obtain (4) for $s = 1$ and (5) for $s > 1$. This completes the proof of the Theorem.

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STACJONARNE ROZKŁADY PRAWDOPODOBIEŃSTWA LICZBY JEDNOSTEK W SYSTEMIE *GI/G/s*

STRESZCZENIE

Przedmiotem naszego zainteresowania w rozważanym systemie obsługi masowej są procesy losowe: liczba $n(t)$ jednostek w systemie w chwili t, czas $Y(t)$ do pierwszego zgłoszenia po chwili t, czas $X_{m(t)}(t)$ do pierwszego wyjścia jednostki z systemu po chwili t. Są to procesy należące m. in. do składowych wektorowego procesu Markowa opisującego stan systemu.

W pracy, przy założeniu stacjonarności wspomnianego procesu Markowa, podane są związki między rozkładami prawdopodobieństwa wymienionych charakterystyk systemu, rozpatrywanych w czasie ciągłym, i rozkładami prawdopodobieństwa włożonych łańcuchów losowych, zdefiniowanych w chwilach zgłoszenia jednostek do systemu i w chwilach wyjścia jednostek z systemu.