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APPLICATIONES MATHEMATICAE
XIV, 4 (1975)

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MINIMAX CHECKING OF REPLACEABLE UNITS

1. FORMULATION OF THE MODEL

In this paper the following situation is considered: At the moment $t = 0$ a system begins to work. The time till the failure of the system (life time) is a random variable X with the unknown distribution function $F(t)$. It is assumed that a system failure is known only by inspecting. Immediately after discovery of a failure, but at least after $T > 0$ time units, the system is replaced by a new one with the same distribution of life time. The inspection-replacement process is continued unlimitedly. Each replacement requires a fixed time d and a fixed cost c_2 ($d, c_2 \geqslant 0$). Each inspection entails a fixed cost c_1 and occurs at a negligible time. On the other hand, a downtime t of the system (i.e. a time between the system failure and the starting of the replaced system) gives rise to a cost $v(t)$, where $v(t) \equiv 0$ for $t \leqslant 0$, and $v(t)$ is continuous and strictly increasing in t, $0 \leqslant t < \infty$.

2. BOUNDED WORKING TIME OF THE SYSTEM ($T < \infty$)

2.1. Derivation of the minimax loss. Let $S_n = \{t_k\}$, $0 < t_1 < t_2 < \ldots < t_n < T$, $n < \infty$, be an inspection strategy (shortly, strategy) prescribing exactly n inspections (at a time t_k occurs the k-th inspection, if no failure of the system has been before detected). For fixed S_n, the t_k are assumed to be constant numbers. Especially, the strategy S_0 means "no inspection". If the number n of inspections is unessential, instead of S_n we only write S. Let γ be the set of all inspection strategies. To derivate the minimax criterion we define functions $u_k(x, y)$, $0 \leqslant x \leqslant y$, by

$$(1) \qquad u_k(x, y) = \frac{(k+1)c_1 + v(y-x+d) + c_2}{y+d}, \qquad k = 0, 1, 2, \ldots$$

Let $S_n = \{t_n\}$. In case where $t_k < X \leqslant t_{k+1} \leqslant T$ ($T < X$), we have the loss $u_k(X, t_{k+1})$ ($u_{n-1}(T, T)$) per unit time. Therefore, the mathematical expectation of the unconditional loss per unit time is given by

$$(2)\qquad K(S_n, F) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} u_k(t, t_{k+1})\,dF(t) + \int_{t_n}^{T} u_{n-1}(t, T)\,dF(t) + \\ + u_{n-1}(T, T)\big(1 - F(T+0)\big).$$

In (2) the intervals of integration are open on the left and closed on the right. Since F is unknown, we consider the expression

$$K(S_n) = \sup_{\{F;\,F(+0)=0\}} K(S_n, F).$$

We will find a strategy $S^* = \{t_k^*\}$ such that

$$K(S^*) = \inf_{\substack{S_n \in \gamma,\\ n=0,1,\ldots}} K(S_n).$$

We call S^* a *minimax inspection strategy*. A similar problem has been treated in [3] (see also [1], [2] and [4]).

THEOREM 1. *For any strategy $S_n = \{t_k\}$, there holds*

$$K(S_n) = \max\{\max_{k=0,1,\ldots,n-1} u_k(t_k, t_{k+1}),\ u_{n-1}(t_n, T)\}.$$

Proof. Evidently, the functions $u_k(x, y)$, $k = 0, 1, \ldots$, have the following property:

(a) for fixed $y > 0$, $u_k(x, y)$ are strictly monotone decreasing in x, $0 \leqslant x \leqslant y$.

Let

$$w_0 = u_0(0, t_1), \quad w_1 = u_1(t_1, t_2), \quad \ldots,$$

$$w_{n-1} = u_{n-1}(t_{n-1}, t_n), \quad w_n = u_{n-1}(t_n, T), \quad w_{n+1} = u_{n-1}(T, T),$$

and let m be defined by

$$w_m = \max_{i=0,1,\ldots,n} w_i.$$

Further, let us write

$$p_k = F(t_{k+1}+0) - F(t_k+0), \qquad k = 0, 1, \ldots, n,$$

$$p_{n+1} = 1 - F(T+0).$$

Then, in view of (a) and $w_n > w_{n+1}$, according to (2), we have

$$K(S_n, F) \leqslant \sum_{k=0}^{n+1} w_k p_k \leqslant w_m.$$

We define now $F_m(t)$ by

$$F_m(t) = \begin{cases} 0, & t \leqslant t_m, \\ 1, & t_m > t. \end{cases}$$

$F_m(t)$ satisfies $K(S_n, F_m) = w_m$. Hence the proof of the theorem is complete.

2.2. Existence and computation of a minimax inspection strategy. In the sequel we assume that $v(t)$ is strictly convex in $[0, \infty)$ [1]. For fixed x, $u_k(x, y)$ are functions in y, $x \leqslant y$, which we denote by $u_{k,x}(y)$. It holds

$$\frac{du_{k,x}(y)}{dy} = \frac{\int\limits_0^{y+d} [v'(y-x+d) - v'(t-x)]dt - (k+1)c_1 - c_2}{(y+d)^2}.$$

In virtue of the convexity of $v(t)$, the integral

$$\int\limits_0^{y+d} [v'(y-x+d) - v'(t-x)]dt$$

is strictly increasing in y.

Hence the function $u_{k,x}(y)$ has exactly one minimum $M_k(x)$. Let $m_k(x)$ be defined by $u_{k,x}[m_k(x)] = M_k(x)$. Then $u_{k,x}(y)$, $k = 0, 1, \ldots,$ have the following property:

(b) For $x < m_k(x)$, the functions $u_{k,x}(y)$ are strictly monotone decreasing in $\lfloor x, m_k(x)\rfloor$. For $m_k(x) < \infty$, the functions $u_{k,x}(y)$ are strictly monotone increasing in $\lfloor m_k(x), \infty\rfloor$.

In view of (b), in case $m_0(0) \geqslant T$, S_0 is the minimax strategy. Hence, in the sequel we assume that $0 \leqslant m_0(0) < T$. Evidently, for $0 \leqslant x_1 \leqslant x_2 < \infty$, $k = 0, 1, \ldots,$ we have

$$u_{k,x_1}(y) > u_{k,x_2}(y), \quad x_2 \leqslant y. \tag{3}$$

Now we assume that a strategy $S_n = \{t_k\}$ has for $n \geqslant 1$ the following property:

$$m_k(t_k) \leqslant t_{k+1}, \quad k = 0, 1, \ldots, n-1. \tag{4}$$

LEMMA 1. *For any $S \epsilon \gamma$, $S \neq S_0$, there exists a strategy $\tilde{S}$ which satisfies condition* (4) *and such that* $K(\tilde{S}) \leqslant K(S)$.

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[1] For concave, especially linear $v(t)$, $t > 0$, the function $u_0(0, x)$ decreases with increasing x, $0 < x$. Therefore, in this case S_0 is the minimax strategy. In view of the assumed continuity of $v(t)$, the convexity of $v(t)$ implies that $v(t)$ is also differentiable.

The straightforward proof of this lemma is omitted (see [3]).

In view of this lemma we can assume, without loss of generality, that a minimax strategy $S^* \neq S_0$ satisfies condition (4). For its precise characterization we need, however, further definitions.

We consider the system of equations

$$u_k(t_k, t_{k+1}) = u_{k+1}(t_{k+1}, t_{k+2}), \qquad k = 0, 1, 2, \ldots \tag{5}$$

By means of (5), a sequence $\{t_k\}$ is recursively generated by each $t_1 > 0$. It breaks off at t_m if the m-th equation of (5) ($k = m-1$) has no solution t_{m+1} with $t_m < t_{m+1} < \infty$, $m \leqslant \infty$. We denote by $\varphi(t_1) = \{t_k\}$ the sequence generated by means of t_1 which satisfies conditions (4) for $k = 0, 1, \ldots, m-1$. The functions $t_k = t_k(t_1)$, given by $\varphi(t_1) = \{t_k\}$, are continuous and, by (a), (b) and (3), strictly increasing in t_1, $t_1 \in [m_0(0), T)$. These properties of $t_k = t_k(t_1)$ we denote by (M).

Furthermore, we define a strategy $S(t_1) = \{t_1, t_2, \ldots, t_n\}$ to be the *partial sequence* of $\varphi(t_1) = \{t_k\}$, $S(t_1) \subseteq \varphi(t_1)$, which breaks off at t_n if either the n-th equation of (5) has no solution t_{n+1} with $t_n < t_{n+1} < T$ and $m_n(t_n) \leqslant t_{n+1}$, or if $u_{n-1}(t_{n-1}, t_n) \geqslant u_{n-1}(t_n, T)$.

Definition. A strategy $S_n = \{t_k\}$ is called *admissible* if

$$u_0(0, t_1) = u_1(t_1, t_2) = \ldots = u_{n-1}(t_{n-1}, t_n) \geqslant u_{n-1}(t_n, T).$$

LEMMA 2. *There exists an admissible strategy $S(t_1)$ for $m_0(0) \leqslant t_1 < T$.*

Proof. We consider the functions $u_{0,0}(t)$ and $u_0(t, T)$ for a variable t, $t \in [m_0(0), T]$. In case where $u_0(m_0(0), T) \leqslant u_{0,0}(m_0(0))$, by (a) and (b) any strategy $S_1 = \{t_1\}$, $m_0(0) \leqslant t_1 < T$, is admissible. Hence let $u_0(m_0(0), T) > u_{0,0}(m_0(0))$. In this case the functions $u_{0,0}(t)$ and $u_0(t, T)$ have exactly one point of intersection in $(m_0(0), T)$. Let $u_{0,0}(\bar{t}) = u_0(\bar{t}, T)$. But then again each strategy $S_1 = \{t_1\}$, $\bar{t} \leqslant t_1 < T$, is admissible. In view of (M), the lemma immediately implies the following

COROLLARY. *There exists a smallest number $t_1 = t_1^*$, $m_0(0) \leqslant t_1^* < T$, with the property that the strategy $S(t_1)$ is admissible.*

THEOREM 2. *$S(t_1^*) =: \{t_1^*, t_2^*, \ldots, t_n^*\}$ is a minimax inspection strategy.*

Proof. Next we consider the case

$$u_0(0, t_1^*) = u_1(t_1^*, t_2^*) = \ldots = u_{n^*-1}(t_{n^*-1}^*, t_{n^*}^*) > u_{n^*-1}(t_{n^*}^*, T). \tag{6}$$

In consequence of (M), under this assumption we have $t_1^* = m_0(0) > 0$. Otherwise, there would exist a number $\bar{t}_1$, $m_0(0) < \bar{t}_1 < t_1^*$ such that $S(\bar{t}_1)$ will also be an admissible strategy; this contradicts to the definition of t_1^*. Therefore, under condition (6) the theorem is already proved.

Now let

(7) $$u_0(0, t_1^*) = u_1(t_1^*, t_2^*) = \ldots = u_{n^*-1}(t_{n^*-1}^*, t_{n^*}^*) = u_{n^*-1}(t_{n^*}^*, T).$$

Then, by (M), for all m, $1 \leqslant m \leqslant n^*$, there exist admissible strategies $\tilde{S}_m := \{t_k^{(m)}\}$ which satisfy the conditions

$$u_0(0, t_1^{(m)}) = u_1(t_1^{(m)}, t_2^{(m)}) = \ldots = u_{m-1}(t_{m-1}^{(m)}, t_m^{(m)}) = u_{m-1}(t_m^{(m)}, T).$$

As in [3], it can be proved that, for any strategy S_m, $K(\tilde{S}_m) \leqslant K(S_m)$ always holds. Further, for $1 \leqslant m \leqslant m' \leqslant n^*$ we have the inequality $t_1^{(m')} \leqslant t_1^{(m)}$, and hence $K(\tilde{S}_{m'}) \leqslant K(\tilde{S}_m)$. It follows that $K(S_{n^*}^*) \leqslant K(\tilde{S}_m) \leqslant K(S_m)$. Therefore, to complete the proof of the theorem, we have still to show that, for any strategy S_n with $n > n^*$, $K(S_{n^*}) \leqslant K(S_n)$ always holds.

Suppose that there exists a strategy $S_n = \{t_k\}$, $n > n^*$, satisfying condition (4) and such that

(8) $$K(S_n) < K(S_{n^*}^*).$$

From (8) we obtain $u_0(0, t_1) < u_0(0, t_1^*)$, and hence we have $t_1 < t_1^*$ $(m_0(0) \leqslant t_1)$. Starting from this, in view of the inequalities $u_k(t_k, t_{k+1}) < u_k(t_k^*, t_{k+1}^*)$, by induction we get $t_k < t_k^*$, $k = 1, 2, \ldots, n^*$. Let $t_{n^*+\nu} < t_{n^*}^* \leqslant t_{n^*+\nu+1}$, $\nu \geqslant 0$. It follows from

$$u_{n^*-1}(t_{n^*-1}^*, t_{n^*}^*) = u_{n^*-1}(t_{n^*}^*, T) > u_{n^*+\nu}(t_{n^*+\nu}, t_{n^*+\nu+1})$$

that the equation $u_{n^*-1}(t_{n^*-1}^*, t_{n^*}^*) = u_{n^*}(t_{n^*}^*, t_{n^*+1}^*)$ has a solution $t_{n^*+1}^*$ with $t_{n^*}^* \leqslant m_{n^*}(t_{n^*}^*) \leqslant t_{n^*+1}^* < T$, $t_{n^*}^* < t_{n^*+1}^*$. We consider the strategy $S_{n^*+1} = \{t_1^*, t_2^*, \ldots, t_{n^*+1}^*\}$ which has the property

$$u_{n^*}(t_{n^*}^*, t_{n^*+1}^*) \leqslant u_{n^*}(t_{n^*+1}^*, T).$$

Therefore, by (M), there exists a strategy $S'_{n^*+1} := \{t'_k\}$, $t'_1 \geqslant t_1$, which satisfies conditions (5) with $n = n^*+1$ and

$$u_0(0, t'_1) = u_1(t'_1, t'_2) = \ldots = u_{n^*+1}(t'_{n^*}, t'_{n^*+1}) = u_{n^*+1}(t'_{n^*+1}, T),$$

such that $K(S_n) < K(S'_{n^*+1})$. In case of $n > n^*+1$ we infer because of (M) and the definition of t_1^*, by repeated application of this argument, that under assumption (8) there exists a strategy $S''_n := \{t''_k\}$ such that conditions (5), the equations

$$u_0(0, t''_1) = u_1(t''_1, t''_2) = \ldots = u_{n-1}(t''_{n-1}, t''_n) = u_n(t''_n, T)$$

and, moreover, the inequality

(9) $$K(S_n) < K(S''_n)$$

are satisfied.

From (9), by induction we get $t_n < t_n''$ as above. It follows that

$$K(S_n) \geqslant u_{n-1}(t_n, T) > u_{n-1}(t_n'', T) = K(S_n''),$$

contradictory to (9). Hence inequality (8) is not possible for $n > n^*$. Thus the proof of the theorem is complete.

By reason of the proof of theorem 2, for the numerical computation of the minimax strategy $S_{n^*}^* = S(t_1^*)$ only the case $m_0(0) < t_1^*$ is interesting. Then, after the computation of $S(m_0(0))$, we immediately decide whether $S(m_0(0))$ $(m_0(0) > 0)$ is admissible (in this case $S(m_0(0))$ is the minimax strategy). Now let $0 \leqslant m_0(0) < t_1^*$. Then $S_{n^*}^* = \{t_k^*\}$ satisfies (7). Hence, in view of theorem 2 and (M), we get $S_{n^*}^*$ by the following procedure. Starting from $m_0(0)$, we let increase t_1 as soon as in $S(t_1) = \{t_1, t_2, \ldots, t_n\}$ for the first time $u_{n-1}(t_{n-1}, t_n) = u_{n-1}(t_n, T)$. The resulting strategy is minimax.

Example. Let $c_1 = 10$, $c_2 = 100$, $v(t) = t^2$, $d = 0$, and $T = 200$. Then we have the minimax inspection times $t_1^* = m_0(0) = 10.5$, $t_2^* = 35.5$, $t_3^* = 73$, $t_4^* = 122.5$, and $t_5^* = 183$. The corresponding minimax loss is given by $K(S^*) = 21$. In this case we have $u_{n^*-1}(t_{n^*}, T) = u_4(183, 200) = 2.2 < K(S^*)$.

3. UNBOUNDED WORKING TIME OF THE SYSTEM ($T = \infty$)

To have a significant difference between the cases $T < \infty$ and $T = \infty$, we assume that

$$\lim_{t\to\infty} \frac{v(t)}{t} = \infty. \tag{10}$$

We define now an inspection strategy S as an unbounded increasing sequence of numbers $S = \{t_k\}$, $0 < t_1 < t_2 < \ldots$ The expected value $K(S, F)$ of the loss cost per unit time, by the application of $S = \{t_k\}$, is given by

$$K(S, F) = \sum_{k=0}^{\infty} \frac{1}{t_{k+1}+d} \int_{t_k}^{t_{k+1}} [(k+1)c_1 + v(t_{k+1} - t + d) + c_2]\, dF(t).$$

Analogously as in the proof of theorem 1, we obtain

$$K(S) = \sup_{\{F; F(+0)=0\}} K(S, F) = \sup_{k=0,1,\ldots} u_k(t_k, t_{k+1}).$$

In view of (10), there exists a $t_1 \geqslant m_0(0)$ such that the sequence of numbers $\varphi(t_1)$ is unlimitedly increasing. Especially, in virtue of (M), there exists a smallest number $t_1 = t_1^*$ with this property. Then, for $t_1 \geqslant t_1^*$, the corresponding sequences $\varphi(t_1)$ are inspection strategies which we again denote by $S(t_1)$.

THEOREM 3. *$S^* := S(t_1^*) := \{t_k^*\}$ is a minimax inspection strategy.*

Proof. Suppose that there exists a strategy $S = \{t_k\}$ satisfying $K(S) < K(S^*)$. Analogously as in section 2, we can assume, without loss of generality, that S satisfies condition (4) for $k = 0, 1, 2, \ldots$ We can also assume, without loss of generality, the existence of the limit

$$a := \lim_{k\to\infty} \frac{k}{t_k}.$$

Further let

$$a^* := \lim_{k\to\infty} \frac{k}{t_k^*}.$$

The inequality $K(S) < K(S^*) = a^* c$ yields $a < a^*$. Hence there exists a $\nu > 0$ with the property $t_\nu^* < t_\nu$. The inequalities $u_k(t_k, t_{k+1}) < u_k(t_k^*, t_{k+1}^*)$ imply, however, $t_k < t_k^*$ for all $k = 1, 2, \ldots$ From this contradiction, the assertion of the theorem follows.

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SEKTION MATHEMATIK
BERGAKADEMIE FREIBERG
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Received on 4. 6. 1973

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MINIMAKSOWA INSPEKCJA ELEMENTÓW W SYSTEMACH Z ODNOWĄ

STRESZCZENIE

W pracy przedstawiony jest matematyczny model optymalnej inspekcji w systemach z odnową, gdy czas życia elementów nie jest znany. Dowodzi się pewnej specyficznej własności minimaksowej strategii inspekcji, pozwalającej wyznaczyć tę strategię w sposób numeryczny.