ZASTOSOWANIA MATEMATYKI
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J. KLEFFE (Berlin)

A MINIMAX PROPERTY OF TRANSLATION INVARIANT ESTIMATORS FOR VARIANCE COMPONENTS

For random or mixed linear models appearing in analysis of variance (ANOVA) it has been common practice to estimate the variances of the random effects by equating the mean squares in the ANOVA table to their expectations. Estimates obtained in this way are unbiased and have variances independent of the possibly unknown vector of means. This property goes back to the so-called translation invariance of these estimates which are functions of the residuals only. Recently, a great variety of new approaches to the estimation of variance and covariance components has been proposed by Rao [13], [14], Lamotte [10], [11], Kleffe and Pincus [8], [9], not to mention the increasing amount of literature concerned with maximum likelihood approaches. In the field of unbiased estimation it has been shown that optimal non-invariant estimators always depend on the vector of expectations. Therefore, Pincus and the author suggested to apply, in some sense, Bayes estimates using a prior distribution over the unknown fixed effects. The use of invariant estimation functions neglects this kind of prior information and, as will be proved, is a minimax decision. In order to obtain this result we develop explicit formulas for Bayes estimates among all unbiased estimators and consider the limiting behaviour of these estimates if prior information tends to zero. This approach is based on the idea of the ∞-MINQUE concept introduced by Focke and Dewess [2].

1. Normal case. First we consider a normal linear regression model

(1)
$$y = X\beta+\varepsilon,$$

where X is a known $(N\times k)$-matrix and the error vector ε is assumed to have a covariance structure

(2)
$$\mathrm{E}\varepsilon\varepsilon' = \theta_1 V_1+\ldots+\theta_p V_p = V(\theta), \qquad \theta\in\mathscr{T},$$

with known symmetric $(N\times N)$-matrices V_i, $i = 1,\ldots,p$. We are interested in the unknown parameter vector $\theta = (\theta_1,\ldots,\theta_p)'$ which varies in a subset $\mathscr{T}$ of the Euclidean p-space R^p given by all $\theta\in R^p$ such that $V(\theta)$ becomes positive-definite. But the family of normal distributions induced by (1) and (2), i.e.

$$\{P^y_{\beta,\theta} = N(X\beta,\, V(\theta)) \mid \beta \in R^k,\, \theta \in \mathcal{T}\},$$

depends additionally on the unknown vector of means $X\beta$. Therefore, we will use prior information about β. We assume that a distribution P^β of the mean vector $X\beta$ is independent of ε and normal with expectation g and covariance matrix G. Then, for fixed g and G, the family of marginal distributions for y is given by

$$\{P^y_{\theta,g,G} = N(g,\, V(\theta)+G) \mid \theta \in \mathcal{T}\}.$$

A measurable function $\hat{\gamma} = f(y)$ is called *unbiased* for $\gamma = f'\theta$ if

$$\int \hat{\gamma}\, dP^y_{\beta,\theta} = \gamma \quad \text{for all } \beta \in R^k,\, \theta \in \mathcal{T}. \tag{3}$$

Pincus [12], as one of many authors, investigated the linear space Γ of all unbiasedly estimable linear parametric functions $\gamma = f'\theta$. Therefore, we refer to the following lemma without proof.

LEMMA 1. *Let $P = X(X'X)^+X'$ and let Q be a $(p\times p)$-matrix given by the elements $q_{ij} = \operatorname{tr}[V_iV_j - PV_iPV_j]$, $i, j = 1, \ldots, p$. Then*

(i) *Γ is the set of all $\gamma = f'\theta$ with $f \in R(Q)$, i.e. f is a member of the linear space generated by the columns of Q (column space).*

(ii) *$R(Q)$ is invariant with respect to linear regular transformations of model* (1).

By (3), for any unbiased estimation we have

$$\int \operatorname*{var}_{\beta,\theta} \hat{\gamma}\, dP^\beta = \int (\hat{\gamma}-\gamma)^2\, dP^y_{\theta,g,G} = \operatorname*{var}_{\theta,g,G} \hat{\gamma}. \tag{4}$$

$\hat{\gamma}_*$ is said to be a *Bayes unbiased estimator* (BAUE) for γ at $\theta_0 \in \mathcal{T}$ if

$$\operatorname*{var}_{\theta_0,g,G} \hat{\gamma}_* \leqslant \operatorname*{var}_{\theta_0,g,G} \hat{\gamma}$$

for all $\hat{\gamma}$ satisfying (3). It is easily verified that a BAUE at θ_0 coincides with the locally minimum variance unbiased estimator at (θ_0, g) if the considered family of distributions is

$$\{P^y_{\theta,g} = N(g,\, V(\theta)+G) \mid \theta \in \mathcal{T},\ g \in R(X)\},$$

where $R(X)$ denotes the column space of X, and G is fixed but such that $R(X)$ is an invariant subspace with respect to the linear mapping of G, i.e. $G = XKX'$ with an arbitrary non-negative-definite symmetric matrix K.

THEOREM 1. *Assume that, for $\theta_0 \in \mathcal{T}$,*

$$\begin{aligned} V &= V(\theta_0), \quad P_V = X(X'V^{-1}X)^+X'V^{-1}, \\ A_i &= (V+G)^{-1}(V_i - P_VV_iP_V')(V+G)^{-1} \end{aligned} \tag{5}$$

and let λ be a solution of the linear equation system

(6) $$S\lambda = f, \quad S = ((s_{ij})), \quad s_{ij} = \mathrm{tr}[A_i V_j], \quad i, j = 1, \ldots, p.$$

Then

(i) *A BAUE for $\gamma = f'\theta \in \Gamma$ at $\theta_0 \in \mathscr{T}$ is uniquely determined by*

(7) $$\hat{\gamma}_* = \sum_i \lambda_i (y' A_i y - 2y' A_i g).$$

(ii) *The variance* (4) *of $\hat{\gamma}_*$ at θ_0 is $2f' S^- f$ for an arbitrary generalized inverse of S.*

Remark. S is singular if $\dim \Gamma < p$ but, in that case, (7) is independent of a special solution λ of (6).

Proof. First we verify the consistency of (6) for $f \in R(Q)$. For this we transform y into $\tilde{y} = (V+G)^{-1/2} y$ such that $\tilde{y}$ follows the linear model (1), (2) with

$$\tilde{X} = (V+G)^{-1/2} X \quad \text{and} \quad \tilde{V}_i = (V+G)^{-1/2} V_i (V+G)^{-1/2}, \quad i = 1, \ldots, p.$$

By (ii) of Lemma 1, $f \in R(Q)$ implies $f \in R(\tilde{Q})$ and the elements of $\tilde{Q}$ are

$$\tilde{q}_{ij} = \mathrm{tr}[\tilde{V}_i \tilde{V}_j - \tilde{P}\tilde{V}_i \tilde{P}\tilde{V}_j], \quad i, j = 1, \ldots, p,$$

where

$$\tilde{P} = (V+G)^{-1/2} X \big(X'(V+G)^{-1} X\big)^+ X'(V+G)^{-1/2}.$$

Using the representation $G = XKX'$ and $P_V X = X$, it is easy to verify

(8) $$(V+G)^{-1} P_V = P_V'(V+G)^{-1},$$

(9) $$\tilde{P}\tilde{V}_j = (V+G)^{-1/2} P_V V_j (V+G)^{-1/2}.$$

On the other hand, by (8) and (9) we have

$$\mathrm{tr}[A_i V_j] = \mathrm{tr}[\tilde{V}_i \tilde{V}_j - (V+G)^{-1/2} P_V V_i P_V' (V+G)^{-1} V_j (V+G)^{-1/2}] = \tilde{q}_{ij}.$$

Then $S = \tilde{Q}$ which proves the existence of $\lambda \in R^p$ with $S\lambda = f$ for $f \in R(Q)$. Because of $X' A_i X = 0$ we obtain

$$\mathrm{E}\hat{\gamma}_* = \sum_i \lambda_i \mathrm{E} y' A_i y = \sum_i \lambda_i \mathrm{tr}\big[A_i\big(V(\theta) + X\beta\beta' X'\big)\big]$$

$$= \sum_i \lambda_i \mathrm{tr}[A_i V(\theta)] = \sum_{i,j} \lambda_i \mathrm{tr}[A_i V_j]\theta_j = \sum_j f_j \theta_j = \gamma,$$

and $\hat{\gamma}_*$ is unbiased.

It remains to show the optimality of $\hat{\gamma}_*$. By the well-known theorem of Rao (see, e.g., [14]), $\hat{\gamma}_*$ is a BAUE for γ at $\theta_0 \in \mathscr{T}$ iff

$$\operatorname*{cov}_{\theta_0, g, G}\big(f(y), \hat{\gamma}_*\big) = 0$$

for all measurable functions $f(y)$ with expectation constant zero and

$$\operatorname*{var}_{\theta_0,g,G} f(y) < \infty.$$

Since g can be expressed as $g = X\bar{g}$, we have

$$\text{(10)} \qquad \mathrm{E}_{\theta_0,g,G} f(y) \,\alpha \int f(y) \exp\{-\tfrac{1}{2}(y - X\bar{g})'(V(\theta)+G)^{-1}(y - X\bar{g})\} dy = 0$$

for all $\bar{g} \in R^k$, $\theta \in \mathscr{T}$. The left-hand side of (10) is a function of $\bar{g}$ and θ, and its derivatives (if they exist) have to vanish. But, using

$$\operatorname*{var}_{\theta_0,g,G} f(y) < \infty \quad \text{and} \quad \left.\frac{\partial(V(\theta)+G)^{-1}}{\partial \theta_i}\right|_{\theta_0} = -(V+G)^{-1}V_i(V+G)^{-1},$$

it is easy to verify the differentiability and the possibility of interchanging the differentiation and integration in equation (10). Differentiation with respect to θ_i yields

$$\int f(y)(y - X\bar{g})'(V+G)^{-1}V_i(V+G)^{-1}(y - X\bar{g})\, dP^y_{\theta_0,g,G} = 0, \qquad i = 1, \ldots, p,$$

which is equivalent to

$$\text{(11)} \quad \operatorname*{cov}_{\theta_0,g,G}\big(f(y),\, y'(V+G)^{-1}V_i(V+G)^{-1}y - 2g'(V+G)^{-1}V_i(V+G)^{-1}y\big) = 0, \qquad i = 1, \ldots, p.$$

Now, forming the second-order partial derivatives with respect to the components of $\bar{g}$, in the same way we obtain

$$\operatorname*{cov}_{\theta_0,g,G}\big(f(y),\, y'(V+G)^{-1}X_iX_j'(V+G)^{-1}y - 2g(V+G)^{-1}X_iX_j'(V+G)^{-1}y\big) = 0, \qquad i, j = 1, \ldots, k,$$

where X_i denotes the i-th column of X. Since $P_V V_i P_V'$ can be expressed by a linear combination of all matrices X_iX_j', $i, j = 1, \ldots, k$, this equation implies

$$\operatorname*{cov}_{\theta_0,g,G}\big(f(y),\, y'(V+G)^{-1}P_V V_i P_V'(V+G)^{-1}y - {} -2g'(V+G)^{-1}P_V V_i P_V'(V+G)^{-1}y\big) = 0.$$

Subtracting this equation from (11), by (5) we obtain

$$\operatorname*{cov}_{\theta_0,g,G}\big(f(y),\, y'A_iy - 2g'A_iy\big) = 0, \qquad i = 1, \ldots, p.$$

Hence $\hat{\gamma}_*$ is BAUE of its expectation for all $\lambda \in R^p$. The estimator $\hat{\gamma}_*$ is also unique and, therefore, independent of the special solution of (6).

In order to prove statement (ii) we use the fact that BAUE coincides with the *Bayes quadratic unbiased estimator* (BAQUE) developed in [8], where a BAQUE was obtained to be the unique solution of a linear equation.

Therefore, it is known that the p matrices $A_i^* = V^{1/2}A_iV^{1/2}$ with A_i given in (5) satisfy

$$A_i^* + A_i^*\tilde{G} + \tilde{G}A_i^* = \tilde{V}_i - \tilde{P}\tilde{V}_i\tilde{P}, \quad i = 1, \ldots, p,$$

where $\tilde{G} = V^{-1/2}GV^{-1/2}$. Multiplying by $V^{1/2}$ on both sides we get

$$VA_iV + VA_iG + GA_iV = V_i - P_VV_iP_V',$$

and hence, multiplying by A_j and forming traces, we obtain

$$\mathrm{tr}[A_iVA_jV] + 2\,\mathrm{tr}[A_iVA_jG] = \mathrm{tr}[V_iA_j].$$

Using the formula

$$\operatorname*{var}_{\theta_0,g,G}(y'Ay - 2y'AX\beta) = 2\,\mathrm{tr}[AVAV] + 4\,\mathrm{tr}[AVAG]$$

(proved, e.g., in [5]) we get

$$\operatorname*{var}_{\theta_0,g,G}\hat{\gamma}_* = 2\sum_{i,j}\lambda_i\lambda_j(\mathrm{tr}[A_iVA_jV] + 2\,\mathrm{tr}[A_iVA_jG])$$

$$= 2\sum_{i,j}\lambda_i\lambda_j\,\mathrm{tr}[A_iV_j] = 2\lambda'S\lambda.$$

But, as stated earlier, we can choose $\lambda = S^-f$, which implies

$$\operatorname*{var}_{\theta_0,g,G}\hat{\gamma}_* = 2f'S^-f.$$

By Theorem 1, a BAUE belongs to the class of quadratic plus linear estimation functions. Such estimators have extensively been studied in the literature (see, e.g., [3], [5], [8], and [9]) and various properties, like linearity, invariance with respect to linear regular transformations of model (1), as well as necessary and sufficient conditions for the independence of BAUE from g, G or even θ have been derived. Now, we investigate the behaviour of BAUE if the prior distribution for $X\beta$ tends to an improper one reflecting the case of no information about β. Therefore, we consider a sequence of covariance matrices rG, where r tends to infinity.

Definition. An estimation function $\hat{\gamma} = f(y)$ is called *invariant* if $f(y + X\beta) = f(y)$ for all $\beta \in R^k$.

There are some degenerated cases of models (1), (2) for which no invariant unbiased estimator for $\gamma = f'\theta$ exists. Introducing the subspace $\Gamma_I \subseteq \Gamma$ of all invariantly and unbiasedly estimable linear functions we proved in [8] the following

LEMMA 2. *Let $M = I - P$ and let Q_I be the $(p \times p)$-matrix with elements*

$$q_{ij}^I = \mathrm{tr}[MV_iMV_j], \quad i, j = 1, \ldots, p.$$

Then

$$\Gamma_I = \{\gamma \in \Gamma \mid f \in R(Q_I)\}.$$

Now we are able to formulate the main result of this section.

THEOREM 2. *Assume that* $\gamma \in \Gamma_I$ *and* $\hat{\gamma}_{*,r}$ *are BAUE at* $\theta_0 \in \mathcal{T}$ *for* $P^\beta \sim N(g, rG)$ *with* $\operatorname{rank}[X] = \operatorname{rank}[G]$. *Then*

(i) *The limit*

$$\hat{\gamma}_{*,\infty} = \lim_{r\to\infty} \hat{\gamma}_{*,r}$$

exists, is independent of G and g, and coincides with the locally minimum variance unbiased estimator among all invariant estimation functions

$$\hat{\gamma}_{*,\infty} = \sum_i \lambda_i y'(MVM)^+ V_i (MVM)^+ y, \tag{12}$$

where $\lambda = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation*

$$\sum_{i=1}^{p} \lambda_i \operatorname{tr}[(MVM)^+ V_i (MVM)^+ V_j] = f_j, \quad j = 1, \ldots, p. \tag{13}$$

(ii) $\hat{\gamma}_{*,\infty}$ *is a minimax estimation function, i.e. a solution of the equation*

$$\min_{\hat{\gamma}\in U} \sup_{\beta\in R^k} \operatorname{var}_{\beta,\theta_0} \hat{\gamma} = \sup_{\beta\in R^k} \operatorname{var}_{\beta,\theta_0} \hat{\gamma}_{*,\infty}, \tag{14}$$

where U stands for the class of all unbiased estimation functions.

Proof. Theorem 1 shows that $\hat{\gamma}_{*,r}$ depends on r by $(V+rG)^{-1}$ only. Using $G = XKX'$ we get

$$(V+rG)^{-1} = V^{-1} - V^{-1}X\left(X'V^{-1}X + \frac{1}{r}K^{-1}\right)^{-1} X'V^{-1},$$

which tends to

$$V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1} = (MVM)^+$$

as r tends to infinity. Substituting the limit in (5) we obtain (12) because of $(MVM)^+X = 0$. The linear equation system (6) takes the form of (13). But $\hat{\gamma}_{*,\infty}$ given by (12) and (13) was proved in [7] to be the locally minimum variance invariant unbiased estimator which exists for all $\gamma \in \Gamma_I$.

In order to prove (ii) we note that

$$\int \operatorname{var}_{\beta,\theta_0} \gamma_{*,r} \, dP_r^\beta \to \operatorname{var}_{\theta_0} \hat{\gamma}_{*,\infty} \quad \text{as } r \to \infty \tag{15}$$

and the limit is independent of β. Following Theorem A.3.43 in [4], $\hat{\gamma}_{*,\infty}$ is a solution of (14) and minimax.

The assumption that P^β is normal is necessary only for the class of all unbiased estimators. Restricting considerations to quadratic plus linear estimation functions, we see that all results derived remain valid if P^{β} is not normal. As we will see in the next section, similar results are also valid if y does not follow a normal distribution.

2. Non-normal models. The purpose of this section is to consider quadratic estimation functions but without reference to the normal distribution of y. In general, besides more specific cases described in the sequel, the probability measure P^{y} is assumed to be such that there is no quadratic plus linear form $\hat{\gamma} = y'Ay + a'y$ with $\operatorname{var}\hat{\gamma} = 0$, except the case $A = 0$ and $a = 0$. This regularity condition ensures the uniqueness of minimum variance quadratic unbiased estimators and is, e.g., met for probability measures being equivalent to the Lebesgue measure in R^N. Generally, the variance of an arbitrary quadratic form also depends on the moments of order three and four of y. To avoid a lot of complications regarding these moments we therefore restrict our attention to models of type (1), where all relations between error components are produced by the experimental design. In other words, ε is assumed to be $U\varepsilon^*$, where U is known, and ε^* has only independently distributed components. This situation appears for all variance component models with stochastically independent random effects. As in the previous papers (see, e.g., [5], [6], and [9]) we consider the linear model

$$y = X\beta + U\varepsilon^* \tag{16}$$

with known $(N \times k)$-matrix X, known $(N \times s)$-matrix U and unknown $(k \times 1)$-vector β of fixed effects. The $s \times 1$ error vector ε^* has expectation zero and covariance structure

$$\mathrm{E}\,\varepsilon^*\varepsilon^{*\prime} = \theta_1F_1 + \ldots + \theta_pF_p, \qquad \theta \in \mathscr{T},$$

where $F_1, \ldots, F_p$ are known $s \times s$ diagonal matrices and $\theta = (\theta_1, \ldots, \theta_p)'$ varies in an open set $\mathscr{T} \subseteq R^p$ such that $V = UF(\theta)U'$ is positive-definite for all $\theta \in \mathscr{T}$. Since we are only working with moments, it is enough to demand that the components of ε^* behave up to moments of order four like stochastically independent random variables. Then introducing

$$\mathrm{E}\,\varepsilon_i^{*2} = \sigma_i^2, \qquad \mathrm{E}\,\varepsilon_i^{*3} = \mu_i\sigma_i^3, \qquad \mathrm{E}\,\varepsilon_i^{*4} = \gamma_i\sigma_i^4, \qquad i = 1, \ldots, s,$$

we are able to calculate the variances of unbiased quadratic plus linear forms (see [9]). The class of all these estimators will be denoted by Q and $\hat{\gamma}_* \in Q$ is said to be *BAQUE* at $\theta_0 \in \mathscr{T}$ if

$$\int\limits_{\beta,\theta_0} \operatorname{var}\hat{\gamma}_*\,dP^{\beta} \leqslant \int\limits_{\beta,\theta_0} \operatorname{var}\hat{\gamma}\,dP^{\beta} \quad \text{for all } \hat{\gamma} \in Q.$$

To ensure the existence of these integrals we assume P^{β} to have finite moments of order one and two:

$$g = \int X\beta\,dP^{\beta} \quad \text{and} \quad G = \int X\beta\beta'X'\,dP^{\beta} - gg'.$$

The following theorem is a special case of a more general one, proved in [9].

THEOREM 3. *Let, for* $\theta_0 \in \mathcal{T}$, $F = F(\theta_0)$, $V = UFU'$, $V_i = UF_iU'$, $i = 1, \ldots, p$,

$$\Gamma = F^2 \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_s \end{pmatrix} \mu = F^{3/2}(\mu_1, \ldots, \mu_s)'.$$

(i) *BAQUE exists for all* $\gamma \in \Gamma$, *and* $\hat{\gamma}_* = y'A^*y + a^*y$ *is BAQUE for* $\gamma = f'\theta$ *iff*

$$(17)\quad 2(VA^*V - PVA^*VP) + 2(MVA^*G + GA^*VM) + U\Gamma[\operatorname{Diag} U'A^*U]U' - PU\Gamma[\operatorname{Diag} U'A^*U]UP +$$
$$+ 2(U[\operatorname{Diag} U'A^*g\mu']U' - PU[\operatorname{Diag} U'A^*g\mu]U'P) + U[\operatorname{Diag} U'a^*\mu']U' - PU[\operatorname{Diag} U'a^*\mu']U'P \in \operatorname{sp}\{V_i - PV_iP,\ i = 1, \ldots, p\},$$

$$(18)\quad M(Va^* + 2VA^*g + U[\operatorname{Diag} U'A^*U]\mu) = 0,$$

$$(19)\quad \operatorname{tr}[A^*V_i] = f_i, \quad i = 1, \ldots, p,$$

$$(20)\quad X'A^*X,\ a^{*\prime}X = 0.$$

(ii) *BAQUE is unique and invariant with respect to linear regular transformations of model* (16).

The symbol $\operatorname{sp}\{V_i - PV_iP,\ i = 1, \ldots, p\}$ stands for the linear space of the matrices involved, and $[\operatorname{Diag} C]$ is used for the matrix obtained in equating all non-diagonal elements of C to zero.

As in Section 1, we first develop explicit expressions for A^* and a^* similar to (7). Two lemmas are necessary to solve the system of linear equations given in Theorem 3.

LEMMA 3. *For arbitrary symmetric non-negative-definite* G *with* $GP = G$, *the linear equation system*

$$(21)\quad A + AG + GA = Q, \quad X'AX = 0$$

has the unique solution

$$A = (I+G)^{-1}Q(I+G)^{-1}.$$

Proof. $PG = G$ and $X'AX = 0$ imply $GAG = 0$, and the left-hand side of (21) turns out to be $(I+G)A(I+G)$. The proof will be complete if we observe that $I+G$ is regular.

LEMMA 4. *Equations* (17), (18), *and* (20) *have unique solutions* (A_i, a_i) *for each* $V_i - PV_iP$ *on the right-hand side of* (17).

Proof. Let us assume that there exists $V^* \in \operatorname{sp}\{V_1, \ldots, V_p\}$ such that the left-hand side of (17) equals $V^* - PV^*P$ for two different pairs

(A_1^*, a_1^*) and (A_2^*, a_2^*) also satisfying (18) and (20). Then the right-hand side of (17) vanishes for $D = A_1^* - A_2^*$ and $d = a_1^* - a_2^*$ (at least one of D and d is different from zero), and multiplication by D and forming traces yield:

(22) $$2\,\mathrm{tr}[DVDV] + 4\,\mathrm{tr}[DVDG] + \mathrm{tr}\big[[\mathrm{Diag}\, UDU]\Gamma[\mathrm{Diag}\, U'DU]\big] + \\ + 2g'DU[\mathrm{Diag}\, U'DU]\mu + d'U[\mathrm{Diag}\, U'DU]\mu = 0.$$

D and d also solve (18), and multiplication by g' from the right and by D from the left yields, after forming traces,

(23) $$\mathrm{tr}[DVdg'] + 2\,\mathrm{tr}[DVDgg'] + g'DU[\mathrm{Diag}\, U'DU]\mu = 0$$

so that (22) and (23) together imply

(24) $$2\,\mathrm{tr}[DVDV] + 4\,\mathrm{tr}[DVD(G+gg')] + 2\,\mathrm{tr}[DVdg'] + \\ + 4g'DU[\mathrm{Diag}\, U'DU]\mu + d'U[\mathrm{Diag}\, U'DU]\mu = 0.$$

On the other hand, by multiplication of (18) by d' from the left we get

(25) $$d'Vd + 2d'VDg + d'U[\mathrm{Diag}\, U'DU]\mu = 0$$

and the sum of (24) and (25) gives

$$\int \operatorname*{var}_{\beta,\theta_0}(y'Dy + d'y)\, dP^{\beta} = 0$$

according to formula 3.2 in [6]. It follows that

$$\operatorname*{var}_{\beta_0,\theta_0}(y'Dy + d'y) = 0$$

for at least one $\beta_0 \in R^k$, which contradicts our assumption concerning the considered class of distributions for y. In order to verify that (17), (18), and (20) have a solution for each $V_i - PV_iP$ on the right-hand side of (17) we use the fact that

$$\dim \mathrm{sp}\{V_i - PV_iP,\ i = 1, \ldots, p\} = \dim \Gamma$$

has to be equal to the number of linearly independent solutions of (17), (18), and (20).

In the sequel we use a not very familiar calculus and introduce:

(i) $\mathrm{diag}\, A$ to be the vector formed by the diagonal elements of A,

(ii) Hadamard's product $A * B = ((a_{ij}b_{ij}))$ of two matrices $A = ((a_{ij}))$ and $B = ((b_{ij}))$ of the same type.

We have

(26) $$\mathrm{diag}\, ADB' = (A * B)\,\mathrm{diag}\, D$$

for arbitrary matrices A, B and any diagonal matrix D appropriate to A and B.

THEOREM 4. *Let $V = UF(\theta_0)U'$ be regular and put*

$$W = U'(V+G)^{-1}U, \quad \overline{W} = U'(V+G)^{-1}P_V U, \tag{27}$$
$$Z = U'(MVM)^{+}U, \quad R = W*W - \overline{W}*\overline{W}, \quad T = \Gamma - Z*\mu\mu'.$$

Moreover, let Q_i^* be the $s\times s$ diagonal matrix with diagonal elements given by

$$\operatorname{diag} Q_i^* = \left(I+\frac{1}{2}RT\right)^{-1} R\operatorname{diag} F_i, \quad i = 1,\dots,p,$$

and let $\lambda = (\lambda_1,\dots,\lambda_p)'$ be a solution of the linear equation system

$$\sum_{i=1}^{p}\lambda_i \operatorname{tr}[Q_i^* F_j] = 2f_j, \quad j = 1,\dots,p. \tag{28}$$

Then BAQUE for $\gamma = f'\theta$ takes the form

$$\hat{\gamma} = \sum_{i=1}^{p}\lambda_i\left(y'A_i y - 2g'A_i y - \frac{1}{2}\mu' Q_i^* U'(MVM)^{+}y\right), \tag{29}$$

where

$$A_i = \frac{1}{2}(V+G)^{-1}Q_i(V+G)^{-1}, \tag{30}$$

$$Q_i = UD_iU' - P_V UD_iU'P_V', \tag{31}$$

$$D_i = F_i - \frac{1}{2}\Gamma Q_i^* + \frac{1}{2}[\operatorname{Diag} ZQ_i^*\mu\mu']. \tag{32}$$

Proof. Regularity of V implies $R(M) = R(MVM)$. Then, multiplying (18) by $U'(MVM)^{+}$ from the left, we get

$$U'a^* + 2UA^*g = Z[\operatorname{Diag} U'A^*U]\mu.$$

Using this result we can write the last two terms together in (17) in the form

$$U[\operatorname{Diag} Z[\operatorname{Diag} U'A^*U]\mu\mu']U' - PU[\operatorname{Diag} Z[\operatorname{Diag} U'A^*U]\mu\mu']U'P.$$

In this way we get an equation for A^* not including a^*. Solving this equation is the main part of the proof. We use the invariance of BAQUE with respect to regular transformations. Therefore, we first consider a special case $V = I$, where the equation of interest takes the form

$$\begin{aligned}2(A+AG+GA) + U\Gamma[\operatorname{Diag} U'AU]U' - PU\Gamma[\operatorname{Diag} U'AU]U'P -\\ - U[\operatorname{Diag} Z[\operatorname{Diag} U'AU]\mu\mu']U' - PU[\operatorname{Diag} Z[\operatorname{Diag} U'AU]\mu\mu']U'P\\ = \sum_i \lambda_i(V_i - PV_iP)\end{aligned} \tag{33}$$

(the asterisk of A is dropped for simplicity). According to Lemma 4, equation (33) has a unique solution A_i for each $V_i - PV_iP$ on the right-hand side and, furthermore, it can easily be seen from (33) that there exists a diagonal matrix D_i with

$$2(A_i+A_iG+GA_i) = UD_iU'-PUD_iU'P. \tag{34}$$

By (31) (for $V = I$) we obtain (30) with the help of Lemma 3. Further, by (27), (30), and (31) we get

$$U'A_iU = \frac{1}{2}(WD_iW-\overline{W}D_i\overline{W}). \tag{35}$$

Now put $Q_i^* = [\mathrm{Diag}\, U'A_iU]$. Applying (30) and (31) to (33) we obtain

$$Q_i^*+\frac{1}{2}U\Gamma Q_i^*U'-\frac{1}{2}PU\Gamma Q_i^*U'P-\frac{1}{2}U[\mathrm{Diag} ZQ_i^*\mu\mu']U'+ \\ +\frac{1}{2}PU[\mathrm{Diag} ZQ_i^*\mu\mu']U'P = V_i-PV_iP. \tag{36}$$

Multiplication by $U'(I+G)^{-1}$ and by its transposed matrix on both sides yields

$$Q_i^*+\frac{1}{2}[\mathrm{Diag}\, W\Gamma Q_iW]-\frac{1}{2}[\mathrm{Diag}\,\overline{W}\Gamma Q_i^*\overline{W}]- \\ -\frac{1}{2}[\mathrm{Diag}\, W[\mathrm{Diag}\, ZQ_i^*\mu\mu']W]+\frac{1}{2}[\mathrm{Diag}\,\overline{W}[\mathrm{Diag} ZQ_i^*\mu\mu']\overline{W}] \\ = [\mathrm{Diag} WF_iW-\overline{W}F_i\overline{W}] \tag{37}$$

if we only compare the diagonal elements on both sides. Applying the operator "diag" to this equation we obtain a usual linear equation system for the vector $\mathrm{diag}Q_i^*$ in the form

$$\left\{I+\frac{1}{2}(W*W-\overline{W}*\overline{W})(\Gamma-Z*\mu\mu')\right\}\mathrm{diag}Q_i^* = (W*W-\overline{W}*\overline{W})\,\mathrm{diag}F_i \tag{38}$$

or, in terms used in Theorem 4 (for $V = I$),

$$\left(I+\frac{1}{2}RT\right)\mathrm{diag}Q_i^* = R\,\mathrm{diag}F_i. \tag{39}$$

Now, starting from an arbitrary solution $\mathrm{diag}Q_i^*$ of (39), by (31), (34), and (36) we can choose D_i such as given by (32). However, it is necessary to show that A_i given by (30), (31), and (32) really solves (33). In order to do this, for arbitrary diagQ_i^* obtained from (39), making use of (37) and (35), we derive

$$Q_i^* = [\mathrm{Diag}\, WD_iW - \overline{W}D_i\overline{W}] = 2\,[\mathrm{Diag}\, U'A_iU]$$

for D_i given by (32).

Now, we obtain the desired result by substituting this expression into (33). But (33) is known to have only one solution and A_i is independent of the choice of Q_i^* if there are different ones. Let us for a moment assume (39) to have different solutions Q_{i1}^* and Q_{i2}^*. It follows from

$$Q_i^* = [\mathrm{Diag}\, U'A_iU] = \frac{1}{2}[\mathrm{Diag}\, U'(I+G)^{-1}Q_i(I+G)^{-1}\,U]$$

that Q_{i1} differs from Q_{i2} and, by (30), also A_{i1} differs from A_{i2}. This contradicts Lemma 4. Thus Q_i^* is unique and $I+\frac{1}{2}RT$ is regular.

The linear equation system (28) follows from (19) by

$$\mathrm{tr}[A_iV_j] = \mathrm{tr}[F_j[\mathrm{Diag}\, U'A_iU]] = \frac{1}{2}\mathrm{tr}[Q_i^*F_j].$$

In the general case $V \neq I$ we apply the regular transformation $V^{-1/2}$ to model (16) and obtain $\tilde{V} = I$. Therefore, in all formulas appearing in Theorem 4 we have to substitute V by $\tilde{V} = I$, U by $\tilde{U} = V^{-1/2}U$, G by $\tilde{G}$, P by $\tilde{P} = V^{-1/2}X(X'V^{-1}X)^-X'V^{-1/2}$, and M by $I-\tilde{P}$. Routine computations lead to the general expressions of Theorem 4 if we multiply the obtained matrices $\tilde{A}_i$ by $V^{-1/2}$ on both sides.

It remains to develop the linear term in (29). However, for known $A^* = A_i$ equation (18) is easily solved by

$$a_i = -2A_ig-(MVM)^+U[\mathrm{Diag}\, U'A_iU]\mu = -2A_ig-\frac{1}{2}(MVM)^+UQ_i^*\mu.$$

Since V is regular, this solution is unique.

Theorem 5. *Assume that* $\gamma \in \Gamma_I$ *and* $\hat{\gamma}_{*,r}$ *are BAQUE at* $\theta_0 \in \mathcal{T}$ *for* $P^\beta \sim N(g, rG)$ *with* $\mathrm{E}X\beta = g$ *and* $\mathrm{cov}\, X\beta = rG$, $\mathrm{rank}[G] = \mathrm{rank}[X]$. *Then*

(i) *The limit*

$$\hat{\gamma}_{*,\infty} = \lim_{r\to\infty}\gamma_{*,r}$$

exists, is independent of G, *and coincides with the locally minimum variance invariant quadratic unbiased estimator at* $\theta_0 \in \mathcal{T}$ *given by*

$$(40)\qquad \hat{\gamma}_{*,\infty} = \sum_i \lambda_i\Big(y'(MVM)^+UD_iU'(MVM)^+y - \frac{1}{2}\mu'Q_i^*\,U'(MVM)^+y\Big),$$

where

$$V = V(\theta_0), \quad D_i = F_i - \frac{1}{2}\Gamma Q_i^* + \frac{1}{2}[\mathrm{Diag} Z Q_i^* \mu\mu'],$$

Q_i^* is a diagonal matrix with

$$\mathrm{diag} Q_i^* = [I + (Z*Z)(\Gamma - Z*\mu\mu')]^{-1}(Z*Z)\,\mathrm{diag} F_i,$$

and $\lambda = (\lambda_1, \ldots, \lambda_p)'$ is a solution of the linear equation system

$$\sum_i \lambda_i \operatorname{tr}[Q_i^* F_j] = f_j, \quad j = 1, \ldots, p.$$

(ii) *$\hat{\gamma}_{*,\infty}$ is a minimax estimation function, i.e. a solution of the equation*

$$\min_{\hat{\gamma} \in Q} \sup_{\beta \in R^k} \operatorname{var}_{\beta,\theta_0} \hat{\gamma} = \sup_{\beta \in R^k} \operatorname{var}_{\beta,\theta_0} \hat{\gamma}_{*,\infty}.$$

The proof follows along the same lines as that of Theorem 2 considering the limits

$$W_r = U'(V + rG)^{-1} U \to Z \quad \text{and} \quad \overline{W}_r = U'(V + rG)^{-1} P_V U \to 0.$$

Expression (40) is the formal limit of (29) as r tends to infinity and can be proved to exist and to be the locally minimum variance invariant quadratic unbiased estimator in starting from Theorem 3.7 in [6] and olving the corresponding equations as in the proof of Theorem 4.

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ZENTRALINSTITUT F. MATHEMATIK UND MECHANIK
AKADEMIE DER WISSENSCHAFTEN DER DDR
108 BERLIN, DDR

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J. KLEFFE (Berlin)

PEWNA WŁASNOŚĆ MINIMAKSOWA ESTYMATORÓW KOMPONENTÓW WARIANCYJNYCH NIEZMIENNICZYCH WZGLĘDEM TRANSLACJI

STRESZCZENIE

Rozważa się model $y = x\beta + \varepsilon$, $\mathrm{E}\varepsilon = 0$, $\mathrm{E}\varepsilon\varepsilon' = \theta_1 V_1 + \ldots + \theta_p V_p$ i podaje jawne wzory na estymatory $(\theta_1, \ldots, \theta_p)$ należące do pewnej klasy estymatorów bayesowskich. Klasa funkcji jest ograniczona do klasy estymatorów nieobciążonych, gdy y ma rozkład normalny, i do klasy nieobciążonych estymatorów kwadratowych — w pozostałych przypadkach. Informacja aprioryczna o niewiadomej β jest wykorzystana przy konstrukcji optymalnych funkcji estymujących. Z wyprowadzonych wzorów wynika, że nieobciążone estymatory niezmiennicze względem translacji są funkcjami estymacji minimaksowej, gdy nie ma informacji o β.
