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QUEUEING SYSTEM WITH MERGING OF TWO INPUT STREAMS

1. Introduction. While analyzing various queueing systems one assumes very often that the input stream is Poissonian or, more generally, that it is a renewal stream. There are, however, cases where this assumption is not satisfied. For instance, it can happen that, besides of the renewal input stream, additional customers arrive to the system in a random, unexpected manner. This may be so, e.g., in a service station model in which machines arrive periodically for regular service and where also additional machines arrive for repair because of their breakdown. A similar input stream was considered in [1] while analyzing the arrivals of customers for the hospital admission.

2. Remarks on the system $(E_L, M)/M/n$ without waiting. Young considered in [5] a system with superposition of two input streams. It was the system $(E_L, M)/M/n$ without waiting, i.e. a system in which it was assumed that the input was a superposition of an L-th Erlangian and an independent Poisson streams, the service time was exponential, and there were n parallel service channels.

Knowing that the interarrival time of customers in the Erlangian stream can be divided in L consecutive phases with exponential duration, the state of the system can be analyzed by the process $Z(t) = [W(t), n(t)]$, where $W(t)$ is the phase number of the Erlang stream and $n(t)$ is the number of occupied service channels.

Young found, for the steady-state probabilities P_{wn} of the process $Z(t)$, a homogeneous system of linear equations which was solved with the additional condition

$$P_{wk} = P_{w+1,k} \quad (w = 1, 2, \ldots, L-1;\ k = 0, 1, \ldots, n). \tag{1}$$

In this way one obtains the solutions

$$P'_k = \sum_{w=1}^{L} P_{wk} = \frac{R^k}{k!} \Big/ \sum_{j=0}^{n} \frac{R^j}{j!} \quad (k = 0, 1, \ldots, n), \tag{2}$$

where $R = (\lambda_1+\lambda_2)/\mu$ and λ_1, λ_2 are the intensities of the Erlang and Poisson streams, respectively, and μ is the service intensity.

However, condition (1) implies only zero solutions (see [4]), it is thus in contradiction with the normalizing condition $\sum_{k=0}^{n} P_k = 1$.

As an example consider the system $(E_2, M)/M/1$. The steady-state probabilities of the system satisfy the following system of equations:

$$\begin{aligned} -(2\lambda_1+\lambda_2)P_{10}+2\lambda_1 P_{20}+\mu P_{11} &= 0,\\ -(2\lambda_1+\lambda_2)P_{20}+\mu P_{21} &= 0,\\ \lambda_2 P_{10}-(2\lambda_1+\mu)P_{11}+2\lambda_1 P_{21} &= 0,\\ 2\lambda_1 P_{10}+\lambda_2 P_{20}+2\lambda_1 P_{11}-(2\lambda_1+\mu)P_{21} &= 0. \end{aligned}$$

The solution of this system under the condition $\sum_{k=0}^{n} P_k = 1$ is equal to

$$P_{10} = \frac{\mu(4\lambda_1+\lambda_2+\mu)}{2(2\lambda_1+\lambda_2+\mu)^2}, \quad P_{20} = \frac{\mu}{2(2\lambda_1+\lambda_2+\mu)},$$

$$P_{11} = \frac{(2\lambda_1+\lambda_2)^2+\lambda_2\mu}{2(2\lambda_1+\lambda_2+\mu)^2}, \quad P_{21} = \frac{2\lambda_1+\lambda_2}{2(2\lambda_1+\lambda_2+\mu)}.$$

Whence we get

$$(3)\qquad \begin{aligned} P_0 &= P_{10}+P_{20} = \frac{\mu(3\lambda_1+\lambda_2+\mu)}{(2\lambda_1+\lambda_2+\mu)^2},\\ P_1 &= P_{11}+P_{21} = \frac{(2\lambda_1+\lambda_2)^2+\mu(\lambda_1+\lambda_2)}{(2\lambda_1+\lambda_2+\mu)^2}. \end{aligned}$$

Thus, the obtained solutions P_0 and P_1 are not of form (2).

3. Analysis of the system $(GI, M)/M/n$ by an extended Markov process. Let us replace the Erlang stream in Young's system by a renewal stream with arbitrary interarrival distribution $G(x)$. In this way we obtain the queueing system $(GI, M)/M/n$.

Introduce the following notation:

λ — input intensity in the Poisson stream;

a — mean number of arrivals in the GI-stream, i.e. $1/a = \int_0^\infty x dG(x)$;

μ — service intensity;

$n(t)$ — number of occupied channels at the moment t, i.e. the state of the system at the moment t;

$X(t)$ — time interval from the moment t to the arrival of the next customer in the GI-stream.

Also, let

$$P_k(t, x) = \Pr\{n(t) = k,\ X(t) < x\} \quad (k = 0, 1, \ldots, n)$$

and

$$P_k(x) = \lim_{t\to\infty} P_k(t, x) \quad (k = 0, 1, \ldots, n).$$

Assume that the considered system is steady-state, i.e. that, independently of the initial conditions, a limit state distribution of the states settles down as sufficient time has elapsed:

$$\lim_{t\to\infty} \Pr\{n(t) = k\} = p_k \quad (k = 0, 1, \ldots, n).$$

Of course, $p_k = P_k(\infty)$.

The steady-state probabilities p_k are found in the following way: first, the dependence between the probabilities p_k and the probabilities of the states of the imbedded chain constructed on the arrival moments of the GI-stream is found, and then these probabilities are expressed explicitly. In the proofs of Theorems 1 and 2 of this paper, the method from [2] is used.

Now, let us define the two-dimensional process $Y(t)$ whose the first component is discrete, and the second one is continuous and linear interval-wise. Thus consider the process $Y(t) = [n(t), X(t)]$, where $n(t)$ is the number of customers in the system at the moment t and $X(t)$ is the time from the moment t to the moment of arrival of the next customer in the GI-stream. This process is a Markov one.

THEOREM 1. *The steady-state probabilities $P_k(x)$ satisfy the following system of differential equations*:

$$\text{(4)}\quad \begin{aligned} &P_0'(x) - \lambda P_0(x) - P_0'(0) + \mu P_1(x) = 0,\\ &P_k'(x) - (\lambda + k\mu)P_k(x) - P_k'(0) + P_{k-1}'(0)G(x) + \lambda P_{k-1}(x) +\\ &\qquad\qquad + (k+1)\mu P_{k+1}(x) = 0 \quad (k = 1, 2, \ldots, n-1),\\ &P_n'(x) - n\mu P_n(x) - P_n'(0) + P_{n-1}'(0)G(x) + \lambda P_{n-1}(x) + P_n'(0)G(x) = 0. \end{aligned}$$

Proof. Let $x > 0$. Consider the event $\{n(t+h) = k, X(t+h) < x\}$ for $k = 1, 2, \ldots, n-1$ and $h > 0$. This event occurs as one of the following mutually exclusive events:

1° In the moment t, the system is in the state k and no arrival occurs and no service is finished up to the moment $t+h$; the probability of this event equals

$$\begin{aligned}\Pr\{n(t) = k, h < X(t) < x+h\}\,(1-\lambda h)\,(1-k\mu h)\\ = [P_k(x+h) - P_k(h)]\,(1-\lambda h - k\mu h) + o(h).\end{aligned}$$

2° In the moment t, the system is in the state $k-1$, and in the time interval from t to $t+h$ only one arrival in the GI-stream occurs, the time elapsing to the next arrival in the GI-stream does not exceed $\theta h+x$ $(0<\theta<1)$, and no service is ended; the probability of this event equals

$$\Pr\{n(t)=k-1,\,X(t)<h\}G(\theta h+x)(1-\lambda h)\big(1-(k-1)\mu h\big)$$
$$=P_{k-1}(h)G(\theta h+x)+o(h).$$

3° In the moment t, the system is in the state $k-1$, and in the time interval from t to $t+h$ only one arrival in the Poisson stream occurs and no service is ended; the probability of this event equals

$$\Pr\{n(t)=k-1,\,h<X(t)<x+h\}\lambda h\big(1-(k-1)\mu h\big)$$
$$=[P_{k-1}(x+h)-P_{k-1}(h)]\lambda h+o(h).$$

4° In the moment t, the system is in the state $k+1$, and in the time interval from t to $t+h$ one service is finished and no arrival occurs; the probability of this event equals

$$\Pr\{n(t)=k+1,\,h<X(t)<x+h\}\,(k+1)\mu h(1-\lambda h)$$
$$=[P_{k+1}(x+h)-P_{k+1}(h)]\,(k+1)\mu h+o(h).$$

5° All events different from 1°-4°; the probability of their occurrence is of order $o(h)$.

Since these events are exclusive, the following equality holds:

$$P_k(x)=[P_k(x+h)-P_k(h)]\,(1-\lambda h-k\mu h)+P_{k-1}(h)G(\theta h+x)+$$
$$+[P_{k-1}(x+h)-P_{k-1}(h)]\lambda h+[P_{k+1}(x+h)-P_{k+1}(h)]\,(k+1)\mu h+o(h).$$

Hence, division by h and taking the limit for $h\to 0$ leads to

$$P_k'(x)-P_k'(0)-(\lambda+k\mu)P_k(x)+(k+1)\mu P_{k+1}(x)+G(x)P_{k-1}'(0)+$$
$$+\lambda P_{k-1}(x)=0.$$

For $k=0$, the event $\{n(t+h)=0,\ X(t+h)<x\}$ can occur only as a realization of events 1° and 4° (with $k=0$); thus, as previously, we have

$$P_0'(x)-\lambda P_0(x)-P_0'(0)+\mu P_1(x)=0.$$

For $k=n$, we have

$$\Pr\{n(t+h)=n,\,X(t+h)<x\}=\Pr\{n(t)=n,\,h<X(t)<x+h\}(1-n\mu h)+$$
$$+\Pr\{n(t)=n,\,X(t)<h\}G(\theta h+x)+\Pr\{n(t)=n-1,\ X(t)<h\}G(\theta h+x)+$$
$$+\Pr\{n(t)=n-1,\,h<X(t)<x+h\}\lambda h+o(h).$$

Hence, as before, we obtain

$$P_n'(x)-n\mu P_n(x)-P_n'(0)+P_{n-1}'(0)G(x)+\lambda P_{n-1}(x)+P_n'(0)G(x) = 0,$$

which completes the proof of Theorem 1.

4. Analysis of the imbedded Markov chain. Let $\{S_r\}$ be the arrival moments in the GI-stream. Consider now the sequence of random variables $\{n_r\}$ given by the formula $n_r = n(S_r-0)$. This sequence forms a Markov chain, since $n_{r+1} = n_r+1+\eta_r$, where η_r is an integer-valued random variable connected with the number of arrivals in the Poisson stream in the interval (S_r, S_{r+1}) and with the number of finished services in the same interval. The random variable depends only upon the length of the interval (S_r, S_{r+1}) and upon n_r, which proves that the chain $\{n_r\}$ is Markovian. The probabilities of the states of the chain are denoted by

$$p_k^*(r) = \Pr\{n_r = k\}, \quad p_k^* = \lim_{r\to\infty} p_k^*(r) \quad (k = 0, 1, \ldots, n).$$

The existence of the limits $\lim\limits_{r\to\infty} p_k^*(r)$ follows from the irreducibility and non-periodicity of the chain $\{n_r\}$, and also from the fact that the number of its states is finite.

The following theorem states the relation between probabilities p_k and p_k^*:

THEOREM 2. *The limit probabilities p_k are related with the limit probabilities p_k^* of the imbedded chain as*

$$(5)\quad \begin{aligned} p_k &= \frac{a}{\mu}\sum_{i=0}^{k-1}\frac{(k-i-1)!}{k!}q^i p_{k-i-1}^* + \frac{q^k}{k!}p_0 \quad (k = 1, 2, \ldots, n), \\ p_0 &= \left(1-\frac{a}{\mu}\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-1}\frac{k!}{(k+i+1)!}q^i p_k^*\right)\Bigg/\sum_{k=0}^{n}\frac{q^k}{k!} \quad (k = 1, 2, \ldots, n), \end{aligned}$$

where $q = \lambda/\mu$ and $1/a = \int\limits_0^\infty x dG(x)$.

Proof. Consider the system of equations (4). From the assumption of the existence of $\lim\limits_{x\to\infty} P_k(x) = p_k$ we have $\lim\limits_{x\to\infty} P_k'(x) = 0$, and also $\lim\limits_{x\to\infty} G(x) = 1$. Using these relations, we obtain, after taking the limit for $x\to\infty$, the system of equations

$$(6)\quad \begin{aligned} -\lambda p_0+\mu p_1-P_0'(0) &= 0, \\ -(\lambda+k\mu)p_k+\lambda p_{k-1}+(k+1)\mu p_{k+1}-P_k'(0)+P_{k-1}'(0) &= 0 \\ (k = 1, 2, \ldots, n-1)&, \\ -n\mu p_n+\lambda p_{n-1}+P_{n-1}'(0) &= 0. \end{aligned}$$

Now we find the distribution of $X(t)$. Evidently,

$$\Pr\{X(t) < x\} = P(x) = \sum_{k=0}^{n} P_k(x).$$

Sidewise summation of all equations in (4) yields

$$\sum_{k=0}^{n} P_k'(x) - \sum_{k=0}^{n} P_k'(0) + \sum_{k=0}^{n} P_k'(0)G(x) = 0.$$

Hence

$$P'(x) = P'(0)\big(1-G(x)\big).$$

Integrating both sides of this equation from 0 to x, we obtain

$$P(x) = P'(0)\int_0^x \big(1-G(u)\big)du.$$

Since $P(\infty) = 1$, we have $P'(0) = a$, and hence

$$P(x) = a\int_0^x \big(1-G(u)\big)du.$$

Thus

$$P_k'(0) = \lim_{x\to 0}\frac{P_k(x)}{x} = \lim_{x\to 0}\frac{1}{x}\Pr\{n(t) = k, X(t) < x\}$$
$$= \lim_{x\to 0}\Pr\{n(t) = k \mid X(t) < x\}\cdot\frac{\Pr\{X(t) < x\}}{x} = p_k^* P'(0) = ap_k^*.$$

Adding now to every equation of (6) all preceding ones, we obtain the recurrence relation

$$p_{k+1} = \frac{\lambda}{\mu(k+1)}p_k + \frac{ap_k^*}{\mu(k+1)} \qquad (k = 0, 1, \ldots, n-1)$$

from which the first formula of (5) follows. From the normalizing condition $\sum_{i=1}^{n} p_i = 1$ we have the formula for p_0. This completes the proof of Theorem 2.

Let $p_{ij} = \Pr\{n_{r+1} = j \mid n_r = i\} = \Pr\{n(S_{r+1}-0) = j \mid n(S_r-0) = i\}$ be the transition probabilities in the chain $\{n_r\}$. We find now their analytical form. Notice that in the interval (S_r, S_{r+1}) a state change occurs only at the arrival moments of the Poisson stream and at the moments of finishing the exponential services. Hence, the process $n(t)$ behaves in the interval (S_r, S_{r+1}), similarly as in the system $M/M/n$ without waiting.

Let $m(t)$ be the number of customers in the system $M/M/n$ without waiting at the moment t. From the previously said it follows that the distribution of n_{r+1} under the conditions $n(S_r-0)=i$ and $S_{r+1}-S_r=t$ is identical with the distribution of $m(S_r+t)$ under the condition

$$m(S_r+0)=\begin{cases} i+1 & \text{if } i\leqslant n-1,\\ i & \text{if } i=n.\end{cases}$$

Since $\{S_{r+1}-S_r\}$ have the distribution $G(t)$, it follows that

$$(7)\qquad \begin{aligned} p_{ij} &= \int_0^\infty P_{i+1,j}(t)\,dG(t) \qquad (i=0,1,\ldots,n-1),\\ p_{nj} &= \int_0^\infty P_{nj}(t)\,dG(t), \end{aligned}$$

where $P_{kj}(t)$ are the transition probabilities in the system $M/M/n$ without waiting. To determine the probabilities p_{ij} in full, one has to know the analytic form of the probabilities $P_{kj}(t)$ in the system $M/M/n$. They are equal to (see [3])

$$P_{ij}(t) = \frac{q^j/j!}{\sum\limits_{k=0}^{n}(q^k/k!)} + q^{n-i}\frac{n!}{j!}\sum_{k=1}^{n}\frac{D_i(r_k,q)D_j(r_k,q)}{r_kD_n(r_k,q)D_n'(r_k+1,q)}\exp[r_k\mu t],$$

where $q=\lambda/\mu$ and

$$D_k(s,q)=\sum_{i=0}^{k}\binom{k}{i}q^{k-1}s(s+1)\ldots(s+i-1),$$

and r_k are the roots of the polynomial $D_n(s+1,q)$.

Coming back to the transition probabilities (7) in the chain $\{n_r\}$, one obtains

$$\begin{aligned} p_{ij} &= \int_0^\infty P_{i+1,j}(t)\,dG(t)\\ &= \frac{q^j/j!}{\sum\limits_{k=0}^{n}(q^k/k!)} + q^{n-i-1}\frac{n!}{j!}\sum_{k=1}^{n}\frac{D_{i+1}(r_k,q)D_j(r_k,q)}{r_kD_n(r_k,q)D_n'(r_k+1,q)}q^*(-r_k\mu)\\ &\qquad\qquad (j=0,1,\ldots,n;\ i=0,1,\ldots,n-1),\\ p_{nj} &= \int_0^\infty P_{nj}(t)\,dG(t) = p_{n-1,j} \qquad (j=0,1,\ldots,n), \end{aligned}$$

where $g^*(s)$ is the Laplace-Stieltjes transform of $G(t)$,

$$g^*(s) = \int_0^\infty e^{-st} dG(t).$$

The steady-state distribution $\{p_k^*\}$ for the chain $\{n_r\}$ is found from the finite linear equation system

$$(8) \qquad \begin{aligned} p_k^* &= \sum_{i=0}^{n} p_i^* p_{ik} \quad (k = 0, 1, \ldots, n), \\ \sum_{k=0}^{n} p_k^* &= 1. \end{aligned}$$

However, the matrix of this system is so complicated that an explicit form of the solution cannot be found.

5. Example. Consider the system $(GI, M)/M/1$. From (8) and (5) it follows

$$(9) \qquad \begin{aligned} p_0^* &= p_{10} = \frac{1}{1+q} - \frac{g^*(\lambda+\mu)}{1+q}, \quad & p_1^* &= p_{11} = \frac{q}{1+q} + \frac{g^*(\lambda+\mu)}{1+q}, \\ p_0 &= \frac{1}{1+q} - \frac{a\big(1-g^*(\lambda+\mu)\big)}{1+q}, \quad & p_1 &= \frac{q}{1+q} + \frac{a\big(1-g^*(\lambda+\mu)\big)}{1+q}. \end{aligned}$$

Formulae (9) are a generalization of formulae (3).

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SYSTEM OBSŁUGI MASOWEJ Z SUPERPOZYCJĄ DWÓCH STRUMIENI ZGŁOSZEŃ

STRESZCZENIE

Przedmiotem pracy jest analiza systemu $(GI, M)/M/n$ bez oczekiwania, tzn. systemu, o którym zakładamy, że strumień zgłoszeń jest superpozycją dwóch strumieni — poissonowskiego M i odnowy GI; czas obsługi ma rozkład wykładniczy, n równoległych kanałów obsługi.

Analiza tego systemu oparta jest na włożonym łańcuchu Markowa, określonym na momentach zgłoszeń jednostek ze strumienia GI, oraz na rozbudowanym procesie Markowa $Y(t) = [X(t), n(t)]$, gdzie $n(t)$ jest liczbą jednostek w systemie w chwili t, a $X(t)$ — czasem od chwili t do chwili zgłoszenia jednostki ze strumienia GI.

W pracy rozpatrujemy prawdopodobieństwa stanów procesu $n(t)$ w warunkach stacjonarnych. Najpierw znajdujemy związki rozpatrywanych prawdopodobieństw w czasie ciągłym z prawdopodobieństwami włożonego łańcucha. Te ostatnie prawdopodobieństwa określone są przez liniowy układ równań o znanej, ale skomplikowanej analitycznie macierzy.