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## MINIMAX CONTROL OF A STOCHASTIC SYSTEM WITH DISTURBANCES BELONGING TO THE EXPONENTIAL FAMILY

1. In the paper a stochastic system defined by the equation (1) is considered, where  $v_n$  is a disturbance of the system at time  $n$ . It is supposed that  $v_n$  are independent and belong to the exponential family and that the distribution of  $v_n$  depends on the unknown parameter  $\lambda$ . Let  $\pi$  be a distribution of  $\lambda$ . In the paper the exact analytical form of a minimax control policy of the system is given under the supposition that  $\pi$  satisfies the condition  $E_\pi(\lambda^2) = m_2$ . Some examples are presented.

The problem of determining of a Bayes control of stochastic system for disturbances belonging to the exponential family was considered in [7] and [9]. There are many papers concerning problems of filtration and control for disturbances distributed normally [2]. Minimax problems of filtration and control are not met often in the literature since determination of a minimax strategy is often complicated [1], [5], [8]. In Sections 2 and 4 problems of determination of Bayes and minimax control are formulated and in Sections 3, 5, 6 their solutions are given, respectively.

2. Let us define the discrete linear system with additive disturbances and exact observations

$$(1) \quad x_{n+1} = ax_n + u_n + cv_n, \quad x_0 = e,$$

where  $x_n$  is the state variable,  $u_n$  is the control, and  $v_0, v_1, v_2, \dots$  are independent random variables with the same distribution dependent on an unknown parameter.

In the next section the following problem is considered: Given the initial state  $e$  and the distribution  $\pi$  of the parameter choose a control  $u_n$ ,  $n = 0, 1, \dots, N$  based on all available data

$$X_n = (x_0, x_1, \dots, x_n) \quad \text{and} \quad U_{n-1} = (u_0, u_1, \dots, u_{n-1}),$$

such that

$$\kappa_n = E \left[ \sum_{i=n}^N (s_i x_i^2 + k_i u_i^2) \mid X_n, U_{n-1} \right] \quad (s_i > 0, k_i > 0)$$

reaches its minimum. If this is done we call  $u_n$  a Bayes control for  $\pi$ .

3. Suppose that the random variables  $v_n$  have the density  $p(v, \lambda)$  with respect to a  $\sigma$ -finite measure  $\mu$  on  $R$ , where  $\lambda$  is a parameter. We assume that  $\mu$  is the Lebesgue measure or the counting measure. We suppose that the random variables  $v_n$  belong to the exponential family, i.e.

$$(2) \quad p(v, \lambda) = S(v) \exp [qA(\lambda) + vB(\lambda)],$$

where the parametrization is chosen to have

$$E_\lambda(v_i) = -q \frac{A'(\lambda)}{B'(\lambda)} = q\lambda$$

and it is assumed that

$$E_\lambda(v_i^2) = q_1 \lambda^2 + q_2 \lambda + q_3$$

(see [4], [6]).

We suppose that the parameter  $\lambda$  has the a priori distribution  $\pi$  with the density

$$(3) \quad g(\lambda; \beta, r) = C(\beta, r) B'(\lambda) \exp [\beta A(\lambda) + rB(\lambda)],$$

$\lambda \in \Lambda \in B^{-1}(\Lambda_0)$  where  $\Lambda_0$  is the natural parameter space. When such an a priori density is assigned to  $\lambda$  then the object of filtering, to determine the Bayes control, is to produce the a posteriori density for  $\lambda$  after any new observation of  $x$ . We change the control after obtaining the new data.

Having observed  $x_1$ , the a posteriori density  $f(\lambda | X_1)$  of the parameter  $\lambda$  may be calculated according to the Bayes rule

$$\begin{aligned} f(\lambda | X_1) &\stackrel{\text{df}}{=} f(\lambda | v_0 = v) = \frac{p(v, \lambda) g(\lambda; \beta, r)}{\int_{\Lambda} p(v, \lambda) g(\lambda; \beta, r) d\lambda} \\ &= \frac{B'(\lambda) \exp [(\beta + q) A(\lambda) + (r + v) B(\lambda)]}{\int_{\Lambda} B'(\lambda) \exp [(\beta + q) A(\lambda) + (r + v) B(\lambda)] d\lambda} \\ &= C(\beta + q, r + v) B'(\lambda) \exp [(\beta + q) A(\lambda) + (r + v) B(\lambda)] \\ &= g(\lambda; \beta_1, r_1), \end{aligned}$$

where

$$\beta_1 = \beta + q, \quad r_1 = r + v_0.$$

This means that the a posteriori density of parameter  $\lambda$  is of the same form as the a priori density and only new parameters  $\beta_1, r_1$  should be computed.

Similarly, after  $x_n$  is measured, the a posteriori density of  $\lambda$  is

$$\begin{aligned} f(\lambda|X_n) &= \frac{p(v_{n-1}, \lambda)g(\lambda; \beta_{n-1}, r_{n-1})}{\int_{\Lambda} p(v_{n-1}, \lambda)g(\lambda; \beta_{n-1}, r_{n-1})d\lambda} \\ &= C(\beta_{n-1} + q, r_{n-1} + v_{n-1})B'(\lambda) \exp [(\beta_{n-1} + q)A(\lambda) + \\ &\quad + (r_{n-1} + v_{n-1})B(\lambda)] \\ &= g(\lambda; \beta_n, r_n), \end{aligned}$$

where

$$(4) \quad \beta_n = \beta_{n-1} + q, \quad r_n = r_{n-1} + v_{n-1}.$$

Given  $X_n$ , the conditional density of random variable  $v_n$  is

$$\begin{aligned} (5) \quad h(v|X_n) &= \int_{\Lambda} p(v, \lambda)g(\lambda; \beta_n, r_n)d\lambda \\ &= \frac{p(v, \lambda)g(\lambda; \beta_n, r_n)}{g(\lambda; \beta_{n+1}, r_{n+1})} = \frac{S(v)C(\beta_n, r_n)}{C(\beta_n + q, r_n + v)}. \end{aligned}$$

Suppose that the moments of random variables  $v_n$  for given  $X_n$  are of the form

$$(6) \quad E(v_n|X_n) = Q^{(n)}r_n,$$

$$(7) \quad E(v_n^2|X_n) = Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)}$$

for some constants  $Q^{(n)}, Q_1^{(n)}, Q_2^{(n)}, Q_3^{(n)}$  dependent on the parameter  $\beta$ .

Formulae (6) and (7) may be used to obtain

$$E(x_{n+1}|X_n) = E(ax_n + u_n + cv_n|X_n) = ax_n + u_n + cQ^{(n)}r_n,$$

$$E(r_{n+1}|X_n) = E(r_n + v_n|X_n) = r_n + Q^{(n)}r_n,$$

$$(8) \quad E(x_{n+1}^2|X_n) = (ax_n + u_n)^2 + 2cQ^{(n)}(ax_n + u_n)r_n + c^2(Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)}),$$

$$\begin{aligned} E(x_{n+1}r_{n+1}|X_n) &= (ax_n + u_n)r_n + Q^{(n)}(ax_n + u_n)r_n + \\ &\quad + cQ^{(n)}r_n^2 + c(Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)}), \end{aligned}$$

$$E(r_{n+1}^2|X_n) = r_n^2 + 2Q^{(n)}r_n^2 + Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)}.$$

To find the Bayes control denote

$$V_n = \min_{u_n, \dots, u_N} \kappa_n = \min_{u_n, \dots, u_N} E \left[ \sum_{i=n}^N (s_i x_i^2 + k_i u_i^2) | X_n, U_{n-1} \right].$$

Then

$$V_n = \min_{u_N} (s_N x_N^2 + k_N u_N^2) = s_N x_N^2$$

and the optimal  $u_N = 0$ .

Moreover, by application of Bellman's dynamic programming optimality principle, we obtain

$$\begin{aligned} V_n &= \min_{u_n, \dots, u_N} E \left[ \sum_{i=n}^N (s_i x_i^2 + k_i u_i^2) \mid X_n, U_{n-1} \right] \\ &= \min_{u_n} \left[ s_n x_n^2 + k_n u_n^2 + \min_{u_{n+1}, \dots, u_N} E \left[ \sum_{i=n+1}^N (s_i x_i^2 + k_i u_i^2) \mid X_n, U_{n-1} \right] \right]. \end{aligned}$$

Since

$$\begin{aligned} \min_{u_{n+1}, \dots, u_N} E \left[ \sum_{i=n+1}^N (s_i x_i^2 + k_i u_i^2) \mid X_n, U_{n-1} \right] \\ &= \min_{u_{n+1}, \dots, u_N} E \left[ E \left[ \sum_{i=n+1}^N (s_i x_i^2 + k_i u_i^2) \mid X_{n+1}, U_n \right] \mid X_n, U_{n-1} \right] \\ &= E(V_{n+1} \mid X_n, U_{n-1}), \end{aligned}$$

we obtain

$$(9) \quad V_n = \min_{u_n} [s_n x_n^2 + k_n u_n^2 + E(V_{n+1} \mid X_n, U_{n-1})].$$

To determine the Bayes control this yields the equation

$$2k_n u_n + \frac{\partial}{\partial u_n} E(V_{n+1} \mid X_n, U_{n-1}) = 0.$$

But, for given  $X_n$  and  $U_{n-1}$ , we have  $x_{n+1} = u_n^* + ax_n + cv_n$ , what implies

$$(10) \quad 2k_n u_n^* + E \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} \mid X_n, U_{n-1} \right) = 0,$$

where  $u_n^*$  is the Bayes control.

We show now that  $V_n$  is of the form

$$(11) \quad V_n = A_n x_n^2 + 2B_n x_n r_n + C_n r_n^2 + D_n r_n + G_n.$$

For  $n = N$  this holds with  $A_n = s_N$ ,  $B_n = C_n = D_n = G_n = 0$ . Then, assuming (11) to be true for  $n+1$ , we obtain

$$\frac{\partial V_{n+1}}{\partial x_{n+1}} = 2A_{n+1} x_{n+1} + 2B_{n+1} r_{n+1}$$

and by (8)

$$(12) \quad \mathbb{E} \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} \middle| X_n, U_{n-1} \right) \\ = 2A_{n+1}(ax_{n+1} + u_n^* + cQ^{(n)}r_n) + 2B_{n+1}(1 + Q^{(n)})r_n.$$

From (10) and (12) we have

$$k_n u_n^* + A_{n+1}(ax_n + u_n^* + cQ^{(n)}r_n) + B_{n+1}(1 + Q^{(n)})r_n = 0$$

or

$$(13) \quad u_n^* = -\frac{A_{n+1}a}{k_n + A_{n+1}}x_n - \frac{A_{n+1}cQ^{(n)} + B_{n+1}(Q^{(n)} + 1)}{k_n + A_{n+1}}r_n.$$

Moreover, from (8) and (11) we get

$$(14) \quad \mathbb{E}(V_{n+1}|X_n) = A_{n+1}\mathbb{E}(x_{n+1}|X_n) + 2B_{n+1}\mathbb{E}(x_{n+1}r_{n+1}|X_n) + \\ + C_{n+1}\mathbb{E}(r_n^2|X_n) + D_{n+1}\mathbb{E}(r_{n+1}|X_n) + G_{n+1} \\ = A_{n+1}[(ax_n + u_n^*)^2 + 2cQ^{(n)}(ax_n + u_n^*)r_n + \\ + c^2(Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)})] + \\ + 2B_{n+1}[(1 + Q^{(n)})(ax_n + u_n^*)r_n + c(Q^{(n)}r_n^2 + Q_1^{(n)}r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)})] + \\ + C_{n+1}[(1 + Q^{(n)} + Q_1^{(n)})r_n^2 + Q_2^{(n)}r_n + Q_3^{(n)}] + D_{n+1}(1 + Q^{(n)}) + G_{n+1}.$$

On the other hand, by (9) and (11), the same value may be expressed as follows:

$$(15) \quad \mathbb{E}(V_{n+1}|X_n) = A_n x_n^2 + 2B_n x_n r_n + C_n r_n^2 + D_n r_n + G_n - s_n x_n^2 - k_n u_n^{*2}.$$

By the substitution of (13) in place of  $u_n^*$  in (14) and (15) and successive comparison of terms containing  $x_n^2$ ,  $x_n r_n$ ,  $r_n^2$ ,  $r_n$ , 1 we obtain the recursive formulae

$$(16) \quad A_n = s_n + \frac{a^2 k_n A_{n+1}}{k_n + A_{n+1}},$$

$$(17) \quad B_n = \frac{ak_n(cQ^{(n)}A_{n+1} + (1 + Q^{(n)})B_{n+1})}{k_n + A_{n+1}},$$

$$C_n = c^2 Q_1^{(n)} A_{n+1} + 2c(Q^{(n)} + Q_1^{(n)})B_{n+1} + (1 + 2Q^{(n)} + Q_1^{(n)})C_{n+1} - \frac{A_{n+1} + k}{a^2 k_n^2} B_n^2,$$

$$D_n = Q_2^{(n)}(c^2 A_{n+1} + 2cB_{n+1} + C_{n+1}) + (1 + Q^{(n)})D_{n+1},$$

$$G_n = Q_3^{(n)}(c^2 A_{n+1} + 2cB_{n+1} + C_{n+1}) + G_{n+1},$$

with the boundary conditions  $A_N = s_N$ ,  $B_N = C_N = D_N = G_N = 0$ .

Thus, all the coefficients  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$  and  $G_n$  can be computed recursively and the exact analytical solution for the Bayes control  $u_n^*$ , given by (13), may be obtained. Notice that only  $A_n$  and  $B_n$  are necessary for the optimal control, the remaining constants are needed for the computation of the Bayes risk  $V_n$ .

Notice, that to determine the optimal control  $u_n^*$  and the Bayes risk  $V_n$ ,  $n = 0, 1, \dots, N$ , only the assumptions (6) and (7) and the equations (1) and (4) are sufficient.

4. Let  $u_n$  be a control. Sometimes we need the notion of a vector  $U = (u_0, u_1, \dots, u_N)$  which we call a control policy and denote by a capital letter. Let  $R(\lambda, U)$  be the risk connected with the control policy  $U$  when the parameter is  $\lambda$

$$R(\lambda, U) = E_\lambda \left[ \sum_{i=0}^N (s_i^2 x_i^2 + k_i u_i^2) | X_0 \right],$$

where the expectation is taken with respect to the distribution of random variables  $v_0, v_1, \dots, v_N$  given by (2). Let  $\Gamma$  be a class of distributions  $\pi$  of parameter  $\lambda$  such that  $E_\pi(\lambda^2) = m_2$ . Let  $\varkappa(\pi, U)$  be the Bayes risk associated with the distribution  $\pi$  and the control policy  $U$

$$\varkappa(\pi, U) = E_\pi(R(\lambda, U)) = E \left[ \sum_{i=0}^N (s_i x_i^2 + k_i u_i^2) | X_0 \right].$$

Let  $D$  be the class of control policies  $U$  for which  $\varkappa(\pi, U)$  exists for each  $\pi \in \Gamma$ . A control  $U_0$  is called a minimax control policy if

$$\sup_{\pi \in \Gamma} \varkappa(\pi, U_0) = \inf_{U \in D} \sup_{\pi \in \Gamma} \varkappa(\pi, U).$$

In the next sections we look for minimax control policies.

5. Let  $u_n$  be a control which is Bayes with respect to a distribution  $\pi$  of parameter  $\lambda$  defined by (3). Denote

$$R_n = E_\lambda \left[ \sum_{i=n}^N (s_i x_i^2 + k_i u_i^{*2}) | X_n, U_{n-1} \right].$$

Let  $U^* = (u_0^*, u_1^*, \dots, u_N^*)$ . Obviously,

$$(18) \quad R_0 = R(\lambda, U^*).$$

Moreover,

$$\begin{aligned} (19) \quad R_n &= E_\lambda \left[ \sum_{i=n}^N (s_i x_i^2 + k_i u_i^{*2}) | X_n, U_{n-1}^* \right] \\ &= s_n x_n^2 + k_n u_n^{*2} + E_\lambda \left[ E_\lambda \left[ \sum_{i=n+1}^N (s_i^2 x_i^2 + k_i u_i^{*2}) | X_{n+1}, U_n^* \right] | X_n, U_{n-1}^* \right] \\ &= s_n x_n^2 + k_n u_n^{*2} + E_\lambda [R_{n+1} | X_n, U_{n-1}^*]. \end{aligned}$$

$R(\lambda, U^*)$  can be determined from the equations (18) and (19). For any control  $u_n$  we have

$$\begin{aligned} E_\lambda(x_{n+1}|X_n) &= E_\lambda(ax_n + u_n + cv_n|X_n) = ax_n + u_n + cq\lambda, \\ E_\lambda(r_{n+1}|X_n) &= E(r_n + v_n|X_n) = r_n + q\lambda, \\ E_\lambda(x_{n+1}^2|X_n) &= (ax_n + u_n)^2 + 2(ax_n + u_n)cq\lambda + c^2(q_1\lambda^2 + q_2\lambda + q_3), \\ E_\lambda(x_{n+1}r_{n+1}|X_n) &= (ax_n + u_n)r_n + (ax_n + u_n)q\lambda + cq\lambda r_n + c(q_1\lambda^2 + q_2\lambda + q_3), \\ E_\lambda(r_{n+1}^2|X_n) &= r_n^2 + 2qr_n\lambda + q_1\lambda^2 + q_2\lambda + q_3. \end{aligned}$$

We prove that  $R_n$  is of the form

$$(20) \quad R_n = a_n x_n^2 + 2b_n x_n r_n + c_n r_n^2 + 2d_n x_n \lambda + e_n \lambda^2 + 2f_n r_n \lambda + g_n x_n + h_n r_n + i_n \lambda + j_n.$$

For  $n = N$  this is satisfied with  $a_N = s_N$ ,  $b_N = \dots = j_N = 0$ . Suppose that (20) holds for  $n+1$ . By (19) it is sufficient to prove that there are constants  $a_n, \dots, j_n$  such that

$$\begin{aligned} (21) \quad & a_n x_n^2 + 2b_n x_n r_n + c_n r_n^2 + 2d_n x_n \lambda + e_n \lambda^2 + 2f_n r_n \lambda + g_n x_n + h_n r_n + i_n \lambda + j_n \\ &= s_n x_n^2 + k_n u_n^{*2} + a_{n+1} [(ax_n + u_n^*)^2 + 2(ax_n + u_n^*)cq\lambda + c^2(q_1\lambda^2 + q_2\lambda + q_3)] + \\ & \quad + 2b_{n+1} [(ax_n + u_n^*)r_n + (ax_n + u_n^*)q\lambda + cq\lambda r_n + c(q_1\lambda^2 + q_2\lambda + q_3)] + \\ & \quad + c_{n+1} [r_n^2 + 2qr_n\lambda + q_1\lambda^2 + q_2\lambda + q_3] + \\ & \quad + 2d_{n+1} [(ax_n + u_n^*)\lambda + cq\lambda^2] + \\ & \quad + e_{n+1} \lambda^2 + 2f_{n+1} (r_n\lambda + q\lambda^2) + g_{n+1} (ax_n + u_n^* + cq\lambda) + \\ & \quad + h_{n+1} (r_n + q\lambda) + i_{n+1} \lambda + j_{n+1}. \end{aligned}$$

Taking into account that

$$(22) \quad ax_n + u_n^* = \frac{ak_n}{k_n + A_{n+1}} x_n - \frac{cQ^{(n)}A_{n+1} + (Q^{(n)} + 1)B_{n+1}}{k_n + A_{n+1}} r_n \stackrel{\text{df}}{=} E_n x_n + F_n r_n$$

and substituting this in (21) we obtain by successive comparison of terms containing  $x_n^2$ ,  $x_n r_n$ ,  $r_n^2$ ,  $x_n \lambda$ ,  $\lambda^2$ ,  $r_n \lambda$ ,  $x_n$ ,  $r_n$ ,  $\lambda$ , 1 the following:

$$(23) \quad \begin{aligned} a_n &= s_n + k_n(E_n - a)^2 + a_{n+1} E_n^2, \\ b_n &= k_n F_n(E_n - a) + E_n F_n a_{n+1} + E_n b_{n+1}, \\ c_n &= k_n F_n^2 + F_n^2 a_{n+1} + 2F_n b_{n+1} + c_{n+1}, \\ d_n &= cqE_n a_{n+1} + qE_n b_{n+1} + E_n d_{n+1}, \\ e_n &= c^2 q_1 a_{n+1} + 2cq_1 b_{n+1} + q_1 c_{n+1} + 2cq d_{n+1} + e_{n+1} + 2qf_{n+1}, \\ f_n &= cqF_n a_{n+1} + q(F_n + c)b_{n+1} + qc_{n+1} + F_n d_{n+1} + f_{n+1}, \end{aligned}$$

$$\begin{aligned}
 g_n &= E_n g_{n+1}, \\
 h_n &= F_n g_{n+1} + h_{n+1}, \\
 (23) \quad i_n &= c^2 q_2 a_{n+1} + 2c q_2 b_{n+1} + q_2 c_{n+1} + c q g_{n+1} + q h_{n+1} + i_{n+1}, \\
 j_n &= q_3 (c^2 a_{n+1} + 2c b_{n+1} + c_{n+1}) + j_{n+1},
 \end{aligned}$$

with the boundary conditions  $a_N = s_N$ ,  $b_N = \dots = j_N = 0$ .

Comparing (16) and the first row of (23) we obtain

$$a_n - A_n = E_n^2 (a_{n+1} - A_{n+1}).$$

Since  $a_N = A_N$  we have that  $a_n = A_n$  for  $n = 0, 1, \dots, N$ . Substituting this into the second row of (23) we obtain  $b_n = 0$ . Moreover, the conditions  $g_N = h_N = 0$  give  $g_n = h_n = 0$  for  $n = 0, 1, \dots, N$ . Then  $R_n$  reduces to the form

$$(24) \quad R_n = a_n x_n^2 + c_n r_n^2 + 2d_n x_n \lambda + e_n \lambda^2 + 2f_n r_n \lambda + i_n \lambda + j_n,$$

where the constants  $a_n, \dots, j_n$  are determined from the equations

$$\begin{aligned}
 a_n &= A_n, \\
 c_n &= F_n^2 (k_n + A_{n+1}) + c_{n+1}, \\
 d_n &= c q A_{n+1} E_n + E_n d_{n+1}, \\
 (25) \quad e_n &= c^2 q_1 A_{n+1} + q_1 c_{n+1} + 2c q d_{n+1} + 2q f_{n+1} + e_{n+1}, \\
 f_n &= c q A_{n+1} F_n + q c_{n+1} + F_n d_{n+1} + f_{n+1}, \\
 i_n &= c^2 q_2 A_{n+1} + q_2 c_{n+1} + i_{n+1}, \\
 j_n &= q_3 (c^2 A_{n+1} + c_{n+1}) + j_{n+1}
 \end{aligned}$$

and

$$a_N = s_N, \quad c_N = d_N = e_N = f_N = i_N = j_N = 0.$$

**6.** We call a control policy  $U$  to be a constant risk control policy if  $\kappa(\pi, U) = \text{const}$  for each  $\pi \in \Gamma$ . Since  $R(\lambda, U^*) = R_0$  and  $E_\pi(\lambda^2) = m_2$  for each  $\pi \in \Gamma$  we obtain from (24) that  $U^*$  is a constant risk control policy if

$$(26) \quad 2d_0 x_0 + 2f_0 r + i_0 = 0.$$

Let  $\pi^*$  be the distribution of parameter  $\lambda$  which has the density (3) and for which the control policy  $U^*$  is a Bayes one. Denote  $k(\beta, r) = E_\pi(\lambda^2)$  for  $\pi$  given by (3). Since  $\pi^* \in \Gamma$ , we obtain that the parameters  $\beta, r$  of the distribution  $\pi^*$  satisfy the condition

$$(27) \quad k(\beta, r) = m_2.$$

But from decision theory a constant risk strategy which is Bayes with respect to some  $\pi \in \Gamma$  is a minimax strategy [10]. Thus we have proved the following

**THEOREM.** Let  $\beta^*, r^*$  be a solution of equations (26) and (27), where  $d_0$  is a constant,  $f_0 = f_0(\beta)$ ,  $i_0 = i_0(\beta)$  are the functions of  $\beta$  determined from the formulae (16), (22) and (25), where  $Q^{(n)} = Q^{(n)}(\beta)$  are defined by (6). The policy  $U$ , Bayes with respect to distribution  $\pi^*$  defined by (3) with  $\beta = \beta^*$ ,  $r = r^*$ , is a minimax control policy. This control policy is given by (13) with  $r_n = r^* + \sum_{i=0}^{n-1} v_i$ ,  $Q^{(n)} = Q^{(n)}(\beta^*)$ .

From the Theorem it follows that the way to determine a minimax control policy is complicated and in our opinion it hardly can be simplified from the nature of the problem.

7. In this section we compute the parameters connected with the filtration for some distributions belonging to the exponential family.

(a) Let the random variables  $v_n$  be distributed according to the Poisson law

$$p(v, \lambda) = \frac{\lambda^v}{v!} e^{-\lambda} \quad (\lambda > 0).$$

Then

$$E(v_n) = \lambda, \quad E(v_n^2) = \lambda^2 + \lambda$$

and  $q = q_1 = q_2 = 1$ ,  $q_3 = 0$ . Moreover, according to (3) the a priori distribution  $\pi$  of parameter  $\lambda$  is of the form

$$g(\lambda; \beta, r) = \begin{cases} \frac{\beta^r}{\Gamma(r)} \lambda^{r-1} e^{-\beta\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

what implies

$$E_\pi(\lambda) = r/\beta, \quad E_\pi(\lambda^2) = r(r+1)/\beta^2.$$

The equation (27) takes now the form

$$(28) \quad r(r+1)/\beta^2 = m_2.$$

The conditional density of the random variable  $v_n$  given  $X_n$  is

$$h(v|X_n) = \frac{S(v)C(\beta_n, r_n)}{C(\beta_n+1, r_n+v)} = \frac{\beta_n^{r_n}}{\Gamma(r_n)} \frac{\Gamma(r_n+v)}{v!} \frac{1}{(\beta_n+1)^{r_n+v}}.$$

Since

$$\sum_{v=0}^{\infty} \frac{\Gamma(r+v)}{v!} \frac{1}{(\beta+1)^{r+v}} = \frac{\Gamma(r)}{\beta^r},$$

then

$$E(v_n | X_n) = r_n / \beta_n, \quad E(v_n(v_n - 1) | X_n) = r_n(r_n + 1) / \beta_n^2,$$

what leads to the equation

$$E(v_n^2 | X_n) = \frac{r_n^2}{\beta_n^2} + \frac{\beta_n + 1}{\beta_n^2} r_n.$$

Then

$$Q^{(n)} = \frac{1}{\beta_n}, \quad Q_1^{(n)} = \frac{1}{\beta_n^2}, \quad Q_2^{(n)} = \frac{\beta_n + 1}{\beta_n}, \quad Q_3^{(n)} = 0.$$

(b) Suppose that the random variables  $v_n$  have the gamma distribution

$$p(v, \lambda) = \begin{cases} \frac{1}{\Gamma(q) \lambda^q} v^{q-1} e^{-v/\lambda}, & \text{if } v > 0, \\ 0, & \text{if } v \leq 0. \end{cases}$$

Then

$$E(v_n) = q\lambda, \quad E(v_n^2) = q(q+1)\lambda^2$$

and  $q_1 = q(q+1)$ ,  $q_2 = q_3 = 0$ .

According to (3) the a priori distribution  $\pi$  of parameter  $\lambda$  is of the form

$$g(\lambda; \beta, r) = \begin{cases} \frac{r^{\beta+1}}{\Gamma(\beta+1)} \cdot \frac{1}{\lambda^{\beta+2}} e^{-r/\lambda}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0. \end{cases}$$

and

$$E_\pi(\lambda) = r/\beta, \quad E_\pi(\lambda^2) = r^2/\beta(\beta-1).$$

Equation (27) takes then the form

$$(29) \quad r^2/\beta(\beta-1) = m_2.$$

The conditional density of random variable  $v_n$  given  $X_n$  is now

$$h(v | X_n) = \begin{cases} \frac{r_n^{\beta_n+1}}{B(q, \beta_n+1)} \cdot \frac{v^{q-1}}{(r_n+v)^{\beta_n+q+1}}, & \text{if } v > 0, \\ 0, & \text{if } v \leq 0, \end{cases}$$

what gives

$$E(v_n | X_n) = q \frac{r_n}{\beta_n}, \quad E(v_n^2 | X_n) = \frac{q(q+1)}{\beta_n(\beta_n-1)} r_n^2.$$

Then

$$Q^{(n)} = \frac{q}{\beta_n}, \quad Q_1^{(n)} = \frac{q(q+1)}{\beta_n(\beta_n-1)}, \quad Q_2^{(n)} = Q_3^{(n)} = 0.$$

(c) Suppose that  $v_n$  have the binomial distribution

$$p(v, \lambda) = \binom{q}{v} \lambda^v (1-\lambda)^{q-v} \quad (0 < \lambda < 1).$$

In this case

$$E_\lambda(v_n) = q\lambda, \quad E_\lambda(v_n^2) = q(q-1)\lambda^2 + q\lambda,$$

i.e.  $q_1 = q(q-1)$ ,  $q_2 = q$ ,  $q_3 = 0$ .

The a priori distribution  $\pi$  is now a beta distribution

$$g(\lambda; \beta, r) = \begin{cases} \frac{1}{B(r, \beta-r)} \lambda^{r-1} (1-\lambda)^{\beta-r-1} & \text{if } 0 < \lambda < 1, \\ 0 & \text{in the other case} \end{cases}$$

and

$$E_\pi(\lambda) = r/\beta, \quad E_\pi(\lambda^2) = r(r+1)/\beta(\beta+1).$$

Then the equation (27) is now of the form

$$(30) \quad r(r+1)/\beta(\beta+1) = m_2 \quad (\beta > 0, 0 < m_2 < 1).$$

To determine the parameters  $Q_i$  we obtain from (5)

$$h(v|X_n) = \binom{q}{v} \frac{B(r_n+v, \beta_n+q-r_n-v)}{B(r_n, \beta_n-r_n)}.$$

Then

$$\begin{aligned} E(v_n|X_n) &= \frac{1}{B(r_n, \beta_n-r_n)} \sum_{v=0}^q v \binom{q}{v} B(r_n+v, \beta_n+q-r_n-v) \\ &= \frac{q}{B(r_n, \beta_n-r_n)} \int_0^1 x^{r_n} (1-x)^{\beta_n-r_n-1} \left( \sum_{v=1}^q \binom{q-1}{v-1} x^{v-1} (1-x)^{q-1} \right) dx \\ &= \frac{q}{B(r_n, \beta_n-r_n)} \int_0^1 x^{r_n} (1-x)^{\beta_n-r_n-1} dx = q \frac{r_n}{\beta_n}. \end{aligned}$$

In a similar way

$$E(v_n(v_n-1)|X_n) = q(q-1) \frac{r_n(r_n+1)}{\beta_n(\beta_n+1)}.$$

Then

$$Q^{(n)} = \frac{q}{\beta_n}, \quad Q_1^{(n)} = \frac{q(q-1)}{\beta_n(\beta_n+1)}, \quad Q_2^{(n)} = \frac{q(\beta_n+q)}{\beta_n(\beta_n+1)}, \quad Q_3^{(n)} = 0.$$

(d) Suppose that the random variables  $v_n$  have the negative binomial distribution

$$p(v, \lambda) = \frac{\Gamma(q+v)}{\Gamma(q)\Gamma(v+1)} \frac{\lambda^v}{(1+\lambda)^{q+v}}.$$

We obtain

$$E(v_n) = q\lambda, \quad E(v_n^2) = q(q+1)\lambda^2 + q\lambda.$$

Then  $q_1 = q(q+1)$ ,  $q_2 = q$ ,  $q_3 = 0$ .

Let  $q$  be a natural number. Let us put  $p = \lambda/(1+\lambda)$ . In this case the distribution  $p(v, \lambda)$  takes the more familiar form

$$p(v, \lambda) = \binom{q+v-1}{v} p^v (1-p)^q.$$

According to (3) we have

$$g(\lambda; \beta, r) = \begin{cases} \frac{1}{B(\beta+1, r)} \frac{\lambda^{r-1}}{(1+\lambda)^{\beta+r+1}}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leq 0, \end{cases}$$

and

$$E_\pi(\lambda) = r/\beta, \quad E_\pi(\lambda^2) = \frac{r(r+1)}{\beta(\beta-1)} \quad (\beta > 1).$$

Then

$$(31) \quad \frac{r(r+1)}{\beta(\beta-1)} = m_2.$$

To determine  $Q_i$  we obtain from (15)

$$h(v|X_n) = \frac{\Gamma(q+v)}{\Gamma(q)\Gamma(v+1)} \cdot \frac{B(\beta_n+q+1, r_n+v)}{B(\beta_n+1, r_n)}$$

and

$$\begin{aligned} E(v_n|X_n) &= \sum_{v=0}^{\infty} v \frac{\Gamma(q+v)}{\Gamma(q)\Gamma(v+1)} \frac{B(\beta_n+q+1, r_n+v)}{B(\beta_n+1, r_n)} \\ &= \frac{q}{B(\beta_n+1, r_n)} \sum_{v=0}^{\infty} \frac{\Gamma(q+1+v)}{\Gamma(q+1)\Gamma(v+1)} B(\beta_n+q+1, r_n+v+1) \end{aligned}$$



$$\begin{aligned}
 &= \frac{q}{B(\beta_n + 1, r_n)} \sum_{v=0}^{\infty} \frac{\Gamma(q+1+v)}{\Gamma(q+1)\Gamma(v+1)} \int_0^1 x^{r_n+v} (1-x)^{\beta_n+q} dx \\
 &= \frac{q}{B(\beta_n + 1, r_n)} \int_0^1 x^{r_n} (1-x)^{\beta_n-1} \left( \sum_{v=0}^{\infty} \frac{\Gamma(q+1+v)}{\Gamma(q+1)\Gamma(v+1)} x^v (1-x)^{q+1} \right) dx \\
 &= \frac{q}{B(\beta_n + 1, r_n)} \int_0^1 x^{r_n} (1-x)^{\beta_n-1} dx = q \frac{r_n}{\beta_n}.
 \end{aligned}$$

Similarly

$$E(v_n(v_n - 1) | X_n) = q(q + 1) \frac{r_n(r_n + 1)}{\beta_n(\beta_n - 1)}.$$

Then

$$Q^{(n)} = \frac{q}{\beta}, \quad Q_1^{(n)} = \frac{q(q + 1)}{\beta_n(\beta_n - 1)}, \quad Q_2^{(n)} = \frac{q(q + \beta_n)}{\beta_n(\beta_n - 1)}, \quad Q_3^{(n)} = 0.$$

(e) Let the random variables  $v_n$  be normally distributed

$$p(v, \lambda) = \frac{1}{\sqrt{2\pi}} e^{-(v-\lambda)^2/2}.$$

Then

$$E(v_n) = \lambda, \quad E(v_n^2) = \lambda^2 + 1$$

and  $q = q_1 = q_3 = 1, q_2 = 0$ .

Let, according to (3), the a priori distribution  $\pi$  of parameter  $\lambda$  be given by the density

$$g(\lambda; \beta, r) = \sqrt{\frac{\beta}{2\pi}} e^{-\beta(\lambda-r/\beta)^2/2},$$

what implies

$$E_{\pi}(\lambda) = \frac{r}{\beta}, \quad E_{\pi}(\lambda^2) = \frac{r^2}{\beta^2} + \frac{1}{\beta} \quad (\beta > 0, -\infty < r < \infty).$$

Then the parameters  $\beta$  and  $r$  satisfy the condition

$$(32) \quad \frac{r^2}{\beta^2} + \frac{1}{\beta} = m_2.$$

The conditional density of the random variable  $v_n$  given  $X_n$  is

$$h(v|X_n) = \frac{S(v)C(\beta_n, r_n)}{C(\beta_n+1, r_n+v)} = \sqrt{\frac{\beta_n+1}{2\pi\beta_n}} \exp\left\{-\frac{\beta_n}{\beta_n+1} \frac{(v-r_n/\beta_n)^2}{2}\right\},$$

what gives

$$E(v_n|X_n) = \frac{r_n}{\beta_n}, \quad E(v_n^2|X_n) = \frac{r_n^2}{\beta_n^2} + \frac{\beta_n+1}{\beta_n}.$$

Then

$$Q^{(n)} = 1/\beta_n, \quad Q_1^{(n)} = 1/\beta_n^2, \quad Q_2^{(n)} = 0, \quad Q_3^{(n)} = (\beta_n+1)/\beta_n.$$

8. Let  $N = 1$ . From (16), (17), (22) and (25) we obtain

$$E_0 = \frac{ak_0}{k_0+s_1}, \quad F_0 = -\frac{cs_1 Q^{(0)}}{k_0+s_1},$$

$$d_0 = cqs \frac{ak_0}{k_0+s_1}, \quad f_0 = -cqs_1 \frac{cs_1 Q^{(0)}}{k_0+s_1}, \quad i_0 = c^2 q_2 s_1$$

and the equation (26) now is

$$(33) \quad 2ak_0 qx_0 + cq_2(k_0+s_1) = 2cs_1 qQ^{(0)}r.$$

Moreover,

$$(34) \quad Q^{(0)} = q/\beta,$$

where  $q = 1$  in the cases (a) and (d) and  $q > 0$  in the cases (b), (c), (d). Solving the equations (28)–(32) with respect to  $r$  and taking into account equation (34) we obtain that the function  $\varphi(\beta) = Q^{(0)}r/q$  takes the values in the intervals

$$(a) (0, \sqrt{m_2}), \quad (b) (0, \sqrt{m_2}), \quad (c) (m_2, \sqrt{m_2}),$$

$$(d) (0, \sqrt{m_2}), \quad (e) (-\sqrt{m_2}, \sqrt{m_2}).$$

Then the equation (33) has a solution only if  $x_0$  satisfies the corresponding inequalities.

For the normal distribution is  $q_2 = 0$  what compared with (25) implies  $i_n = 0$ ,  $n = 0, 1, \dots, N$ . The equation (26) takes in this case a simple form

$$d_0 x_0 + f_0 r = 0.$$

At the end, let us notice that the method presented in the paper can be applied also in the case when the coefficients  $a$  and  $c$  in (1) depend on  $n$ .

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