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EXTENDED PIECEWISE MARKOV PROCESSES IN DISCRETE TIME

1. Definitions and notation. This paper is a continuation of [2], where we investigated extended piecewise Markov processes in continuous time. The definition of an extended piecewise Markov process in discrete time is analogous to the definition of such a process in continuous time. Also, the obtained relations are analogous to the relations given in theorems 2 and 3 in [2]. The method of proving the theorems is that of extension to a Markov process. The obtained results are applied to the investigation of generalized Moran's models of a dam.

Let us introduce the notation obligatory in the whole paper: $\mathscr{R}_+ = (0, \infty)$, $\mathscr{B}_+$ stands for the σ-algebra of Borel subsets of $\mathscr{R}_+$, $\mathscr{B}_{\mathscr{N}}$ for the σ-algebra of all subsets of $\mathscr{N} = \{1, 2, \ldots\}$, $\mathscr{B}_1$ for the σ-algebra of all subsets of the set $\{0, 1\}$.

Assume that on the measurable space $(\mathscr{X}, \mathscr{F})$ there are given:

a set of measurable stochastic kernels $\{P_{a,y}(x, A), x \in \mathscr{X}, A \in \mathscr{F}\}$, $a = 0, 1$, $y \in \mathscr{X}$;

two stochastic kernels $\{Q_a(x, A), x \in \mathscr{X}, A \in \mathscr{F}\}$, $a = 0, 1$;

a set of probability distributions $\{f_{a,y}(n), n = 0, 1, \ldots\}$, $a = 0, 1$, $y \in \mathscr{X}$, such that, for every $y \in \mathscr{X}$, $f_{1,y}(0) = 0$.

Definition 1. A stochastic process $\{X(t),\ t = 0, 1, \ldots\}$, defined on a probability space $(\Omega, \sigma, \Pr)$, valued in the state space $(\mathscr{X}, \mathscr{F})$, is said to be an *extended piecewise Markov process* if the following conditions (a)-(d) are fulfilled:

(a) There exists a sequence of random variables $0 = \tau_0 \leqslant \tau_1 < \tau_2 \leqslant \tau_3 < \ldots$ defined on $(\Omega, \sigma, \Pr)$, with integer values, such that the process

$$\{X(t), \tau_{2m+a} \leqslant t < \tau_{2m+a+1}, t = 0, 1, \ldots\}, \quad a = 0, 1, \ m = 0, 1, \ldots \quad (\tau_{2m+a+1} - \tau_{2m+a} \geqslant 1)$$

is a homogeneous Markov chain with transition probabilities $P_{a,y}(x, A)$ dependent on a and on the condition $\{X(\tau_{2m+a}) = y\}$.

(b) At the moments τ_{2m+a}, $a = 0, 1$, $m = 0, 1, \ldots$, for which $\tau_{2m+a} \neq \tau_{2m+a+1}$, the state change of the process is a composition of two consecutive independent jumps: a Markovian one with the stochastic kernel $P_{a,y}(x, A)$ dependent on the condition $\{X(\tau_{2m+a-1}) = y\}$ and a regenerative one with the stochastic kernel $Q_a(y, A)$.

(c) At the moments τ_{2m}, $m = 0, 1, \ldots$, for which $\tau_{2m} = \tau_{2m+1}$ the state change of the process is a composition of three independent jumps: a Markovian one with the stochastic kernel $P_{1,y}(x, A)$ dependent on the condition $\{X(\tau_{2m-1}) = y\}$ and two regenerative ones with stochastic kernels $Q_1(x, A)$ and $Q_0(x, A)$.

(d) For arbitrary $a = 0, 1$, $m = 0, 1, \ldots$, $f_{a,y}(n)$ is the probability distribution of the distance between τ_{2m+a+1} and τ_{2m+a} dependent on a and on the condition $\{X(\tau_{2m+a}) = y\}$.

Let X_m^-, $m = 1, 2, \ldots$, denote the state of the process at the moment τ_m before the last regenerative transition. Consider four sequences of random variables

$$\{X_{2m-a}^-, m = 1, 2, \ldots\}, \quad \{X(\tau_{2m+a}), m = 0, 1, \ldots\}, \quad a = 0, 1.$$

It is easy to see that they are Markov chains, since τ_{2m+a}, $a = 0, 1$, $m = 0, 1, \ldots$, are regenerative moments of the process $\{X(t), t = 0, 1, \ldots\}$.

Assume that there exists a probability measure N_0^- being invariant for the chain $\{X_{2m}^-, m = 1, 2, \ldots\}$. Hence it is easy to find the invariant measures N_1^-, N_0^+, N_1^+ for the chains

$$\{X_{2m-1}^-, m = 1, 2, \ldots\}, \quad \{X(\tau_{2m}), m = 0, 1, \ldots\},$$
$$\{X(\tau_{2m+1}), m = 0, 1, \ldots\},$$

respectively.

In the sequel of this paper we find relations between the stationary probability distribution of the process $\{X(t), t = 0, 1, \ldots\}$ and the probability measures N_a^-, N_a^+, $a = 0, 1$. Sufficient conditions for the ergodicity of Markov chains on a general state space are given in [4].

2. Invariant measure for an extended Markov process. Let $\{X(t), t = 0, 1, \ldots\}$ be an extended piecewise Markov process with regenerative moments $\{\tau_m, m = 0, 1, \ldots\}$. Assume that

$$\sup_{m \geq 0} \tau_m = +\infty \quad \text{Pr-almost everywhere}$$

and for $t = 0, 1, \ldots$ define the following processes:

$$\left.\begin{aligned} Y(t) &= X(\tau_{2m+a}), \\ Z(t) &= \tau_{2m+a+1} - t, \\ a(t) &= a, \end{aligned}\right\} \quad \tau_{2m+a} \leq t < \tau_{2m+a+1}, \; a = 0, 1, \; m = 0, 1, \ldots$$

The process $\{Y(t), t = 0, 1, \ldots\}$ is said to be a *semi-Markov process*, $\{Z(t), t = 0, 1, \ldots\}$ a *residual-time process* and $\{a(t),\ t = 0, 1, \ldots\}$ a *breakdown process.*

Let us introduce the following notation:

$$\overline{\mathscr{X}} = \mathscr{X}^2 \times \mathscr{N} \times \{0, 1\}, \qquad \bar{x} = (x, y, z, i) \in \overline{\mathscr{X}},$$

$$\overline{\mathscr{F}} = \mathscr{F}^2 \times \mathscr{B}_{\mathscr{N}} \times \mathscr{B}_1, \qquad \bar{A} = A \times B \times \{j\} \times \{a\} \in \overline{\mathscr{F}},$$

$$\bar{X}(t) = \big(X(t),\ Y(t),\ Z(t),\ a(t)\big),$$

$$\bar{P}(\bar{x}, \bar{A}) = \Pr\big(\bar{X}(t+1) \in \bar{A} \mid \bar{X}(t) = \bar{x}\big), \qquad t = 0, 1, \ldots$$

THEOREM 1. *The process $\{\bar{X}(t), t = 0, 1, \ldots\}$ is a homogeneous Markov chain with transition probabilities $\bar{P}(\bar{x}, \bar{A})$ defined by the formula*

$$\begin{aligned}(1)\quad \bar{P}(\bar{x}, \bar{A}) &= P_{i,y}(x, A) I_B(y)\, \delta_{z-1,j}\, \delta_{i,a} + \\ &+ \delta_{z,1} \delta_{1-i,a} \int_{\mathscr{X}} P_{i,y}(x, du) \int_A Q_i(u, ds) I_B(s) f_{1-i,s}(j) + \\ &+ \delta_{z,1} \int_{\mathscr{X}} P_{i,y}(x, du) \int_{\mathscr{X}} Q_i(u, ds) f_{1-i,s}(0) \int_A Q_{1-i}(s, dz) I_B(z) f_{i,z}(j), \\ &\qquad\qquad \bar{x} \in \overline{\mathscr{X}},\ \bar{A} \in \overline{\mathscr{F}}.\end{aligned}$$

Proof. From definition 1 it follows immediately that this process is homogeneous. If $z \geqslant 2$, then the moment $t+1$ is not regenerative and we have

$$\Pr\big(\bar{X}(t+1) \in \bar{A} \mid \bar{X}(t) = \bar{x}\big) = P_{i,y}(x, A) I_B(y)\, \delta_{z-1,j}\, \delta_{i,a}.$$

If $z = 1$, then the moment $t+1$ is regenerative and the next distance between regenerative moments may be equal to $j \geqslant 1$ with probability $f_{1-i,s}(j)$ dependent on the state of the process at the moment $t+1$ or may be equal to zero with probability $f_{1-i,s}(0)$ (in that case we have two regenerative transitions at the moment $t+1$). Hence we have

$$\begin{aligned}&\Pr\big(\bar{X}(t+1) \in \bar{A} \mid \bar{X}(t) = \bar{x}\big) \\ &= \delta_{z,1} \Big[\delta_{1-i,a} \int_{\mathscr{X}} P_{i,y}(x, du) \int_A Q_i(u, ds) I_B(s) f_{1-i,s}(j) + \\ &\quad + \delta_{i,a} \int_{\mathscr{X}} P_{i,y}(x, du) \int_{\mathscr{X}} Q_i(u, ds) f_{1-i,s}(0) \int_A Q_{1-i}(s, dz) I_B(z) f_{i,z}(j)\Big],\end{aligned}$$

where the second component on the right-hand side is equal to zero if $i = 0$.

THEOREM 2. *If the probability measures N_a^+ are invariant for the Markov chains $\{X(\tau_{2m+a}),\ m = 0, 1, \ldots\}$, $a = 0, 1$, then the measure $\bar{N}$ defined by the formula*

$$(2)\quad \bar{N}(\bar{A}) = v \sum_{k=0}^{\infty} \int_{\mathscr{X}} N_a^+(dx) I_B(x) f_{a,x}(j+k) P_{a,x}(k, x, A), \qquad \bar{A} \in \overline{\mathscr{F}},$$

where

$$\frac{1}{v} = \sum_{a=0}^{1} \int_{\mathscr{X}} N_a^+(ds) m_{a,s}, \quad m_{a,y} = \sum_{n=1}^{\infty} n f_{a,y}(n), \quad a = 0, 1, \; y \in \mathscr{X},$$

is an invariant measure for the transition probabilities $\bar{P}(\bar{x}, \bar{A})$, *provided the right-hand side of* (2) *is finite.*

Proof. By the definition of the invariant measure and by formulas (2) and (1) we have

$$(3) \quad \int_{\bar{\mathscr{X}}} \bar{N}(d\bar{x})\bar{P}(\bar{x}, \bar{A}) = v \sum_{k=1}^{\infty} \int_{\mathscr{X}} N_a^+(ds) I_B(s) f_{a,s}(j+k) P_{a,s}(k, s, A) +$$

$$+ v \int_A \Big[\Big(\int_{\mathscr{X}} N_{1-a}^+(ds) \sum_{k=1}^{\infty} f_{1-a,s}(k) \int_{\mathscr{X}} P_{1-a,s}(k, s, dx) + \int_{\mathscr{X}} N_a^+(ds) \sum_{k=1}^{\infty} f_{a,s}(k) \times$$

$$\times \int_{\mathscr{X}} P_{a,s}(k, s, du) \int_{\mathscr{X}} Q_a(u, dx) f_{1-a,x}(0)\Big) Q_{1-a}(x, dw)\Big] I_B(w) f_{a,w}(j).$$

By the definitions of the measures N_a^+ and N_a^- we have

$$N_1^-(A) = \int_{\mathscr{X}} N_0^+(ds) \sum_{k=1}^{\infty} f_{0,s}(k) P_{0,s}(k, s, A) +$$

$$+ \int_{\mathscr{X}} N_1^+(ds) \sum_{k=1}^{\infty} f_{1,s}(k) \int_{\mathscr{X}} P_{1,s}(k, s, du) \int_A Q_1(u, dz) f_{0,z}(0),$$

$$N_0^-(A) = \int_{\mathscr{X}} N_1^+(ds) \sum_{k=1}^{\infty} f_{1,s}(k) P_{1,s}(k, s, A).$$

Hence the expression in the square brackets on the right-hand side of (3) equals $N_a^+(dw)$. Thus we obtain

$$\int_{\bar{\mathscr{X}}} \bar{N}(d\bar{x})\bar{P}(\bar{x}, \bar{A}) = v \sum_{k=1}^{\infty} \int_{\mathscr{X}} N_a^+(ds) I_B(s) f_{a,s}(j+k) P_{a,s}(k, s, A) +$$

$$+ v \int_{\mathscr{X}} N_a^+(dw) P(0, w, A) I_B(w) f_{a,w}(j) = \bar{N}(\bar{A}).$$

The constant v is computed from the condition

$$\sum_{a=0}^{1} \bar{N}(\mathscr{X}^2 \times \mathscr{N} \times \{a\}) = 1 = v \sum_{a=0}^{1} \int_{\mathscr{X}} N_a^+(ds) m_{a,s}.$$

COROLLARY 1. *The marginal measure*

$$N(A) = \sum_{a=0}^{1} \overline{N}(A \times \mathscr{X} \times \mathscr{N} \times \{a\})$$

and the measures N_a^+ *fulfil the relations*

$$(4) \qquad N(A) = v \sum_{k=0}^{\infty} \sum_{a=0}^{1} \int_{\mathscr{X}} N_a^+(dx) F_{a,x}(k) P_{a,x}(k, x, A), \qquad A \in \mathscr{F},$$

where

$$F_{a,y}(k) = \sum_{j=k+1}^{\infty} f_{a,y}(j), \qquad a = 0, 1, \ y \in \mathscr{X},$$

provided the right-hand side of (4) *is finite.*

Formula (4) gives relations analogous to relations (8) in [2]. The following theorem is analogous to theorem 3 in [2] for the stationary process $\{\overline{X}(t), t = 0, 1, \ldots\}$ (for the definition of a stationary process see [1], p. 165).

THEOREM 3. *The marginal measure*

$$\tilde{N}(A \times B \times \{a\}) = \overline{N}(A \times B \times \mathscr{N} \times \{a\})$$

and the measures N_a^- *and* N_a^+ *fulfil the relations*

$$(5) \qquad \tilde{N}(A \times \mathscr{X} \times \{a\}) - \int_{\mathscr{X}^2} \tilde{N}(dx \times dy \times \{a\}) \tilde{\Pi}_a\big((x, y), A \times \mathscr{X}\big) = v\big(N_{1-a}^-(A) - N_a^+(A)\big), \qquad a = 0, 1, \ A \in \mathscr{F},$$

where

$$\tilde{\Pi}_a\big((x, y), A \times B\big) = P_{a,y}(x, A) I_B(y), \qquad a = 0, 1, \ x, y \in \mathscr{X}, \ A, B \in \mathscr{F}.$$

Proof. From the definition of the invariant measure $\overline{N}$ and from formula (1) we obtain the equation

$$(6) \qquad \overline{N}(\overline{A}) = \int_{\mathscr{X} \times B} \overline{N}(dx \times dy \times \{j+1\} \times \{a\}) P_{a,y}(x, A) + \int_{\mathscr{X}^2} \overline{N}(dx \times dy \times \{1\} \times \{1-a\}) \int_{\mathscr{X}} P_{1-a,y}(x, ds) \times \int_{A \cap B} Q_{1-a}(s, du) f_{a,u}(j) + \int_{\mathscr{X}^2} \overline{N}(dx \times dy \times \{1\} \times \{a\}) \times \int_{\mathscr{X}} P_{a,y}(x, ds) \int_{\mathscr{X}} Q_a(s, du) f_{1-a,s}(0) \int_{A \cap B} Q_{1-a}(u, dz) f_{a,z}(j).$$

Thus for $\nu(j,a) = \bar{N}(\mathscr{X}^2 \times \{j\} \times \{a\})$ we have

$$\nu(j,a) = \nu(j+1,a) + \int_{\mathscr{X}^2} \bar{N}(dx \times dy \times \{1\} \times \{1-a\}) \times \tag{7}$$
$$\times \int_{\mathscr{X}} P_{1-a,y}(x, ds) \int_{\mathscr{X}} Q_{1-a}(s, du) f_{a,u}(j) +$$
$$+ \int_{\mathscr{X}^2} \bar{N}(dx \times dy \times \{1\} \times \{a\}) \int_{\mathscr{X}} P_{a,y}(x, ds) \times$$
$$\times \int_{\mathscr{X}} Q_a(s, du) f_{1-a,s}(0) \int_{\mathscr{X}} Q_{1-a}(u, dz) f_{a,z}(j).$$

Transforming the right-hand side of (7) it is possible to verify that this equation has a solution of the form

$$\nu(j,a) = v \int_{\mathscr{X}} N_a^+(ds) F_{a,s}(j-1), \quad j \geqslant 1.$$

The constant v is determined by the condition

$$\sum_{j=1}^{\infty} \sum_{a=0}^{1} \nu(j,a) = 1 = v \sum_{a=0}^{1} \int_{\mathscr{X}} N_a^+(ds) m_{a,s}.$$

Introducing the notation

$$\tilde{N}_a^*(A \times B) = \bar{N}(A \times B \times \{a\} \mid 1)$$
$$= \Pr\big((X(t), Y(t), a(t)) \in A \times B \times \{a\} \mid Z(t) = 1\big),$$

we have

$$\bar{N}(A \times B \times \{1\} \times \{a\}) = v \int_{\mathscr{X}} N_a^+(ds) F_{a,s}(0) \tilde{N}_a^*(A \times B). \tag{8}$$

By the definition of the measures N_a^- we have

$$\int_{\mathscr{X}} N_{1-a}^+(ds)\big(1 - f_{1-a,s}(0)\big) \int_{\mathscr{X}^2} \tilde{N}_{1-a}^*(dx \times dy) P_{1-a,y}(x, A) + \tag{9}$$
$$+ \int_{\mathscr{X}} N_a^+(ds) F_{a,s}(0) \int_{\mathscr{X}^2} \tilde{N}_a^*(dx \times dy) \int_{\mathscr{X}} P_{a,y}(x, ds) \int_{A \cap B} Q_a(s, du) f_{1-a,u}(0)$$
$$= N_a^-(A), \quad a = 0, 1, \ A \in \mathscr{F}.$$

Summing (6) over j from 1 to infinity and using (8) and (9) we obtain (5).

In the case of a discrete state space, from theorem 3 we obtain the proposition of theorem 3.1 in [3].

If $f_{0,y}(0) = 1$ for every $y \in \mathscr{X}$, then the process $\{X(t), t = 0, 1, \ldots\}$ is a piecewise Markov process and from theorems 2 and 3 we obtain the following theorems:

THEOREM 4. *If the probability measure N^+ is invariant for the Markov chain $\{X(\tau_m),\ m = 0, 1, \ldots\}$, then the measure $\bar{N}$ defined by the formula*

$$(10)\qquad \bar{N}(\bar{A}) = v\sum_{k=0}^{\infty}\int_{\mathscr{X}} N^+(dx)I_B(x)f_x(j+k)P_x(k, x, A), \qquad \bar{A} \in \bar{\mathscr{F}},$$

where

$$\frac{1}{v} = \int_{\mathscr{X}} N^+(ds)m_s, \qquad m_y = \sum_{n=1}^{\infty} nf_y(n), \qquad P_y(0, x, A) = I_A(x),$$

$$x, y \in \mathscr{X},\ A \in \mathscr{F},$$

is an invariant measure for the transition probabilities $\bar{P}(\bar{x}, \bar{A})$, provided the right-hand side of (10) *is finite.*

THEOREM 5. *The marginal measure $\tilde{N}(A \times B) = \bar{N}(A \times B \times \mathscr{N})$ and the measures N^- and N^+ fulfil the relation*

$$(11)\qquad \tilde{N}(A \times \mathscr{X}) - \int_{\mathscr{X}^2} \tilde{N}(dx \times dy)\tilde{\Pi}\big((x, y), A \times \mathscr{X}\big) = v\big(N^-(A) - N^+(A)\big),$$

$$A \in \mathscr{F},$$

where

$$\tilde{\Pi}\big((x, y), A \times B\big) = P_y(x, A)I_B(y), \qquad x, y \in \mathscr{X},\ A, B \in \mathscr{F}.$$

3. Applications. Consider a dam with finite capacity k. At discrete moments $t = 0, 1, \ldots$, into the dam there flows a water stream being a mixture of two streams. In the first input stream the quantities $\{A(t),\ t = 0, 1, \ldots\}$ flowing in at consecutive moments are independent random variables with identical distribution functions $G(y)$, where $G(y) = 0$ for $y \leqslant 0$. In the second input stream the flow appears at random moments $0 = \tau_0 < \tau_1 < \ldots$ with probability distribution of the distances $\{f(n),\ n = 1, 2, \ldots\}$ and forms independent random variables $B(t)$ with identical distribution function $H(y)$, where $H(y) = 0$ for $y \leqslant 0$.

We denote by $X(t)$ the water level in the dam at the moment t including the input of this moment. Since the capacity of the dam is finite, for $X(t) > k$ a quantity $X(t) - k$ overflows. After an overflow, at every moment there flows out of the dam a constant quantity of water $c < k$, unless $X(t) < c$ in which case the quantity equals $X(t)$. This can be expressed by the formula

$$X(t+1) = \begin{cases} \big[\min\big(X(t), k\big) - c\big]_+ + A(t+1), & t+1 \neq \tau_m, \\ \big[\min\big(X(t), k\big) - c\big]_+ + A(t+1) + B(t+1), & t+1 = \tau_m, \end{cases}$$

where $m = 0, 1, \ldots$, and $[a]_+$ denotes $\max(a, 0)$.

From the description of the model it follows that the chain $\{X(t),\ t = 0, 1, \ldots\}$ is a piecewise Markov process with state space $(\mathscr{R}_+, \mathscr{B}_+)$

and with regenerative moments $\{\tau_m,\ m = 0, 1, \ldots\}$. Between regenerative moments the process $\{X(t), \tau_m < t < \tau_{m+1}\}$ is a Markov chain with transition probabilities defined by the formula

$$P\big(x, (0, y]\big) = \begin{cases} G(y), & 0 \leqslant x \leqslant c, \\ G(y-x+c), & c < x < k, \\ G(y-k+c), & k \leqslant x, \end{cases}$$

where $y > 0$.

At the regenerative moments $t = \tau_m,\ m = 0, 1, \ldots,$ besides the Markovian transition with kernel $P\big(x, (0, y]\big)$ we have a regenerative transition with kernel given by the formula

$$Q\big(x, (0, y]\big) = H(y-x), \quad x, y > 0.$$

Using (10) and (11) and writing shortly

$$N(y) = N\big((0, y]\big), \quad N^-(y) = N^-\big((0, y]\big), \quad N^+(y) = N^+\big((0, y]\big)$$

we obtain the following theorem:

THEOREM 6. *In Moran's model of a dam with an additional input stream, the distribution functions $N(y)$, $N^-(y)$, and $N^+(y)$ of the stationary probability distributions defined for the process $\{X(t), t = 0, 1, \ldots\}$ fulfil the relations*

$$N(y) = v\sum_{n=0}^{\infty} F(n) \int_{\mathscr{R}_+} P\big(n, x, (0, y]\big) dN^+(x), \tag{12}$$

$$N(y) - \int_{\mathscr{R}_+} P\big(x, (0, y]\big) dN(x) = v \int_{(0,y]} [1-H(y-x)]\, dN^-(x), \tag{13}$$

where $y > 0$.

The considered model of a dam may be modified in the following way. Assume that there exist random variables $\{\tau_{2m+a}, a = 0, 1, m = 0, 1, \ldots\}$ with probability distributions of the distances denoted by $\{f_a(n),\ n = 0, 1, \ldots\}$, $a = 0, 1$, in which the water demand changes deterministically. In the intervals $\tau_{2m} < t < \tau_{2m+1},\ m = 0, 1, \ldots,$ the demands are equal to c_0 (the output is $\min\big(c_0, X(t)\big)$), and in the intervals $\tau_{2m+1} < t < \tau_{2m+2},\ m = 0, 1, \ldots,$ the demands are equal to $c_1 \neq c_0$. In a particular case $c_0 = 0$, in the intervals $\tau_{2m} < t < \tau_{2m+1},\ m = 0, 1, \ldots,$ there is no output but only accumulation of water.

Under these assumptions, the process $\{X(t), t = 0, 1, \ldots\}$ is an extended piecewise Markov process with regenerative moments $\{\tau_{2m+a}, a = 0, 1, m = 0, 1, \ldots\}$. Theorems 2 and 3 enable us to write relations analogous to (12) and (13).

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UOGÓLNIONE PROCESY PRZEDZIAŁAMI MARKOWA W CZASIE DYSKRETNYM

STRESZCZENIE

Definicja uogólnionych procesów przedziałami Markowa w czasie dyskretnym, metoda ich badania i otrzymane związki są analogiczne do definicji, metody i związków zawartych w [2]. Otrzymane wyniki stosuje się do badania pewnych modyfikacji modelu tamy, podanego przez Morana.
