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MINIMAX CONTROL OF A MULTIVARIATE TIME-CONTINUOUS LINEAR STOCHASTIC SYSTEM

In the paper the problem of minimax control is considered for the linear, time-continuous stochastic system defined in (24) when the loss function is given by (9).

1. Processes with independent increments. Processes belonging to the exponential class with quadratic variance function. Let $\zeta(t)$, $t \geq 0$, be a multidimensional, homogeneous, stochastically continuous process with independent increments and finite moments of the second order. This process admits the representation

$$(1) \quad \zeta(t) = Mw(t) + \int_{R^l} y\tilde{v}(t, dy) + \mu(t) = \zeta_c(t) + \zeta_n(t) + \mu(t),$$

where M is a matrix with constant entries, $w(t)$, $t \geq 0$, is a multidimensional Wiener process,

$$(2) \quad \tilde{v}(t, A) = v(t, A) - tq(A),$$

$v(t, A)$ is a Poisson measure on (R^l, \mathcal{B}_0^l) , \mathcal{B}_0^l is a σ -field consisting of Borel sets in R^l such that their closures do not contain the point 0,

$$(3) \quad tq(A) = E(v(t, A)).$$

The processes $\zeta_c(t)$ and $\zeta_n(t)$ are independent.

Obviously, $\mu(t) = E(\zeta(t))$.

The meaning of the measure $v(t, A)$ is the following: $v(t, A)$ is equal to the number of jumps of the process $\zeta(t)$ with the values in A in the interval of time $(0, t]$.

Let \mathcal{B}^k be the σ -field of Borel sets in R^k , $\mathcal{B}^1 = \mathcal{B}$.

The process $\zeta(t)$, $t \geq 0$, belongs to the exponential class if

(a) $\zeta(t) = \{\zeta_1(t), \dots, \zeta_k(t)\}$, where the processes $\zeta_1(t), \dots, \zeta_k(t)$ are independent and take their values in (R^1, \mathcal{B}) ;

(b) $\zeta_i(t)$, $t \geq 0$, are homogeneous, stochastically continuous processes with independent increments, $P_i^{(0)}(\{0\}) = 1$, where $P_i^{(t)}(\cdot)$ are the measures induced by the random variables $\zeta_i(t)$, respectively, $E(\zeta_i^2(t)) < \infty$;

(c) there are σ -finite measures μ_i defined on (R^1, \mathcal{B}) such that

$$\frac{dP_i^{(t)}}{d\mu_i}(z_i) = S_i(z_i, t) \exp \{A_i(\lambda)t + B_i(\lambda)z_i\},$$

where $\lambda \in \Lambda$ is a parameter.

It is assumed that a natural parametrization is chosen for all measures $P_i^{(t)}$ and that the variances $D^2(\zeta_i(t))$ are quadratic in λ . Then we can assume that

$$(4) \quad E(\zeta_i(t)) = r_i \lambda t, \quad D^2(\zeta_i(t)) = (\alpha_{1i} \lambda^2 + \alpha_{2i} \lambda + \alpha_{3i})t \quad (i = 1, 2, \dots, k),$$

where $r_i > 0$, α_{1i} , α_{2i} , α_{3i} are constants.

Random variables and processes belonging to the exponential class are considered in [1], [2], [4] and [6].

In [6] it is proved that there are only five infinitely divisible distributions (and linear transformations of them) belonging to the exponential class for which the variance is a quadratic function of the mean. Thus there are only five one-dimensional processes belonging to the exponential class with quadratic variance function and also processes belonging to this class obtained from the above by a linear transformation of the phase space. (By a *linear transformation* we mean here a transformation of the form $y = ax + b$ in R^1 , where a and b are constants.) These five processes are the following (the measure μ with respect to which the densities $dP^{(t)}(z)/d\mu$ are given is the Lebesgue or the counting measure):

(a) The Wiener process with drift:

$$\frac{dP^{(t)}}{d\mu}(z) = \frac{1}{\sqrt{2\pi t}} \exp \left[-\frac{(z - \lambda t)^2}{2t} \right].$$

By a *Wiener process* we mean the process $w(t) = w_1(t) - \lambda t$, where the distribution of the process $w_1(t)$ is defined above. Almost all sample functions of a Wiener process are continuous.

(b) The Poisson process:

$$\frac{dP^{(t)}}{d\mu}(z) = \frac{(r\lambda t)^z}{z!} e^{-r\lambda t}.$$

Almost all sample functions of the Poisson process are nondecreasing, integer-valued and they increase by jumps of magnitude 1. This is the only process with independent increments which has these properties.

(c) The negative binomial process:

$$\frac{dP^{(t)}}{d\mu}(z) = \frac{\Gamma(rt + z)}{\Gamma(rt) z!} \frac{\lambda^z}{(1 + \lambda)^{rt + z}}.$$

Almost all sample functions of this process are nondecreasing and integer-valued but the magnitudes of jumps may be different from 1.

(d) The gamma process:

$$\frac{dP^{(t)}}{d\mu}(z) = \frac{1}{\Gamma(rt) \lambda^{rt}} z^{rt-1} e^{-z/\lambda} I_{(0, \infty)}(z),$$

where I_A is the characteristic function of the set A . Almost all sample functions of this process are nondecreasing functions of jumps.

(e) The generalized exponential hyperbolic secant process (the GEHS process):

$$\frac{dP^{(t)}}{d\mu}(z) = \frac{1}{(1 + \lambda^2)^{rt/2}} \exp(z \operatorname{arctg} \lambda) S(z, t),$$

where

$$\begin{aligned} S(z, t) &= \frac{2^{rt-2}}{\pi} B\left(\frac{rt}{2} - i\frac{z}{2}, \frac{rt}{2} + i\frac{z}{2}\right) \\ &= \frac{2^{rt-2} \Gamma^2(rt/2)}{\pi \Gamma(rt)} \prod_{k=0}^{\infty} \left(1 + \frac{z^2}{(rt + 2k)^2}\right)^{-1} \end{aligned}$$

Almost all sample functions of this process are functions of jumps which may be positive and negative.

For the processes listed in (b)–(e) the measure q defined in (3) is the following:

(b) $q(\{z\}) = r I_{\{1\}}(z),$

(c) $q(\{z\}) = r \left(\frac{\lambda}{1 + \lambda}\right)^z \frac{1}{z} I_N(z),$ where $N = \{1, 2, \dots\},$

(d) $q(dz) = r \frac{e^{-z/\lambda}}{z} I_{(0, \infty)}(z) dz,$

(e) $q(dz) = r \frac{\exp(z \operatorname{arctg} \lambda)}{2z \sinh \pi z/2} dz.$

The measure q in (b) and (c) is discrete, and in (d) and (e) is absolutely continuous.

From (d) and (e) it follows that for gamma and GEHS processes, the expected number of jumps in an interval of time of positive length is infinite.

2. Stochastic system. Let us consider a time-continuous stochastic system defined by the stochastic differential equation

(5) $d\xi = a(t) \xi dt + \tilde{a}(t) u dt + d\eta + \varphi(t) dt, \quad \xi(0) = \xi_0,$

where ξ is the state process, u is the control,

$$(6) \quad d\eta = \sum_{i=1}^m b_i(t) dw_i(t) + \int_{R^d} c(t, y) \tilde{\nu}(dt, dy) = d\eta_1 + d\eta_2, \quad \eta(0) = 0,$$

$b_i(t)$ and $c(t, y)$ are vector functions, $w_i(t)$ are standard Wiener processes, $\tilde{\nu}(t, A)$ is the random measure of the process $\zeta_n(t)$ defined in (1) and (2),

$$(7) \quad \varphi(t) = \sum_{i=1}^m b_i(t) \lambda + \int_{R^d} c(t, y) q(dy),$$

$q(A)$ is defined in (3), and λ is a (known) parameter.

Here $a(t)$ and $\tilde{a}(t)$ are $(d \times d)$ - and $(d \times \tilde{d})$ -matrices, respectively, and the column vectors $b_i(t)$ and $c(t, y)$ have dimension d .

It is assumed that the process $\eta(t)$ has finite moments of the second order, and the matrices $a(t)$, $\tilde{a}(t)$ and $\varphi(t)$ have entries uniformly bounded and belonging to the space D .

Assume that $\eta(t)$ is a standard process. Let \mathcal{F}_t be the least σ -field with respect to which all $\eta(s)$, $s \leq t$, are measurable. We say that the function $u(t)$ belongs to \mathcal{C} if it is a Borel function and equation (5) has for the control $u(t)$ a unique strong solution $\xi(t, u(\cdot))$ such that almost all realizations of the process $\xi(t, u(\cdot))$, $0 \leq t \leq T$, belong to the space D . Denote by $\{\mathcal{G}_t\}$ the current of σ -fields, $\mathcal{G}_t \subset \mathcal{F}_t$, generated by the variables $\xi(s, u(\cdot))$, $s \leq t$, for all possible $u(\cdot) \in \mathcal{C}$. Any family of functionals

$$u(t, x(\cdot)) = \{u_1(t, x(\cdot)), \dots, u_{\tilde{d}}(t, x(\cdot))\}, \quad t \in [0, T], \quad x(\cdot) \in D,$$

with values in $U = R^{\tilde{d}}$ is called a *control policy* if

- (a) for each $x(\cdot) \in D$, $u(t, x(\cdot))$ is a Borel function of t ;
- (b) for each $t \in [0, T]$, $u_i(t, x(\cdot))$ depend only on $x(s)$ for $s \leq t$;
- (c) the process $u(t, \xi(\cdot)) = \{u_1(t, \xi(\cdot)), \dots, u_{\tilde{d}}(t, \xi(\cdot))\}$, $t \in [0, T]$, ξ being the solution of (5) given $u(t, \xi(\cdot))$, is defined, the random variables $u(t, \xi(\cdot))$ are measurable with respect to $\mathcal{G}_{t-0} = \bigcup_{s < t} \mathcal{G}_s$, the processes $u_i(t, \xi(\cdot))$, $t \in [0, T]$, $i = 1, \dots, \tilde{d}$, are measurable and the random variables $u_i(t, \xi(\cdot))$ are \mathcal{F}_t^{ξ} -measurable⁽¹⁾;
- (d) we have

$$(8) \quad E \int_0^T \sum_{j=1}^{\tilde{d}} (u_j(t, \xi(\cdot)))^4 dt < \infty.$$

We call also the process $u(t, \xi(\cdot))$ a *control policy*.

Control policies $u = u(t, \xi(\cdot))$ for which equation (5) has a unique strong solution $\xi(t)$ are called *admissible*.

⁽¹⁾ \mathcal{F}_t^{ξ} is the least σ -field with respect to which all $\xi(s)$, $s \leq t$, are measurable.

Put $\gamma(t) = u(t, \zeta(\cdot))$. A control policy $\gamma(t)$, $t \in [0, T]$ is called a *step control policy* if there exist $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ such that $\gamma(t) = \gamma_k$ for $t \in (t_k, t_{k+1}]$, where γ_k is a \mathcal{G}_{t_k} -measurable random variable with values in U . The class of all admissible step control policies with given t_1, \dots, t_n is denoted by $\mathcal{U}(t_1, \dots, t_n)$.

The class of all admissible control policies is denoted by \mathcal{U} , the class of all admissible step control policies by \mathcal{U}_0 , and the class of all admissible control and step control policies in the interval $[t, T]$ by $\mathcal{U}[t, T]$ and $\mathcal{U}_0[t, T]$, respectively. Obviously,

$$\mathcal{U}_0 = \bigcup_{(t_1, \dots, t_n)} \mathcal{U}(t_1, \dots, t_n).$$

The matrix transposed to the matrix A is denoted by A' .

Define the loss function in the interval $[t, T]$ as

$$(9) \quad F_t(x(\cdot), u(\cdot)) = [x(T)', \lambda] S [x(T)', \lambda]' + \int_t^T [[x(s)', \lambda] F(s) [x(s)', \lambda]' + u(s)' \tilde{F}(s) u(s)] ds \\ \stackrel{\text{df}}{=} [x(T)', \lambda] S [x(T)', \lambda]' + \int_t^T f(t, x, u) ds,$$

where $u(s)$ is the control, $[x(s)', \lambda]$ is the vector $x(s)'$ with the coordinate λ added.

We assume that S is a $((d+1) \times (d+1))$ -matrix nonnegative definite, $F(s)$ and $\tilde{F}(s)$ are $((d+1) \times (d+1))$ - and $(\bar{d} \times \bar{d})$ -matrices, respectively, nonnegative definite, with entries uniformly bounded and continuous. Moreover, we assume that the matrix $\tilde{F}(t)$ is positive definite uniformly in t ,

$$(10) \quad (\tilde{F}(t)u, u) \geq c|u|^2 \quad \text{for each } t \in [0, T], u \in R^{\bar{d}},$$

where (a, b) is the scalar product of the (column) vectors a and b , and $c > 0$ is a constant.

For stochastic equations and systems see [3] and [5].

3. Optimal control. Define the risk function for control policy u as

$$R(\lambda, u) = E(F_0(\zeta(\cdot), u(t, \zeta(\cdot))))$$

where ζ is the solution of (5) given u .

We say that the control policy u belongs to Δ ($u \in \Delta$) if it is admissible for each $\lambda \in \Lambda$. From (8) it follows that if $u \in \Delta$, then $R(\lambda, u)$ exists for all $\lambda \in \Lambda$.

Let us introduce the function being the optimal cost of control in the interval $[t, T]$:

$$(11) \quad Z(t, x(\cdot)) = \text{ess inf}_{\gamma \in \mathcal{U}[t, T]} E(F_t(\zeta_t^{(\gamma)}(\cdot), \gamma(\cdot))),$$

where $\gamma(t)$ is the control and $\xi_t^{(\gamma)}(s) \equiv x(s)$, $s \in [0, t]$, $\xi_t^{(\gamma)}(\cdot)$ being the solution of (5) for the control γ .

From the assumptions it follows that

$$Z(t, x(\cdot)) = \operatorname{ess\,inf}_{\gamma \in \mathcal{U}_0[t, T]} E(F_t(\xi_t^{(\gamma)}(\cdot), \gamma(\cdot)))$$

and that there is a Markovian control $\gamma(t) = u(t, \xi(t))$ such that the essential infimum in (11) is attained for $\gamma(t)$, and $Z(t, x) = Z(t, x(t))$ satisfies the generalized Bellman equation

$$(12) \quad -\frac{\partial Z(t, x)}{\partial t} = \inf_{u \in U} \{ \hat{L}_u Z(t, x) + f(t, x, u) \},$$

$$Z(T, x) = (x', \lambda) S(x', \lambda)',$$

assuming this equation has a solution. \hat{L}_u denotes here the infinitesimal operator of the process $Z(t, \xi(t))$ corresponding to the process $\xi(t)$ for a given u (see [3], p. 180).

This theorem is applicable for the control system with $\varphi(t) = 0$, but putting $\tilde{a}\hat{u} = \tilde{a}u + \varphi$, i.e. $\hat{u} = u + \tilde{a}^{-} \varphi$, in equation (5) and in the loss function (\tilde{a}^{-} is a pseudoinverse matrix to the matrix \tilde{a}) we can apply this theorem and get the equation for $\varphi \neq 0$ defined in (7), u being the control under consideration,

$$(13) \quad L_0(Z) + (x', \lambda) F(x', \lambda)' + \inf_u [(\nabla Z, \tilde{a}u) + u' \tilde{F}u] = 0,$$

where

$$L_0(Z) = L_0^{(1)}(Z) + K(t) + (\nabla Z, \varphi),$$

$$L_0^{(1)}(Z) = \frac{\partial Z}{\partial t} + (\nabla Z, ax) + \frac{1}{2} \operatorname{Sp}(B \nabla^2 Z),$$

$$K(t) = \int_{R^d} [Z(t, x + c(t, y)) - Z(t, x) - (\nabla Z(t, x), c(t, y))] q(dy),$$

∇Z is the vector with coordinates $\partial Z / \partial x_i$, $x = \{x_1, \dots, x_d\}$, $\nabla^2 Z$ is the matrix with entries $\partial^2 Z / \partial x_i \partial x_j$, and

$$B(t) = \sum_{i=1}^m b_i(t) b_i(t)'$$

We seek the solution of equation (13) in the form

$$(14) \quad Z(t, x) = x' D(t) x + 2E(t) x \lambda + G(t) \lambda^2 + H(t) \lambda + I(t),$$

where $D(t)$ is a symmetric matrix nonnegative definite. Under this assumption the expression $(\nabla Z, \tilde{a}u) + u' \tilde{F}u$ attains its infimum in u if

$$\tilde{a} (Dx + E' \lambda) + \tilde{F}u = 0,$$

which gives

$$(15) \quad u = -Px - Q\lambda,$$

where

$$(16) \quad P = \tilde{F}^{-1} \tilde{a} D, \quad Q = \tilde{F}^{-1} \tilde{a}' E'.$$

Introducing expressions (14) and (15) into (13) we obtain

$$(17) \quad x' \frac{dD}{dt} x + 2 \frac{dE}{dt} x \lambda + \frac{dG}{dt} \lambda^2 + \frac{dH}{dt} \lambda + \frac{dI}{dt} + \\ + x' (a' D + Da) x + 2Eax\lambda + Sp(DB) + \\ + 2\varphi' Dx + 2E\varphi\lambda + \int_{R^d} c(t, y)' D(t) c(t, y) q(dy) + \\ + x' F^{(1)} x + 2fx\lambda + f_0 \lambda^2 - \\ - 2(x' D\tilde{a}Px + Q' \tilde{a}' Dx\lambda + E\tilde{a}Px\lambda + E\tilde{a}Q\lambda^2) + \\ + x' P' \tilde{F} Px + 2Q' \tilde{F} Px\lambda + Q' \tilde{F} Q\lambda^2 = 0,$$

where

$$F = \begin{bmatrix} F^{(1)} & f' \\ f & f_0 \end{bmatrix},$$

f_0 being a scalar.

The boundary condition (12) should be adjoined to equation (17).

Equation (17) holds surely if $D(t)$ satisfies the matrix equation

$$\frac{dD}{dt} + a' D + Da + F^{(1)} - 2D\tilde{a}P + P' \tilde{F} P = 0$$

and the expressions by $x\lambda$, λ^2 , λ , 1 in (17) are equal to 0. (The function φ and the integral in (17) depend on λ . It is assumed that φ is a linear function of λ and the integral is a quadratic function of this variable.)

Taking into account (16) we obtain

$$(18) \quad \frac{dD}{dt} + a' D + D' a - D\tilde{a}\tilde{F}^{-1} \tilde{a}' D + F^{(1)} = 0$$

with the boundary condition obtained from (12) and (14):

$$(19) \quad D(T) = S^{(1)},$$

where

$$(20) \quad S = \begin{bmatrix} S^{(1)} & s' \\ s & s_0 \end{bmatrix},$$

s_0 being a scalar. Thus any solution $D(t)$ of equation (18) with the boundary condition (19) is symmetric.

Equation (18) is very familiar in control theory. It is well known that the solution of this equation with the boundary condition (19) exists and is nonnegative definite.

Now we give the condition under which the process $\eta_2(t)$ in (6) surely satisfies the assumptions

$$(21a) \quad \varphi(t) = \sum_{i=1}^m b_i(t) \lambda + \int_{R^d} c(t, y) q(dy) = \varphi_1(t) + \varphi_2(t) \text{ is linear in } \lambda,$$

$$(21b) \quad \int_{R^d} c(t, y)' D(t) c(t, y) q(dy) \text{ is quadratic in } \lambda.$$

Let us put $y = ex$, where e is a $(d \times l)$ -matrix with constant entries, and $c(t, y) = c(t) y$, where $c(t)$ is a $(d \times d)$ -matrix. Then

$$(22) \quad \begin{aligned} \eta_2(t) &= \int_0^t \int_{R^d} c(t, y) \tilde{v}(dt, dy) \\ &= \int_0^t \int_{R^l} c(t) ex \frac{d\tilde{v}}{d\tilde{v}^*}(t, x) \tilde{v}^*(dt, dx). \end{aligned}$$

Let v be such that

$$(i) \quad (d\tilde{v}/d\tilde{v}^*)(t, x) = 1;$$

(ii) the measure $\tilde{v}^*(t, \cdot) = \tilde{v}_1(t, \cdot) \cup \dots \cup \tilde{v}_l(t, \cdot)$ is concentrated on the subspace

$$(R^1 \times 0 \times \dots \times 0) \cup (0 \times R^1 \times \dots \times 0) \cup \dots \cup (0 \times 0 \times \dots \times R^1)$$

(0 is the null element in R^1); moreover, if

$$tq^*(A \times B) = E(v^*(t, A \times B)), \quad \tilde{v}^*(t, C) = v^*(t, C) - tq^*(C),$$

then

$$tq^*(A_1 \times 0 \times \dots \times 0) = tq_1(A_1) \quad \text{if } A_1 \in \mathcal{B}_0^1,$$

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$$tq^*(0 \times 0 \times \dots \times A_l) = tq_l(A_l) \quad \text{if } A_l \in \mathcal{B}_0^l;$$

(iii) $\tilde{v}_i(t, A_i) = \tilde{v}_i(t, 0 \times \dots \times A_i \times \dots \times 0)$, $\tilde{v}_i(t, A_i) = v_i(t, A_i) - tq_i(A_i)$, $v_i(t, A_i)$ is a Poisson measure for the one-dimensional process $\zeta_i(t)$ belonging

to the exponential class with parameter λ satisfying conditions (4), $\zeta_i(t)$, $i = 1, \dots, l$, are independent,

$$\zeta_n(t) = (\zeta_1(t) - r_1 \lambda t, \dots, \zeta_l(t) - r_l \lambda t).$$

Under these assumptions, from (22) we obtain

$$d\eta_2 = c(t) e \int_{R^l} x \bar{v}^*(dt, dx) = c(t) e d\zeta_n \stackrel{\text{df}}{=} \bar{c}(t) d\zeta_n$$

and

$$(23) \quad \varphi_2(t) = c(t) e \int_{R^l} x q^*(dx) = c(t) e r \lambda \stackrel{\text{df}}{=} \hat{c}(t) \lambda.$$

Here $r = [r_1, \dots, r_l]'$, where the r_i are defined in (4). Equation (5) takes the form

$$(24) \quad d\xi = a(t) \xi dt + \bar{a}(t) u dt + \sum_{i=1}^m b_i(t) dw_i(t) + \bar{c}(t) d\zeta_n(t) + \left(\sum_{i=1}^m b_i(t) + \bar{c}(t) r \right) \lambda dt, \quad \xi(0) = \xi_0.$$

Moreover,

$$(25) \quad \int_{R^d} c(t, y)' D(t) c(t, y) q(dy) = \int_{R^l} x' \text{diag}(\bar{c}(t)' D(t) \bar{c}(t)) x q^*(dx) = \alpha'_{1,1/2} \text{diag}(\bar{c}(t)' D(t) \bar{c}(t)) \alpha_{1,1/2} \lambda^2 + \alpha'_{2,1/2} \text{diag}(\bar{c}(t)' D(t) \bar{c}(t)) \alpha_{2,1/2} \lambda + \alpha'_{3,1/2} \text{diag}(\bar{c}(t)' D(t) \bar{c}(t)) \alpha_{3,1/2},$$

where

$$\alpha_{k,1/2} = \begin{bmatrix} \sqrt{\alpha_{k1}} \\ \dots \\ \sqrt{\alpha_{kl}} \end{bmatrix} \quad (k = 1, 2, 3),$$

α_{ki} are defined in (4).

Comparing the terms by $x\lambda$, λ^2 , λ , 1 in (17) and taking into account equations (21), (23) and (25), we obtain

$$(26a) \quad \frac{dE}{dt} + E(a - \bar{a}P) + \sum_{i=1}^m (b_i + \hat{c}) D + f = 0,$$

$$(26b) \quad \frac{dG}{dt} - E\bar{a}Q + 2E \left(\sum_{i=1}^m b_i + \hat{c} \right) + f_0 + \alpha'_{1,1/2} \text{diag}(\bar{c} D \bar{c}) \alpha_{1,1/2} = 0,$$

$$(26c) \quad \frac{dH}{dt} + \alpha'_{2,1/2} \text{diag}(\tilde{c}' D \tilde{c}) \alpha_{2,1/2} = 0,$$

$$(26d) \quad \frac{dI}{dt} + \alpha'_{3,1/2} \text{diag}(\tilde{c}' D \tilde{c}) \alpha_{3,1/2} + \text{Sp}(DB) = 0$$

with the boundary conditions obtained from (12) as

$$(27) \quad E(T) = s, \quad G(T) = s_0, \quad H(T) = I(t) = 0.$$

Equations (26b), (26c) and (26d) have solutions for any E and D . Equation (18) has a solution. From [3], pp. 196–199, it follows that equation (17) has a solution. Thus also equation (26a) has a solution.

From this discussion it follows that the function $Z(t, x)$ given by (14) with D , E , G , H and I defined by (18) and (26) and by the boundary conditions (19) and (27) is the optimal cost of control of the system (24) and that the control policy obtained by (15) and (16) is the optimal control policy.

Obviously, if u is the optimal control policy for a given parameter λ , then for this parameter we have $R(\lambda, u) = Z(0, \xi_0)$.

4. Determining the risk. Let $u_{\lambda_0} = -Px - Q\lambda_0$ and let $\lambda \in \Lambda$. In this section we consider the problem of determining the risk $R(\lambda, u_{\lambda_0})$.

Let

$$Z_\lambda(t, x) = E(F_t(\xi_t^{(\gamma)}(\cdot), \gamma(\cdot)))$$

for $\gamma(t) = -P(t)\xi(t) - Q(t)\lambda_0$, $\xi_t^{(\gamma)}(t) = x$, $\xi_t^{(\gamma)}(\cdot)$ being the solution of (5) for the control γ . We have

$$(28) \quad R(\lambda, u_{\lambda_0}) = Z_\lambda(0, \xi_0)$$

and $Z_\lambda(t, x)$ satisfies the equation

$$L_0(Z_\lambda) + (x', \lambda) F(x', \lambda)' + (\nabla Z_\lambda, \tilde{a}u_{\lambda_0}) + u'_{\lambda_0} \tilde{F} u_{\lambda_0} = 0.$$

In the same way as in the previous section we prove that

$$(29) \quad Z_\lambda(t, x) = x' \hat{D}(t)x + 2\hat{E}(t)x\lambda + \hat{G}(t)\lambda^2 + \hat{H}(t)\lambda + \hat{I}(t) + \\ + \hat{J}(t)\lambda_0^2 + 2\hat{K}(t)\lambda_0\lambda,$$

where

$$\frac{d\hat{D}}{dt} + a' \hat{D} + \hat{D}a - \hat{D}\tilde{a}\tilde{F}^{-1}\tilde{a}' \hat{D} + F^{(1)} = 0,$$

$$\frac{d\hat{E}}{dt} + \hat{E}(a - \tilde{a}P) + \left(\sum_{i=1}^m b_i + \tilde{c}' \right) \hat{D} + f = 0,$$

$$(30) \quad \begin{aligned} \frac{d\hat{G}}{dt} + 2\hat{E} \left(\sum_{i=1}^m b_i + \hat{c} \right) + f_0 + \alpha'_{1,1/2} \text{diag}(\bar{c}' \hat{D} \bar{c}) \alpha_{1,1/2} &= 0, \\ \frac{d\hat{H}}{dt} + \alpha'_{2,1/2} \text{diag}(\bar{c}' \hat{D} \bar{c}) \alpha_{2,1/2} &= 0, \end{aligned}$$

$$\frac{d\hat{I}}{dt} + \alpha'_{3,1/2} \text{diag}(\bar{c}' \hat{D} \bar{c}) \alpha_{3,1/2} + \text{Sp}(\hat{D}B) = 0,$$

$$\frac{d\hat{J}}{dt} + Q' \hat{F} Q = 0, \quad \frac{d\hat{K}}{dt} - \hat{E} \bar{a} Q = 0$$

and

$$(31) \quad \begin{aligned} \hat{D}(T) &= S^{(1)}, \quad \hat{E}(T) = s, \quad \hat{G}(T) = s_0, \\ \hat{H}(T) &= \hat{I}(T) = \hat{J}(T) = \hat{K}(T) = 0. \end{aligned}$$

Then

$$\hat{D}(t) = D(t), \quad \hat{E}(t) = E(t)$$

and the functions $\hat{G}(t)$, $\hat{H}(t)$, $\hat{I}(t)$, $\hat{J}(t)$ and $\hat{K}(t)$ are the corresponding integrals.

From (28) and (29) it follows that

$$(32) \quad \begin{aligned} R(\lambda, u_{\lambda_0}) &= \hat{G}(0) \lambda^2 + (2E(0) \zeta_0 + \hat{H}(0) + 2\hat{K}(0) \lambda_0) \lambda + \\ &\quad + \zeta'_0 D(0) \zeta_0 + \hat{I}(0) + \hat{J}(0) \lambda_0^2 \\ &\stackrel{\text{df}}{=} \hat{G}(0) \lambda^2 + Y_1(\lambda_0) \lambda + Y_2(\lambda_0). \end{aligned}$$

5. Small Horizon Minimax Law. Let the parameter $\lambda \in \Lambda_0 \subset \Lambda$ be unknown. Let Δ_0 be the set of all control policies admissible for all $\lambda \in \Lambda_0$. A control policy $u^{(0)} \in \Delta_0$ is *minimax* (for Λ_0) if

$$\sup_{\lambda \in \Lambda_0} R(\lambda, u^{(0)}) = \inf_{u \in \Delta_0} \sup_{\lambda \in \Lambda_0} R(\lambda, u).$$

The parameter λ sometimes is a random variable with the a priori distribution π . Let \mathcal{B}_Λ be the set of Borel sets in Λ . Assume that for given $u \in \Delta$ the function $R(\lambda, u)$ is measurable with respect to \mathcal{B}_Λ . The functional

$$r(\pi, u) = \int R(\lambda, u) \pi(d\lambda)$$

is called the *Bayes risk*.

We sometimes have the information that $\pi \in \Gamma$, where Γ is known. Let Δ^* be the set of these control policies $u \in \Delta$ for which the risk $R(\lambda, u)$ is measurable with respect to \mathcal{B}_Λ . The control policy $u^{(1)}$ for which

$$\sup_{\pi \in \Gamma} r(\pi, u^{(1)}) = \inf_{u \in \Delta^*} \sup_{\pi \in \Gamma} r(\pi, u)$$

and for which this supremum is finite is called a Γ -*minimax control policy*.

In this paper Γ is the set of all a priori distributions π for which $E_\pi(\lambda^2) \leq m_2$, where $m_2 > 0$ is given.

Let (i_1, i_2, \dots, i_k) be a permutation of $(1, 2, \dots, k)$. The process

$$\begin{aligned}
 (33) \quad \zeta(t) &= \{\zeta_n(t), w_1(t), \dots, w_m(t)\} \\
 &= \{\zeta_{i_1}(t) - r_{i_1} \lambda t, \dots, \zeta_{i_l}(t) - r_{i_l} \lambda t, w_1(t), \dots, w_m(t)\} \\
 &= \{\zeta_{i_1}(t) - r_{i_1} \lambda t, \dots, \zeta_{i_l}(t) - r_{i_l} \lambda t, \zeta_{i_{l+1}}^{(0)} - \lambda t, \dots, \zeta_{i_k}^{(0)} - \lambda t\} \\
 &= \{\zeta_{i_1}(t) - r_{i_1} \lambda t, \dots, \zeta_{i_k}(t) - r_{i_k} \lambda t\}
 \end{aligned}$$

in (24) belongs to \mathcal{E}_1 if the processes $\zeta_1(t), \dots, \zeta_k(t)$ are independent, $\zeta_i(t)$ ($i = 1, \dots, k$) are the Wiener processes with drift or the GEHS processes (some of them are Wiener, some GEHS).

The process $\zeta(t)$ belongs to \mathcal{E}_2 if the processes $\zeta_1(t), \dots, \zeta_k(t)$ are independent, $\zeta_i(t)$ ($i = 1, \dots, k$) is one of the five processes mentioned in Section 1. (For different i the processes $\zeta_i(t)$ may be different, for example, $\zeta_1(t)$ may be the Poisson process and $\zeta_2(t)$ the Wiener process with drift.)

Consider a zero-sum two-person game (A, B, W) in the normal form, where A and B are the sets of strategies of Players I and II, respectively, and $W(a, b)$ is the payoff function defined on the product $A \times B$ (Player I pays W to Player II). The strategy a_0 is *relatively optimal with respect to* b_0 if

$$W(a_0, b_0) = \min_{a \in A} W(a, b_0).$$

Similarly, the strategy b_0 is *relatively optimal with respect to* a_0 if

$$W(a_0, b_0) = \max_{b \in B} W(a_0, b).$$

Considering the game (Δ^*, Γ, r) , where r is the Bayes risk, we obtain the following theorem:

SMALL HORIZON MINIMAX LAW. (i) *If the process $\zeta(t)$ in (24) defined by (33) belongs to \mathcal{E}_1 and $Y_1(\lambda) \geq 0$ for all $\lambda \in [-\sqrt{m_2}, \sqrt{m_2}]$, then the Γ -minimax control policy is $u_{m_2^{1/2}}$.*

(ii) *If $\zeta(t) \in \mathcal{E}_2$, $\lambda \in (0, \infty)$ and $Y_1(\lambda) \geq 0$ for all $\lambda \in (0, m_2]$, then the Γ -minimax control policy is $u_{m_2^{1/2}}$.*

(iii) *If $\zeta(t) \in \mathcal{E}_1$ and $Y_1(\lambda) \leq 0$ for all $\lambda \in [-\sqrt{m_2}, \sqrt{m_2}]$, then the Γ -minimax control policy is $u_{-m_2^{1/2}}$.*

Proof. From (32) we obtain

$$R(\lambda, u_{m_2^{1/2}}) = \hat{G}(0) \lambda^2 + Y_1(\sqrt{m_2}) \lambda + Y_2(\sqrt{m_2}),$$

where \hat{G} , Y_1 and Y_2 do not depend on λ . But $\hat{G}(0) \geq 0$, since otherwise $R(\lambda, u_{m_2^{1/2}}) \rightarrow -\infty$ when $\lambda \rightarrow \infty$, which is impossible because the loss

function is nonnegative. Assume that case (i) or (ii) occurs. Then for $\pi \in \Gamma$ we have

$$\begin{aligned} r(\pi, u_{m_2^{1/2}}) &= \hat{G}(0) E_\pi(\lambda^2) + Y_1(\sqrt{m_2}) E_\pi(\lambda) + Y_3(\sqrt{m_2}) \\ &\leq R(\sqrt{m_2}, u_{m_2^{1/2}}), \end{aligned}$$

which proves that the distribution $\pi_{m_2^{1/2}}$ of the parameter λ concentrated at the point $\lambda = \sqrt{m_2}$ is relatively optimal with respect to $u_{m_2^{1/2}}$. Since the control policy $u_{m_2^{1/2}}$ is optimal if $\lambda = \sqrt{m_2}$, it is minimax.

Assume that case (iii) occurs. We have

$$\begin{aligned} r(\pi, u_{-m_2^{1/2}}) &= \hat{G}(0) E_\pi(\lambda^2) + Y(-\sqrt{m_2}) E_\pi(\lambda) + Y_3(-\sqrt{m_2}) \\ &\leq R(-\sqrt{m_2}, u_{-m_2^{1/2}}) \end{aligned}$$

and $u_{-m_2^{1/2}}$ is minimax by the same arguments.

From (10), (30) and (31) it follows that if T is very small and λ is not great, then $Y_1(\lambda) \simeq 2E(0)\xi_0$, and if $E(0)\xi_0 \neq 0$ and m_2 is not great, then the function $Y_1(\lambda)$ is positive or negative for all $\lambda \in [-\sqrt{m_2}, \sqrt{m_2}]$. The name of the Theorem follows from this.

COROLLARY. (i) If the process $\zeta(t) \in \mathcal{E}_1$, $\lambda \in [-k, k]$ and $Y_1(\lambda) \geq 0$ for all $\lambda \in [-k, k]$, then the minimax control policy is u_k .

(ii) If $\zeta(t) \in \mathcal{E}_2$, $\lambda \in (0, k]$ and $Y_1(\lambda) \geq 0$ for all $\lambda \in (0, k]$, then the minimax control policy is u_k .

(iii) If $\zeta(t) \in \mathcal{E}_1$, $\lambda \in [-k, k]$ and $Y_1(\lambda) \leq 0$ for all $\lambda \in [-k, k]$, then the minimax control policy is u_{-k} .

Proof. Consider the game (Δ_0, Λ_0, R) , where $\Lambda_0 = [-k, k]$ in cases (i) and (iii), and $\Lambda_0 = (0, k]$ in case (ii). The strategies considered in the proof of the Theorem remain relatively optimal here (for $k = \sqrt{m_2}$). Thus the Corollary is proved.

For problems of minimax control of discrete-time stochastic systems with disturbances belonging to the exponential class see [7] and [8].

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