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ADMISSIBILITY OF HENDERSON'S ESTIMATORS FOR THREE VARIANCE COMPONENTS

It is proved that Henderson's estimators for three variance components in a random model are admissible in the class of all unbiased quadratic estimators.

1. Introduction. Let y be an $n\times 1$ random vector of the form

$$y = \sum_{i=1}^{k} X_i \beta_i,$$

where the matrices $X_1, \ldots, X_k$ are known, $r(X_1, \ldots, X_{k-1}) < n$, and β_i $(i = 1, \ldots, k)$ are random vectors.

Assume that β_i $(i = 1, \ldots, k)$ are stochastically independent, have multivariate normal densities with the expected value zero and covariance matrix $\sigma_i I_i$, where $\sigma_1, \ldots, \sigma_k$ are the unknown parameters. Moreover, suppose that the matrices $V_1 = X_1 X_1', \ldots, V_k = X_k X_k'$ are linearly independent, commute and $V_k = I$. We prove the admissibility of Henderson's estimators for $k = 3$ variance components. This is done as follows. First, we show that the minimal sufficient statistics for the family of distributions induced by y is of the form

$$z = (y'A_1 y, \ldots, y'A_s y)',$$

where the matrices $A_1, \ldots, A_s$ form a basis of the minimal quadratic subspace containing the space $\mathscr{V} = \mathrm{sp}\{V_1, \ldots, V_k\}$. The explicit formulas for $A_1, \ldots, A_s$ are given in Lemma 1. It is also shown that Henderson's estimators of σ_i are of the form $a_i' z$ with a_i $(i = 1, 2, 3)$ as in formulas (1)-(3) below. Now, applying the La Motte characterization of admissible linear unbiased estimators [1], we obtain the admissibility of Henderson's estimators. This admissibility for two variance components was considered by Olsen et al. [3].

2. Construction of the minimal sufficient statistics for the family of distributions of y. Let

$$\mathscr{V} = \mathrm{sp}\{V_1, \ldots, V_k\},$$

$$\Omega = \{\sigma = (\sigma_1, \ldots, \sigma_k)';\ \sigma_i \geqslant 0 \text{ for } i = 1, \ldots, k-1,\ \sigma_k > 0\},$$

and let $\bar{\Omega}$ be the closure of Ω. Let $\lambda_{1i}, \ldots, \lambda_{ni}$ be the characteristic roots of the matrix V_i, $i = 1, \ldots, k$. Since the matrices $V_1, \ldots, V_k$ commute, there exists an orthogonal matrix P such that

$$PV_iP' = \mathrm{diag}(\lambda_{1i}, \ldots, \lambda_{ni}), \quad i = 1, \ldots, k.$$

Since $V(\sigma) = \sum_{i=1}^{k} \sigma_i V_i$, we have

$$PV(\sigma)P' = \mathrm{diag}\big(d_1(\sigma), \ldots, d_n(\sigma)\big),$$

where

$$d_j(\sigma) = \sum_{i=1}^{k} \sigma_i \lambda_{ji}, \quad j = 1, \ldots, n,\ \sigma \in \bar{\Omega}.$$

Assume that s is the number of distinct functions in the sequence $d_1(\sigma), \ldots, d_n(\sigma)$ and let $d_{i_1}(\sigma), \ldots, d_{i_s}(\sigma)$ be these distinct functions.

The following lemma is a generalization of the spectral theorem for a symmetric matrix.

LEMMA 1. *If matrices $V_1, \ldots, V_k$ are symmetric, linearly independent, and commute, then there exists a unique set of symmetric matrices $A_1, \ldots, A_s$ satisfying the following conditions*:

(a) $A_i^2 = A_i$, $i = 1, \ldots, s$;

(b) $A_iA_j = 0$, $i \neq j$, $i, j = 1, \ldots, s$;

(c) $V(\sigma) = \sum_{j=1}^{s} d_{i_j}(\sigma) A_j$;

(d) $\sum_{i=1}^{s} A_i = I$;

(e) $\mathrm{sp}\{A_1, \ldots, A_s\}$ *is the minimal quadratic subspace containing* $\mathscr{V}$.

From Lemma 1 and the factorization theorem (see, e.g., [2]) we obtain

THEOREM 1. *If matrices $A_1, \ldots, A_s$ satisfy conditions* (a)-(e) *of Lemma* 1, *then*

(a) *the statistics $y'A_1y, \ldots, y'A_sy$ are stochastically independent*;

(b) *the statistic $z = (y'A_1y, \ldots, y'A_sy)'$ is a minimal sufficient statistic for the family of distributions of y*;

(c) *the statistic z is a minimal complete sufficient statistic if and only if $s = k$.*

3. Admissibility of Henderson's I estimators for three variance components. Throughout this section we restrict ourselves to the case of $k = 3$ components, since for larger k the formulas become rather complicated.

Let

$$\sigma^{(1)} = (1, 0, 0)', \quad \sigma^{(2)} = (0, 1, 0)',$$

$$S_i = \{j;\ d_{i_j}(\sigma^{(i)}) \neq 0\}, \quad \text{and} \quad P_i = \sum_{j \in S_i} A_j \quad \text{for } i = 1, 2,$$

where $d_{i_j}(\sigma)$ are distinct functions in the sequence $d_1(\sigma), \ldots, d_n(\sigma)$, as in Section 2.

Let

$$D_{ij} = \operatorname{diag}(\lambda_{1i}\lambda_{1j}r_1, \ldots, \lambda_{si}\lambda_{sj}r_s) \quad \text{for } i, j = 1, 2, 3,$$

where $r_i = \operatorname{tr} A_i$.

Put

$$H = \begin{bmatrix} \lambda_{11}r_1 & \lambda_{12}r_1 & r_1 \\ \cdots & \cdots & \cdots \\ \lambda_{s-1,1}r_{s-1} & \lambda_{s-1,2}r_{s-1} & r_{s-1} \\ \lambda_{s1}r_s & \lambda_{s2}r_s & r_s \end{bmatrix}.$$

We use throughout $R(A)$, $N(A)$, $r(A)$, A', and A^+ to denote the range, null space, rank, transpose, and Moore-Pensore inverse, respectively, of a matrix A.

Lemma 2. *If $r(X_1; X_2) < n$, then*

$$N(D_{12}) \cap N(D_{13}) \cap N(H') = \{0\}.$$

We use this lemma to prove Theorem 2. Applying Henderson's I method (analysis of variance) [4], we obtain estimators $\hat{\sigma}_i$ of variance components of the form

$$\hat{\sigma}_i = a_i' z \quad \text{for } i = 1, 2, 3,$$

where

$$(1) \qquad a_1 = \alpha_1 D_{13} D_{13}^+ a_1 + \beta_1 D_{23}^+ (I - D_{13} D_{13}^+)(\lambda_{12}r_1, \ldots, \lambda_{s2}r_s)' + \\ + \gamma_1 \big(I - (D_{13} + D_{23})(D_{13} + D_{23})^+\big)(0, \ldots, 0, 1)',$$

while

$$\alpha_1 = \frac{1}{\operatorname{tr} V_1}, \quad \beta_1 = \frac{1}{\operatorname{tr} V_1}\left(1 - \frac{\operatorname{tr} V_2}{\operatorname{tr} V_2(I - P_1)}\right),$$

$$\gamma_1 = \frac{1}{\operatorname{tr} V_1}\left(\frac{\operatorname{tr} V_2 \operatorname{tr} P_2(I - P_1)}{\operatorname{tr} V_2(I - P_1)(n - t)} - \frac{t}{n - t}\right), \quad t = r(X_1; X_2);$$

$$(2) \qquad a_2 = \alpha_2 D_{23}^{+}(I - D_{13}D_{13}^{+})(\lambda_{12}r_1, \ldots, \lambda_{s2}r_s)' + \\ + \beta_2\big(I - (D_{13}+D_{23})(D_{13}+D_{23})^{+}\big)(0, 0, \ldots, 0, 1)',$$

while

$$\alpha_2 = \frac{1}{\operatorname{tr} V_2(I - P_1)}, \quad \beta_2 = \frac{1}{\operatorname{tr} V_2(I - P_1)}\left(-\frac{\operatorname{tr} P_2(I - P_1)}{n-t}\right);$$

$$(3) \qquad a_3 = \frac{1}{(n-t)r_s}\big(I - (D_{13}+D_{23})(D_{13}+D_{23})^{+}\big)(r_1, \ldots, r_s)'.$$

THEOREM 2. *Henderson's* I *estimators* $\hat{\sigma}_i$ *are admissible in the class of all unbiased quadratic estimators of variance components* σ_i *for* $i = 1, 2, 3$.

Proof. Let

$$c_0 = \beta_1 D_{23}^{+}(I - D_{13}D_{13}^{+})(\lambda_{12}r_1, \ldots, \lambda_{s2}r_s)' + \\ + \gamma_1\big(I - (D_{13}+D_{23})(D_{13}+D_{23})^{+}\big)(0, \ldots, 0, 1)'.$$

Then $c_0 \in N(D_{13})$ and

$$D_{23}c_0 = \beta_1 D_{23}D_{23}^{+}(I - D_{13}D_{13}^{+})(\lambda_{12}r_1, \ldots, \lambda_{s2}r_s)' \\ = \beta_1(\lambda_{12}r_1, \ldots, \lambda_{s2}r_s)' - \beta_1 \sum_{i \in S_1 \cap S_2} \lambda_{i2}r_i e_i,$$

where e_i $(i = 1, \ldots, s)$ is the standard basis in R^s. Hence $D_{23}c_0 \in R(H) + R(D_{13})$. Therefore

$$a_1 = \alpha_1 D_{13}D_{13}^{+} a_1 + c_0$$

and

$$c_0 \in N(D_{13}) \cap D_{23}^{-1}\big(R(H) + R(D_{13})\big).$$

Since

$$D_{13}^{-1}\big(R(H)\big) \cap D_{23}^{-1}\big(R(H) + R(D_{13})\big) \\ = D_{13}^{+}\big(R(H) \cap R(D_{13})\big) + N(D_{13}) \cap D_{23}^{-1}\big(R(H) + R(D_{13})\big),$$

we have

$$a_1 \in D_{13}^{-1}\big(R(H)\big) \cap D_{23}^{-1}\big(R(H) + R(D_{13})\big).$$

Let $\mathscr{K} = D_{13}^{-1}\big(R(H)\big)$. Since the matrix D_{13} is symmetric and non-negative definite, we get

$$R(H) + \mathscr{K}^{\perp} = R(H) + R(D_{13}).$$

Hence $a_1 \in \mathscr{K} \cap D_{23}^{-1}\big(R(H) + \mathscr{K}^{\perp}\big)$. Finally, we infer that $a_1'z$ is D_{23}-best in $\mathscr{K}$. Lemma 2 implies

$$N(D_{23}) \cap \mathscr{K} \cap N(H') = N(D_{23}) \cap N(D_{13}) \cap N(H') = \{0\}.$$

Thus, in view of Lemma 3.4 (iii) from [3], $a_1'D_{23}a_1 < a'D_{23}a$ for all $a \in \mathscr{K}$ such that $\mathrm{E}a'z = \sigma_1$. Consequently, $a_1'z$ must be admissible for the parametric function σ_1 in the class of all unbiased estimators of the form $a'z$ with $a \in \mathscr{K}$. By Theorem 1 from [1] we conclude that $a_1'z$ is admissible in the class of all unbiased quadratic estimators of σ_1.

The admissibility of estimators $\hat{\sigma}_2$ and $\hat{\sigma}_3$ is proved in an analogous way.

References

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