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BILINEAR PROGRAMMING

1. Introduction. The *bilinear programming problem* (problem BLP) belongs to a class of mathematical programming problems having many local maxima. It may be stated as follows:

PROBLEM BLP. Maximize

$$F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}+\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \tag{1.1}$$

subject to

$$\boldsymbol{x} \in X = \{\boldsymbol{x} \in R^n \mid A\boldsymbol{x} = \boldsymbol{c}, \boldsymbol{x} \geqslant \boldsymbol{0}\}, \tag{1.2}$$

$$\boldsymbol{y} \in Y = \{\boldsymbol{y} \in R^m \mid B\boldsymbol{y} = \boldsymbol{d}, \boldsymbol{y} \geqslant \boldsymbol{0}\}, \tag{1.3}$$

where Q, A, B are matrices of dimensions $m \times n$, $k \times n$, $l \times m$, respectively, and $\boldsymbol{a} \in R^n$, $\boldsymbol{b} \in R^m$, $\boldsymbol{c} \in R^k$, $\boldsymbol{d} \in R^l$.

Let us notice that problem (1.1)-(1.3) may be written in the following form:

Maximize

$$G(\boldsymbol{z}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{z}+\boldsymbol{z}^{\mathrm{T}}C\boldsymbol{z}$$

subject to

$$\boldsymbol{z} \in Z = \{\boldsymbol{z} \in R^{n+m} \mid D\boldsymbol{z} = \boldsymbol{h}, \boldsymbol{z} \geqslant \boldsymbol{0}\},$$

where

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \frac{1}{2}Q^{\mathrm{T}} \\ \frac{1}{2}Q & 0 \end{bmatrix}, \quad D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix}.$$

Thus problem BLP is a special case of the nonconvex quadratic programming problem, the objective function of which is neither convex nor concave. However, the matrix C has a special structure.

The class of nonconvex quadratic programming problems with a nonnegative definite matrix C can be solved, as shown in [3], by some modification of an algorithm solving the problem BLP.

The set $X \times Y$ of feasible solutions of problem BLP is, in general, a polyhedral convex set (not necessarily bounded). Each vertex of $X \times Y$ is an ordered pair of vertices of X and Y, respectively.

In most methods for solving the problem BLP known from literature (see [5], [6], [8]-[10]) it is assumed that the set of feasible solutions is bounded (this condition should be verified before applying them). Obviously, the boundedness of $X \times Y$ implies the existence of an optimal solution. But such a solution may exist even though $X \times Y$ is unbounded. This is clear from the following trivial example:

Example 1. Maximize $F(x, y) = -xy$ subject to $x \geqslant 0$, $y \geqslant 0$.

The authors of [1] and [2] do not mention the assumption that the set $X \times Y$ is bounded. Thus the methods proposed fail to be proper.

Altman [1] states that the sufficient condition of optimality (given by Theorem 4 of our paper) is also necessary but this statement is false (see Example 3 and Remark 8). Besides, the claim of [1] that it is possible to solve problem BLP only by standard linear programming methods seems to be extremely optimistic.

Remarks concerning the Cabot and Francis paper [2] are given in the introduction of paper [3] when a nonconvex quadratic problem is discussed.

The present paper deals with the general problem BLP defined by (1.1)-(1.3) without any additional assumptions.

The most important properties of problem BLP are given in Section 2.

In Section 3 we deal with the theoretical foundations of the method for solving the problem BLP. It should be stressed that the theory and comments concerning the procedure of the algorithm are presented in this section according to the structure of the algorithm.

The algorithm for finding an optimal solution is described in Section 4 and is illustrated by simple numerical examples in Section 5.

2. Some general properties of problem BLP. The subject of our considerations is the general problem BLP defined by (1.1)-(1.3). We assume that the problem is *consistent*, i.e. the set of feasible solutions is nonempty.

Let us notice that the function $F(\boldsymbol{x}, \boldsymbol{y})$ is continuous on the closed set $X \times Y$. If the set $X \times Y$ is bounded, then it is compact. Thus, due to the Weierstrass theorem, we have the following

Remark 1. If the set of feasible solutions of problem BLP is bounded, then an optimal solution exists.

Let us notice that the polyhedral convex sets X and Y can be written in the form

$$X = \{\boldsymbol{x} = \boldsymbol{x}^{\mathrm{p}} + \lambda \boldsymbol{x}^{\mathrm{c}} \mid \boldsymbol{x}^{\mathrm{p}} \in X_{\mathrm{p}},\ \boldsymbol{x}^{\mathrm{c}} \in X_{\mathrm{c}},\ \lambda \geqslant 0\},$$

$$Y = \{\boldsymbol{y} = \boldsymbol{y}^{\mathrm{p}} + \delta \boldsymbol{y}^{\mathrm{c}} \mid \boldsymbol{y}^{\mathrm{p}} \in Y_{\mathrm{p}},\ \boldsymbol{y}^{\mathrm{c}} \in Y_{\mathrm{c}},\ \delta \geqslant 0\},$$

where X_{p} and Y_{p} denote convex hulls of all vertices of X and Y, respectively, and X_{c} and Y_{c} are convex polyhedral cones of the form

$$X_{\mathrm{c}} = \{\boldsymbol{x} \in R^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{x} \geqslant \boldsymbol{0}\}, \qquad Y^{\mathrm{c}} = \{\boldsymbol{y} \in R^m \mid B\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{y} \geqslant \boldsymbol{0}\}.$$

Thus the function $F(\boldsymbol{x}, \boldsymbol{y})$ can be presented as follows:

$$\begin{aligned}(2.1)\quad F(\boldsymbol{x}, \boldsymbol{y}) &= F(\boldsymbol{x}^{\mathrm{p}}+\lambda\boldsymbol{x}^{\mathrm{c}}, \boldsymbol{y}^{\mathrm{p}}+\delta\boldsymbol{y}^{\mathrm{c}}) = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{p}}+\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^{\mathrm{p}}+(\boldsymbol{y}^{\mathrm{p}})^{\mathrm{T}}Q\boldsymbol{x}^{\mathrm{p}}+\\ &+\lambda[\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}^{\mathrm{c}}+(\boldsymbol{y}^{\mathrm{p}})^{\mathrm{T}}Q\boldsymbol{x}^{\mathrm{c}}]+\delta[\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^{\mathrm{c}}+(\boldsymbol{y}^{\mathrm{c}})^{\mathrm{T}}Q\boldsymbol{x}^{\mathrm{p}}]+\lambda\delta(\boldsymbol{y}^{\mathrm{c}})^{\mathrm{T}}Q\boldsymbol{x}^{\mathrm{c}}.\end{aligned}$$

THEOREM 1. *The function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set of feasible solutions of problem BLP if and only if the following conditions hold*:

(i) $\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \leqslant 0$ *for* $(\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{c}}\times Y_{\mathrm{c}}$,
(ii) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \leqslant 0$ *for* $(\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{c}}\times Y_{\mathrm{p}}$,
(iii) $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}+\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \leqslant 0$ *for* $(\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{p}}\times Y_{\mathrm{c}}$.

Proof. Let us start with the necessity.

Suppose there exists a point $(\bar{\boldsymbol{x}}^{\mathrm{c}}, \bar{\boldsymbol{y}}^{\mathrm{c}}) \in X_{\mathrm{c}}\times Y_{\mathrm{c}}$ such that $(\bar{\boldsymbol{y}}^{\mathrm{c}})^{\mathrm{T}}Q\bar{\boldsymbol{x}}^{\mathrm{c}} > 0$. Then we have

$$\begin{aligned}&\lim_{\lambda\to+\infty}\lim_{\delta\to+\infty} F(\boldsymbol{x}^{\mathrm{p}}+\lambda\bar{\boldsymbol{x}}^{\mathrm{c}}, \boldsymbol{y}^{\mathrm{p}}+\delta\bar{\boldsymbol{y}}^{\mathrm{c}}) = F(\boldsymbol{x}^{\mathrm{p}}, \boldsymbol{y}^{\mathrm{p}})+\\ &+\lim_{\lambda\to+\infty}\lim_{\delta\to+\infty}\lambda\delta\left[\frac{1}{\delta}(\boldsymbol{a}^{\mathrm{T}}\bar{\boldsymbol{x}}^{\mathrm{c}}+(\boldsymbol{y}^{\mathrm{p}})^{\mathrm{T}}Q\bar{\boldsymbol{x}}^{\mathrm{c}})+\frac{1}{\lambda}(\boldsymbol{b}^{\mathrm{T}}\bar{\boldsymbol{y}}^{\mathrm{c}}+(\bar{\boldsymbol{y}}^{\mathrm{c}})^{\mathrm{T}}Q\boldsymbol{x}^{\mathrm{p}})+(\bar{\boldsymbol{y}}^{\mathrm{c}})^{\mathrm{T}}Q\bar{\boldsymbol{x}}^{\mathrm{c}}\right]\\ &= +\infty,\end{aligned}$$

which is in contradiction with the assumption that the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above.

To prove the necessity of (ii) suppose, to the contrary, that the point $(\bar{\boldsymbol{x}}^{\mathrm{c}}, \bar{\boldsymbol{y}}^{\mathrm{p}}) \in X_{\mathrm{c}}\times Y_{\mathrm{p}}$ satisfies the inequality $\boldsymbol{a}^{\mathrm{T}}\bar{\boldsymbol{x}}^{\mathrm{c}}+(\bar{\boldsymbol{y}}^{\mathrm{p}})^{\mathrm{T}}Q\bar{\boldsymbol{x}}^{\mathrm{c}} > 0$. This implies

$$\lim_{\lambda\to+\infty} F(\boldsymbol{x}^{\mathrm{p}}+\lambda\bar{\boldsymbol{x}}^{\mathrm{c}}, \bar{\boldsymbol{y}}^{\mathrm{p}}) = F(\boldsymbol{x}^{\mathrm{p}}, \bar{\boldsymbol{y}}^{\mathrm{p}})+\lim_{\lambda\to+\infty}\lambda[\boldsymbol{a}^{\mathrm{T}}\bar{\boldsymbol{x}}^{\mathrm{c}}+(\bar{\boldsymbol{y}}^{\mathrm{p}})^{\mathrm{T}}Q\bar{\boldsymbol{x}}^{\mathrm{c}}] = +\infty$$

and we obtain a contradiction again.

An analogous reasoning shows the necessity of (iii).

Now, let us notice, according to (2.1), that conditions (i), (ii), (iii) imply the inequality

$$F(\boldsymbol{x}^{\mathrm{p}}+\lambda\boldsymbol{x}^{\mathrm{c}}, \boldsymbol{y}^{\mathrm{p}}+\delta\boldsymbol{y}^{\mathrm{c}}) \leqslant F(\boldsymbol{x}^{\mathrm{p}}, \boldsymbol{y}^{\mathrm{p}}), \quad \text{where } (\boldsymbol{x}^{\mathrm{p}}, \boldsymbol{y}^{\mathrm{p}}) \in X_{\mathrm{p}}\times Y_{\mathrm{p}}.$$

Hence

$$\max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X\times Y\} = \max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{p}}\times Y_{\mathrm{p}}\}.$$

Then the sufficiency of (i), (ii), (iii) is an immediate consequence of Remark 1.

From the proof of Theorem 1 we obtain immediately the following corollaries:

COROLLARY 1. *If the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set of feasible solutions of problem BLP, then there exists an optimal solution of the problem.*

COROLLARY 2. *If condition* (i) *of Theorem* 1 *is satisfied, then*

$$\max\{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_c \times Y\} < +\infty$$

and

$$\max\{\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}+\boldsymbol{y}^{\mathrm{T}}Q\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y_c\} < +\infty.$$

Without loss of generality we may assume that $\operatorname{rank} A = k \leqslant n$ and $\operatorname{rank} B = l \leqslant m$.

Let $\alpha = \{j_1, j_2, \ldots, j_k\} \subset \{1, 2, \ldots, n\}$ denote the set of indices of the basic variables corresponding to the basis $A_\alpha = [A_{\{j_1\}}, A_{\{j_2\}}, \ldots, A_{\{j_k\}}]$ constructed by means of linearly independent columns of the matrix A spanning R^k and let $\beta = \{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, m\}$ be the set of indices of the basic variables corresponding to the basis $B_\beta = [B_{\{i_1\}}, B_{\{i_2\}}, \ldots, B_{\{i_l\}}]$ of the space R^l. Then $\xi = \{1, 2, \ldots, n\} - \alpha$ and $\eta = \{1, 2, \ldots, m\} - \beta$ are the sets of indices of nonbasic variables corresponding to A_α and B_β, respectively. By $\boldsymbol{x}_\alpha$, $\boldsymbol{x}_\xi$, $\boldsymbol{y}_\beta$ and $\boldsymbol{y}_\eta$ we denote subvectors of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ consisting of coordinates with indices belonging to the sets α, ξ, β and η, respectively. Thus A_ξ and B_η denote submatrices of A and B consisting of columns of A and B pointed out by the sets ξ and η, respectively.

A point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) \in X \times Y$ is called a *basic feasible solution* of problem BLP relative to bases A_α and B_β if $\boldsymbol{x}^\alpha_\xi = \boldsymbol{0}$ and $\boldsymbol{y}^\beta_\eta = \boldsymbol{0}$.

THEOREM 2. *If problem BLP has an optimal solution, then it has a basic optimal solution.*

Proof. Let $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in X \times Y$ denote an optimal solution of problem BLP, i.e. let

$$\max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y\} = F(\boldsymbol{x}^0, \boldsymbol{y}^0).$$

Let us notice that $\max\{F(\boldsymbol{x}, \boldsymbol{y}^0) \mid \boldsymbol{x} \in X\}$ is a *linear programming problem* (problem LP). It is known that if $\boldsymbol{x}^0$ is not a vertex of X, then there exists a basic optimal solution $\boldsymbol{x}^\alpha \in X$ of this problem LP. Then, clearly, $F(\boldsymbol{x}^\alpha, \boldsymbol{y}^0) = F(\boldsymbol{x}^0, \boldsymbol{y}^0)$. Thus the feasible solution $(\boldsymbol{x}^\alpha, \boldsymbol{y}^0)$ is an alternative optimal solution of problem BLP.

Repeating this reasoning we can show that if $\boldsymbol{y}^0$ is not a vertex of Y, then there exists a basic optimal solution $\boldsymbol{y}^\beta \in Y$ of the following problem LP: $\max\{F(\boldsymbol{x}^\alpha, \boldsymbol{y}) \mid \boldsymbol{y} \in Y\}$. Obviously, $F(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) = F(\boldsymbol{x}^0, \boldsymbol{y}^0)$. This completes the proof.

A feasible solution $(\boldsymbol{x}^0, \boldsymbol{y}^0)$ is called an *equilibrium point* of problem BLP if

$$F(\boldsymbol{x}^0, \boldsymbol{y}^0) = \max\{F(\boldsymbol{x}, \boldsymbol{y}^0) \mid \boldsymbol{x} \in X\} = \max\{F(\boldsymbol{x}^0, \boldsymbol{y}) \mid \boldsymbol{y} \in Y\}.$$

An immediate consequence of the proof of Theorem 2 is the following

COROLLARY 3. *If a feasible solution* $(\boldsymbol{x}^0, \boldsymbol{y}^0)$ *of problem BLP is an optimal solution, then it is an equilibrium point.*

3. Theoretical foundations of the algorithm. Due to Theorem 2 and Corollary 3 it is possible to find an optimal solution of problem BLP among basic feasible solutions being equilibrium points of the problem. We should also examine whether the function $F(x, y)$ is bounded from above on $X \times Y$ (see Corollary 1). However, the conditions of Theorem 1 are of the form of bilinear programming problems, so we try to avoid solving these problems as long as possible. In the latter situation we introduce some other sufficient conditions not so difficult from the numerical point of view.

Let us define the function

$$\Delta(x, y) = F(x, y) - F(x^0, y^0), \tag{3.1}$$

where a feasible solution (x^0, y^0) of problem BLP is fixed.

The following facts are evident and important for further considerations:

Remark 2. $\Delta(x, y) > 0$ (or $\Delta(x, y) \geqslant 0$) if and only if $F(x, y) > F(x^0, y^0)$ (or $F(x, y) \geqslant F(x^0, y^0)$).

Remark 3. A feasible solution (x^0, y^0) of problem BLP is its optimal solution if and only if

$$\max\{\Delta(x, y) \mid (x, y) \in X \times Y\} = 0.$$

Let $(x^\alpha, y^\beta) \in X \times Y$ denote the basic feasible solution relative to bases A_α and B_β. Rewriting the systems of linear equations from (1.2) and (1.3) in the equivalent basic form

$$x_\alpha = x_\alpha^\alpha - E_\xi x_\xi, \qquad y_\beta = y_\beta^\beta - H_\eta y_\eta,$$

where $x_\alpha^\alpha = A_\alpha^{-1} c$, $y_\beta^\beta = B_\beta^{-1} d$, $E_\xi = A_\alpha^{-1} A_\xi$, $H_\eta = B_\beta^{-1} B_\eta$, we get

$$\Delta(x, y) = p_\xi^{\mathrm{T}} x_\xi + q_\eta^{\mathrm{T}} y_\eta + y_\eta^{\mathrm{T}} V_{\eta\xi} x_\xi, \tag{3.2}$$

where

$$p_\xi = a_\xi + Q_\xi^{\mathrm{T}} y^\beta - E_\xi^{\mathrm{T}}(a_\alpha + Q_\alpha^{\mathrm{T}} y^\beta), \tag{3.3}$$

$$q_\eta = b_\eta + [(Q^{\mathrm{T}})_\eta]^{\mathrm{T}} x^\alpha - H_\eta^{\mathrm{T}}\big(b_\beta + [(Q^{\mathrm{T}})_\beta]^{\mathrm{T}} x^\alpha\big), \tag{3.4}$$

$$V_{\eta\xi} = H_\eta^{\mathrm{T}}(Q_{\beta\alpha} E_\xi - Q_{\beta\xi}) - Q_{\eta\alpha} E_\xi + Q_{\eta\xi}. \tag{3.5}$$

It should be noticed that $[(Q^{\mathrm{T}})_\eta]^{\mathrm{T}}$ and $[(Q^{\mathrm{T}})_\beta]^{\mathrm{T}}$ are submatrices of Q formed from rows of Q corresponding to the sets η and β, respectively. Moreover, let us stress that the function (3.2) depends only on nonbasic variables.

Since the difference $F(x, y) - \Delta(x, y)$ is constant, Theorem 1 is equivalent to the following

Remark 4. The function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set $X \times Y$ if and only if the following conditions hold:

(i) $\max\{\boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi} \boldsymbol{x}_\xi \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{c}} \times Y_{\mathrm{c}}\} = 0$,

(ii) $\max\{\boldsymbol{p}_\xi^{\mathrm{T}} \boldsymbol{x}_\xi + \boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi} \boldsymbol{x}_\xi \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{c}} \times Y_{\mathrm{p}}\} = 0$,

(iii) $\max\{\boldsymbol{q}_\eta^{\mathrm{T}} \boldsymbol{y}_\eta + \boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi} \boldsymbol{x}_\xi \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{p}} \times Y_{\mathrm{c}}\} = 0$.

Now, let us define the sets ξ^0 and η^0 as follows:

$$\xi^0 = \{j \in \xi \mid E_{\{j\}} \leqslant 0\}, \qquad \eta^0 = \{i \in \eta \mid H_{\{i\}} \leqslant 0\}.$$

PROPOSITION 1. *If $p_j > 0$ for some $j \in \xi^0$ (or $q_i > 0$ for some $i \in \eta^0$), then the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.*

Proof. Let us consider the set of points $\boldsymbol{x}(\lambda)$ of the form

$$(*) \qquad \boldsymbol{x}(\lambda) = \lambda \hat{\boldsymbol{x}},$$

where $\hat{\boldsymbol{x}}_\alpha = -E_{\{j\}}$, $\hat{x}_j = 1$, $\hat{\boldsymbol{x}}_{\xi-\{j\}} = \boldsymbol{0}$, $\lambda \geqslant 0$.

Clearly, $\boldsymbol{x}(\lambda) \in X_{\mathrm{c}}$ and the set $\{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{x}^\alpha + \boldsymbol{x}(\lambda)\}$ is an infinite edge of X (i.e. a half line emanating from $\boldsymbol{x}^\alpha$ in the direction $\hat{\boldsymbol{x}}$).

For $(\boldsymbol{x}(\lambda), \boldsymbol{y}^\beta) \in X_{\mathrm{c}} \times Y_{\mathrm{p}}$ we obtain

$$[\boldsymbol{p}_\xi^{\mathrm{T}} + (\boldsymbol{y}_\eta^\beta)^{\mathrm{T}} V_{\eta\xi}] \boldsymbol{x}_\xi(\lambda) = p_j \lambda > 0 \qquad \text{whenever } \lambda > 0.$$

This relation and condition (ii) of Remark 4 imply that the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on $X \times Y$.

In a similar manner we can prove the second part of the proposition introducing the set of points $\boldsymbol{y}(\delta)$ of the form

$$(**) \qquad \boldsymbol{y}(\delta) = \delta \hat{\boldsymbol{y}},$$

where $\hat{\boldsymbol{y}}_\beta = -H_{\{i\}}$, $\hat{y}_i = 1$, $\hat{\boldsymbol{y}}_{\eta-\{i\}} = \boldsymbol{0}$, $\delta \geqslant 0$, and using (iii) of Remark 4.

Using the proof of Proposition 1 and computing the values (according to (3.2))

$$\Delta(\boldsymbol{x}^\alpha + \boldsymbol{x}(\lambda), \boldsymbol{y}^\beta) = p_j \lambda \quad \text{and} \quad \Delta(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta + \boldsymbol{y}(\delta)) = q_i \delta$$

we get

COROLLARY 4. *If $p_j = 0$ for some $j \in \xi^0$, then*

$$F(\boldsymbol{x}^\alpha + \boldsymbol{x}(\lambda), \boldsymbol{y}^\beta) = F(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta).$$

Analogously, if $q_i = 0$ for some $i \in \eta^0$, then

$$F(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta + \boldsymbol{y}(\delta)) = F(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta).$$

Let $j \in \xi - \xi^0$ and $i \in \eta - \eta^0$. Denoting by α' and β' new sets of indices of basic variables such that $\alpha' - \alpha = \{j\}$ and $\beta' - \beta = \{i\}$ we can compute (as in the simplex method for problem LP) the following values:

$$(3.6) \qquad \nu_j = \min\{x_t^\alpha / E_{\{t\}\{j\}} \mid E_{\{t\}\{j\}} > 0,\ t \in \alpha\},$$

$$(3.7) \qquad \mu_i = \min\{y_t^\beta / H_{\{t\}\{i\}} \mid H_{\{t\}\{i\}} > 0,\ t \in \beta\}.$$

Then the adjacent basic feasible solutions $\boldsymbol{x}^{\alpha'}$ and $\boldsymbol{y}^{\beta'}$ of $\boldsymbol{x}^{\alpha}$ and $\boldsymbol{y}^{\beta}$, respectively, are determined by

$$(3.8) \qquad x_j^{\alpha'} = \nu_j, \quad \boldsymbol{x}_{\xi-\{j\}}^{\alpha'} = \boldsymbol{0}, \quad \boldsymbol{x}_{\alpha}^{\alpha'} = \boldsymbol{x}_{\alpha}^{\alpha} - \nu_j \boldsymbol{E}_{\{j\}},$$

$$(3.9) \qquad y_i^{\beta'} = \mu_i, \quad \boldsymbol{y}_{\eta-\{i\}}^{\beta'} = \boldsymbol{0}, \quad \boldsymbol{y}_{\beta}^{\beta'} = \boldsymbol{y}_{\beta}^{\beta} - \mu_i \boldsymbol{H}_{\{i\}}.$$

According to the bases $A_{\alpha'}$ and $B_{\beta'}$ we can obtain adjacent basic feasible solutions of $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$. These are $(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}^{\beta})$ and $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta'})$, respectively.

PROPOSITION 2. *If $p_j > 0$ for some $j \in \xi - \xi^0$, then $F(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}^{\beta}) \geqslant F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$. Moreover, $F(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}^{\beta}) > F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ whenever $\boldsymbol{x}^{\alpha}$ is nondegenerate.*

Proof. Due to (3.6) we have $\nu_j \geqslant 0$. If $\boldsymbol{x}^{\alpha}$ is nondegenerate, then $\nu_j > 0$. Taking into account $\Delta(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}^{\beta}) = p_j \nu_j$ (which follows from (3.8) and (3.2)) and Remark 2 we prove the proposition.

COROLLARY 5. *If $p_j = 0$ for some $j \in \xi - \xi^0$, then $F(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}^{\beta}) = F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$. Moreover, $F\big(\boldsymbol{x}^{\alpha} + \lambda(\boldsymbol{x}^{\alpha'} - \boldsymbol{x}^{\alpha}), \boldsymbol{y}^{\beta}\big) = F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ whenever $0 \leqslant \lambda \leqslant 1$.*

PROPOSITION 3. *If $q_i > 0$ for some $i \in \eta - \eta^0$, then $F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta'}) \geqslant F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$. Moreover, $F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta'}) > F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ if $\boldsymbol{y}^{\beta}$ is nondegenerate.*

Proof. The proposition is implied by formulae (3.7) and (3.9) in an analogous way to Proposition 2.

COROLLARY 6. *If $q_i = 0$ for some $i \in \eta - \eta^0$, then $F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta'}) = F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$. Moreover, $F\big(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta} + \delta(\boldsymbol{y}^{\beta'} - \boldsymbol{y}^{\beta})\big) = F(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ whenever $0 \leqslant \delta \leqslant 1$.*

THEOREM 3. *For a basic feasible solution $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ to be an equilibrium point it is sufficient (and necessary if $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is nondegenerate) that*

$$(3.10) \qquad \boldsymbol{p}_{\xi} \leqslant \boldsymbol{0},$$

$$(3.11) \qquad \boldsymbol{q}_{\eta} \leqslant \boldsymbol{0}.$$

Proof. Since $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is the basic feasible solution, then according to formula (3.2) we obtain $\Delta(\boldsymbol{x}, \boldsymbol{y}^{\beta}) = \boldsymbol{p}_{\xi}^{\mathrm{T}} \boldsymbol{x}_{\xi}$ and $\Delta(\boldsymbol{x}^{\alpha}, \boldsymbol{y}) = \boldsymbol{q}_{\eta}^{\mathrm{T}} \boldsymbol{y}_{\eta}$. Using the first part of Proposition 1 and Proposition 2 for (3.10), and the second part of Proposition 1 and Proposition 3 for (3.11) we complete the proof.

Remark 5. Konno [6] has shown that the basic feasible solution $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ satisfying (3.10) and (3.11) is a Kuhn-Tucker stationary point.

The sufficient but not necessary condition for a point $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ to be optimal is given in the following

THEOREM 4. *If a basic feasible solution $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is an equilibrium point of problem BLP and $V_{\eta\xi} \leqslant [\boldsymbol{0}]$ (i.e. all elements of the matrix $V_{\eta\xi}$ defined by (3.5) are nonpositive), then it is an optimal solution of the problem.*

Proof. Since $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is a basic equilibrium point, then due to Theorem 3 any point $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$ satisfies the inequality $\boldsymbol{p}_{\xi}^{\mathrm{T}} \boldsymbol{x}_{\xi} + \boldsymbol{q}_{\eta}^{\mathrm{T}} \boldsymbol{y}_{\eta} \leqslant 0$. This fact and the assumption $V_{\eta\xi} \leqslant [\boldsymbol{0}]$ imply $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$. Hence

$$\max\{\Delta(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y\} = 0.$$

Since, in particular, $\Delta(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta}) = 0$, we infer that $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is an optimal solution of problem BLP.

Let us notice that the assumptions of Theorem 4 make any verification of conditions (i)-(iii) of Remark 4 useless. In the remainder of the paper we consider a problem, the matrix $V_{\eta\xi}$ of which contains some positive elements (i.e. $V_{\eta\xi} \not\leqslant [\boldsymbol{0}]$). However, this may lead to an unboundedness of the objective function as is shown, for example, in the following proposition:

PROPOSITION 4. *Assume that at least one of the following conditions holds*:

(i) $V_{\{i\}\{j\}} > 0$ *for some* $i \in \eta^0$ *and* $j \in \xi^0$,

(ii) $p_j + \mu_i V_{\{i\}\{j\}} > 0$ *for some* $i \in \eta - \eta^0$ *and* $j \in \xi^0$,

(iii) $q_i + \nu_j V_{\{i\}\{j\}} > 0$ *for some* $i \in \eta^0$ *and* $j \in \xi - \xi^0$.

Then the function $F(\boldsymbol{x}, \boldsymbol{y})$ *is unbounded from above on the set* $X \times Y$.

Proof. Let us consider the points $\boldsymbol{x}(\lambda) \in X_c$ and $\boldsymbol{y}(\delta) \in Y_c$ defined by (*) and (**) in the proof of Proposition 1, respectively.

To prove (i) let us notice that $V_{\{i\}\{j\}} > 0$ implies

$$[\boldsymbol{y}_{\eta}(\delta)]^{\mathrm{T}} V_{\eta\xi} \boldsymbol{x}_{\xi}(\lambda) = \lambda\delta V_{\{i\}\{j\}} > 0 \quad \text{whenever } \lambda > 0 \text{ and } \delta > 0.$$

Hence by (i) of Remark 4 the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on $X \times Y$.

For the proof of (ii) let us notice that the points $(\boldsymbol{x}(\lambda), \boldsymbol{y}^{\beta'}) \in X_c \times Y_p$ satisfy the inequality

$$(\boldsymbol{p}_{\xi}^{\mathrm{T}} + (\boldsymbol{y}_{\eta}^{\beta'})^{\mathrm{T}} V_{\eta\xi}) \boldsymbol{x}_{\xi}(\lambda) = (p_j + \mu_i V_{\{i\}\{j\}})\lambda > 0 \quad \text{whenever } \lambda > 0.$$

Then due to (ii) of Remark 4 the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

In an analogous way, using condition (iii) of Remark 4 for points $(\boldsymbol{x}^{\alpha'}, \boldsymbol{y}(\delta))$, we prove condition (iii) of the proposition.

THEOREM 5. *If a basic feasible solution* $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ *is an optimal solution of problem BLP, then*

$$\max\{V_{\eta\{j\}}^{\mathrm{T}} \boldsymbol{y}_{\eta} \mid \boldsymbol{y} \in Y\} \leqslant -p_j \quad \textit{for each } j \in \xi^0, \tag{3.12}$$

$$\max\{V_{\{i\}\xi} \boldsymbol{x}_{\xi} \mid \boldsymbol{x} \in X\} \leqslant -q_i \quad \textit{for each } i \in \eta^0, \tag{3.13}$$

$$\max\{(\boldsymbol{q}_{\eta} + \nu_j V_{\eta\{j\}})^{\mathrm{T}} \boldsymbol{y}_{\eta} \mid \boldsymbol{y} \in Y\} \leqslant -p_j \nu_j \quad \textit{for each } j \in \xi - \xi^0, \tag{3.14}$$

$$\max\{(\boldsymbol{p}_{\xi}^{\mathrm{T}} + \mu_i V_{\{i\}\xi}) \boldsymbol{x}_{\xi} \mid \boldsymbol{x} \in X\} \leqslant -q_i \mu_i \quad \textit{for each } i \in \eta - \eta^0. \tag{3.15}$$

Proof. To prove (3.12) let us suppose, to the contrary, that there exists $\bar{\boldsymbol{y}} \in Y$ satisfying $V_{\eta\{j\}}^{\mathrm{T}} \bar{\boldsymbol{y}}_{\eta} > -p_j$ for some $j \in \xi^0$. Then taking the points $\boldsymbol{x}(\lambda) \in X_c$ defined by (*) in the proof of Proposition 1 we get, according to (3.2),

$$\lim_{\lambda\to+\infty} \Delta(\boldsymbol{x}^{a}+\boldsymbol{x}(\lambda), \bar{\boldsymbol{y}}) = \lim_{\lambda\to+\infty} [\lambda(p_j+V^{\mathrm{T}}_{\eta\{j\}}\bar{\boldsymbol{y}}_\eta)]+\boldsymbol{q}^{\mathrm{T}}_\eta\bar{\boldsymbol{y}}_\eta = +\infty.$$

This formula and (3.1) imply that the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on $X\times Y$, which contradicts the assumption of the theorem.

An analogous reasoning related to $\boldsymbol{y}(\delta)$ defined by (**) in the proof of Proposition 1 shows the necessity of (3.13).

To prove (3.14) let us suppose, to the contrary, that there exists $\bar{\boldsymbol{y}}\in Y$ satisfying $(\boldsymbol{q}_\eta+\nu_j V_{\eta\{j\}})^{\mathrm{T}}\bar{\boldsymbol{y}}_\eta > -p_j\nu_j$ for some $j\in\xi-\xi^0$. Using this assumption and computing, according to (3.2), the value $\Delta(\boldsymbol{x}^{a'}, \bar{\boldsymbol{y}}) = p_j\nu_j+(\boldsymbol{q}_\eta+\nu_j V_{\eta\{j\}})^{\mathrm{T}}\bar{\boldsymbol{y}}_\eta$ we infer by Remark 2 that $F(\boldsymbol{x}^{a'}, \bar{\boldsymbol{y}}) > F(\boldsymbol{x}^{a}, \boldsymbol{y}^{\beta})$. Thus we have a contradiction again.

Using the value $\Delta(\bar{\boldsymbol{x}}, \boldsymbol{y}^{\beta'}) = q_i\mu_i+(\boldsymbol{p}^{\mathrm{T}}_\xi+\mu_i V_{\{i\}\xi})\bar{\boldsymbol{x}}_\xi$ we prove (3.15) in a similar manner.

From the proof of Theorem 5 we get the following

COROLLARY 7. *Inequalities* (3.12), (3.13) *and*

$$\max\{(\boldsymbol{q}_\eta+\nu_j V_{\eta\{j\}})^{\mathrm{T}}\boldsymbol{y}_\eta \mid \boldsymbol{y}\in Y\} < +\infty \quad \textit{for each } j\in\xi-\xi^0,$$

$$\max\{(\boldsymbol{p}^{\mathrm{T}}_\xi+\mu_i V_{\{i\}\xi})\boldsymbol{x}_\xi \mid \boldsymbol{x}\in X\} < +\infty \quad \textit{for each } i\in\eta-\eta^0$$

are necessary conditions for the function $F(\boldsymbol{x}, \boldsymbol{y})$ to be bounded from above on the set $X\times Y$.

Let us notice that the basic equilibrium point $(\boldsymbol{x}^{a}, \boldsymbol{y}^{\beta})$ of problem BLP satisfying (3.12)-(3.15) is only a local maximum. The knowledge of such a point does not imply the existence of the optimal solution of problem BLP till we do not check if the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set $X\times Y$.

Let us introduce, in the latter situation, the vectors $\boldsymbol{z}_\xi$ and $\boldsymbol{u}_\eta$ whose coordinates are defined by

$$z_j = \max\{V^{\mathrm{T}}_{\eta\{j\}}\boldsymbol{y}_\eta \mid \boldsymbol{y}\in Y\} \quad \text{for } j\in\xi, \tag{3.16}$$

$$u_i = \max\{V_{\{i\}\xi}\boldsymbol{x}_\xi \mid \boldsymbol{x}\in X\} \quad \text{for } i\in\eta. \tag{3.17}$$

THEOREM 6. *Given a basic equilibrium point $(\boldsymbol{x}^{a}, \boldsymbol{y}^{\beta})$ of problem BLP, $z_j < +\infty$ for each $j\in\xi$ and $u_i < +\infty$ for each $i\in\eta$, the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set $X\times Y$.*

Proof. Clearly, if $\max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x}\in X\} < +\infty$, then $\max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x}\in X_{\mathrm{c}}\} = 0$. Taking then

$$\bar{z}_j = \max\{V^{\mathrm{T}}_{\eta\{j\}}\boldsymbol{y}_\eta \mid \boldsymbol{y}\in Y_{\mathrm{c}}\} \quad \text{for } j\in\xi$$

and

$$\bar{u}_i = \max\{V_{\{i\}\xi}\boldsymbol{x}_\xi \mid \boldsymbol{x}\in X_{\mathrm{c}}\} \quad \text{for } i\in\eta$$

we get

$$\boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi}\boldsymbol{x}_\xi \leqslant \bar{\boldsymbol{z}}_\xi^{\mathrm{T}}\boldsymbol{x}_\xi = 0 \quad \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{c}},$$

$$\boldsymbol{p}_\xi^{\mathrm{T}}\boldsymbol{x}_\xi + \boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi}\boldsymbol{x}_\xi \leqslant \boldsymbol{p}_\xi^{\mathrm{T}}\boldsymbol{x}_\xi + \bar{\boldsymbol{u}}_\eta^{\mathrm{T}}\boldsymbol{y}_\eta \leqslant 0 \quad \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{p}}$$

due to (3.10) and

$$\boldsymbol{q}_\eta^{\mathrm{T}}\boldsymbol{y}_\eta + \boldsymbol{y}_\eta^{\mathrm{T}} V_{\eta\xi}\boldsymbol{x}_\xi \leqslant \boldsymbol{q}_\eta^{\mathrm{T}}\boldsymbol{y}_\eta + \bar{\boldsymbol{z}}_\xi^{\mathrm{T}}\boldsymbol{x}_\xi \leqslant 0 \quad \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X}_{\mathrm{p}} \times \boldsymbol{Y}_{\mathrm{c}}$$

due to (3.11).

Applying Remark 4 to the above inequalities we complete the proof.

COROLLARY 8. *If* $z_j < +\infty$ *for each* $j \in \xi$ (*or* $u_i < +\infty$ *for each* $i \in \eta$), *then conditions* (i) *and* (iii) (*or* (i) *and* (ii)) *of Remark* 4 *hold.*

It should be stressed that even if the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the unbounded set $\boldsymbol{X} \times \boldsymbol{Y}$, then some z_j for $j \in \xi - \xi^0$ or u_i for $i \in \eta - \eta^0$ defined by (3.16) and (3.17), respectively, may not exist([1]) (the existence of z_j for any $j \in \xi^0$ and u_i for any $i \in \eta^0$ is guaranteed by Corollary 7). In this case we have to examine problems (i)-(iii) defined in Remark 4. According to Corollary 2 it is advised to start with problem (i). However, due to Corollary 8, in some special cases it is sufficient to require either (ii) or (iii) only.

PROPOSITION 5. *If* $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) \in \boldsymbol{X} \times \boldsymbol{Y}$ *is an equilibrium point of problem BLP, then* $(\boldsymbol{0}, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{c}}$, $(\boldsymbol{0}, \boldsymbol{y}^\beta) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{p}}$ *and* $(\boldsymbol{x}^\alpha, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{p}} \times \boldsymbol{Y}_{\mathrm{c}}$ *are equilibrium points of problems* (i)-(iii), *respectively.*

Proof. The idea of the proof is based on the following observation: if $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an equilibrium point, then, due to Theorem 3, $\boldsymbol{p}_\xi \leqslant \boldsymbol{0}$ and $\boldsymbol{q}_\eta \leqslant \boldsymbol{0}$.

PROPOSITION 6. *If conditions* (3.12)-(3.15) *of Theorem* 5 *hold for a basic feasible solution* $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ *of problem BLP, then those conditions are also fulfilled for the points* $(\boldsymbol{0}, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{c}}$, $(\boldsymbol{0}, \boldsymbol{y}^\beta) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{p}}$ *and* $(\boldsymbol{x}^\alpha, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{p}} \times \boldsymbol{Y}_{\mathrm{c}}$ *in problems* (i)-(iii) *of Remark* 4, *respectively.*

Proof. Let us notice that the points $\boldsymbol{0} \in \boldsymbol{X}_{\mathrm{c}}$ and $\boldsymbol{0} \in \boldsymbol{Y}_{\mathrm{c}}$ are the only vertices of $\boldsymbol{X}_{\mathrm{c}}$ and $\boldsymbol{Y}_{\mathrm{c}}$, respectively, and they are degenerate. The thesis of the proposition follows directly from conditions (3.12)-(3.15) after taking into account that

$$\max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{X}\} < +\infty \quad \text{implies} \quad \max\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{X}_{\mathrm{c}}\} = 0.$$

According to Proposition 6 a solution method for problems (i)-(iii) of Remark 4 reduces to checking if $(\boldsymbol{0}, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{c}}$, $(\boldsymbol{0}, \boldsymbol{y}^\beta) \in \boldsymbol{X}_{\mathrm{c}} \times \boldsymbol{Y}_{\mathrm{p}}$ and $(\boldsymbol{x}^\alpha, \boldsymbol{0}) \in \boldsymbol{X}_{\mathrm{p}} \times \boldsymbol{Y}_{\mathrm{c}}$ are the global maxima, respectively.

([1]) Compare with Remark 8.

Without loss of generality we can restrict ourselves to subsets $\bar{X}_c \subset X_c$ and $\bar{Y}_c \subset Y_c$ defined by

$$(3.18) \qquad \bar{X}_c = \Big\{x \in R^n \mid x_\alpha + E_\xi x_\xi = 0,\ \sum_{j\in\xi} x_j = 1,\ x \geqslant 0\Big\},$$

$$(3.19) \qquad \bar{Y}_c = \Big\{y \in R^m \mid y_\beta + H_\eta y_\eta = 0,\ \sum_{i\in\eta} y_i = 1,\ y \geqslant 0\Big\}.$$

Thus we avoid difficulties of the degeneration. Problems (i)-(iii) are now transformed to those for which the sets of feasible solutions are bounded. Then it is sufficient to check if

(i′) $\max\{y_\eta^{\mathrm{T}} V_{\eta\xi} x_\xi \mid (x, y) \in \bar{X}_c \times \bar{Y}_c\} \leqslant 0$,
(ii′) $\max\{p_\xi^{\mathrm{T}} x_\xi + y_\eta^{\mathrm{T}} V_{\eta\xi} x_\xi \mid (x, y) \in \bar{X}_c \times Y_p\} \leqslant 0$,
(iii′) $\max\{q_\eta^{\mathrm{T}} y_\eta + y_\eta^{\mathrm{T}} V_{\eta\xi} x_\xi \mid (x, y) \in X_p \times \bar{Y}_c\} \leqslant 0$.

However, the sets X_p and Y_p are not defined explicitly by systems of linear conditions (equations and inequalities). In order to construct such systems one has to find all vertices of X and Y, respectively, which would be an obvious disadvantage. That disadvantage can be overcome by finding the values

$$(3.20) \qquad z_j^{\mathrm{p}} = \max\{V_{\eta\{j\}}^{\mathrm{T}} y_\eta \mid y \in Y_p\} \quad \text{for } j \in \{j \in \xi \mid z_j = +\infty\},$$

$$(3.21) \qquad u_i^{\mathrm{p}} = \max\{V_{\{i\}\xi} x_\xi \mid x \in X_p\} \quad \text{for } i \in \{i \in \eta \mid u_i = +\infty\}$$

by the method proposed in [4]. Evidently, from (3.16) and (3.17) we get

$$z_j^{\mathrm{p}} = z_j \quad \text{whenever } z_j < +\infty,$$

$$u_i^{\mathrm{p}} = u_i \quad \text{whenever } u_i < +\infty.$$

Now, let us introduce

$$(3.22) \qquad f(x) = p_\xi^{\mathrm{T}} x_\xi,$$

$$(3.23) \qquad g(y) = q_\eta^{\mathrm{T}} y_\eta,$$

$$(3.24) \qquad \sigma_f = \max\{(q_\eta + u_\eta^{\mathrm{p}})^{\mathrm{T}} y_\eta \mid y \in Y_p\},$$

$$(3.25) \qquad \sigma_g = \max\{(p_\xi + z_\xi^{\mathrm{p}})^{\mathrm{T}} x_\xi \mid x \in X_p\}.$$

Let X^* and Y^* denote the sets of all vertices of X and Y, respectively, ordered in such a manner that $f(x^i) \geqslant f(x^j)$ and $g(y^i) \geqslant g(y^j)$ whenever $i < j$ for $x^i, x^j \in X^*$ and $y^i, y^j \in Y^*$ (obtained by the algorithm from [7]).

THEOREM 7. *Let the function $F(x, y)$ be bounded from above on the set $X \times Y$. Then an equilibrium point (x^α, y^β) of problem BLP is optimal if at least one of the following conditions holds:*

$$(3.26) \qquad p_\xi + z_\xi^{\mathrm{p}} \leqslant 0;$$

$$(3.27) \qquad q_\eta + u_\eta^{\mathrm{p}} \leqslant 0;$$

(3.28) *the sequence* $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^r \in X^*$ (*with* $\boldsymbol{x}^1 = \boldsymbol{x}^a$) *satisfies the inequalities* $f(\boldsymbol{x}^r) \leqslant -\sigma_f$ *and*

$$\max\{\Delta(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in \{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^{r-1}\} \times Y\} \leqslant 0;$$

(3.29) *the sequence* $\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^s \in Y^*$ (*with* $\boldsymbol{y}^1 = \boldsymbol{y}^\beta$) *satisfies the inequalities* $g(\boldsymbol{x}^s) \leqslant -\sigma_g$ *and*

$$\max\{\Delta(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times \{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^{s-1}\}\} \leqslant 0.$$

Proof. If the function $F(\boldsymbol{x}, \boldsymbol{y})$ is bounded from above on the set $X \times Y$, then (²)

$$\max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y\} = \max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X_{\mathrm{p}} \times Y_{\mathrm{p}}\}.$$

From (3.20) and (3.2) we get

$$\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant (\boldsymbol{p}_\xi + \boldsymbol{z}_\xi^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{x}_\xi + \boldsymbol{q}_\eta^{\mathrm{T}} \boldsymbol{y}_\eta \quad \text{for any } (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y.$$

But $(\boldsymbol{x}^a, \boldsymbol{y}^\beta)$ is an equilibrium point which, by Theorem 3, implies $\boldsymbol{q}_\eta^{\mathrm{T}} \boldsymbol{y}_\eta \leqslant 0$ for $\boldsymbol{y} \in Y$. This inequality and (3.26) mean that $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$ for any $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$. Since, in addition, $\Delta(\boldsymbol{x}^a, \boldsymbol{y}^\beta) = 0$, we infer, according to Remark 3, that $(\boldsymbol{x}^a, \boldsymbol{y}^\beta)$ is an optimal solution.

In a similar manner we can prove the sufficiency of (3.27).

To prove the sufficiency of (3.28) let us observe that $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant f(\boldsymbol{x}) + \sigma_f$ for any $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$. Thus $f(\boldsymbol{x}^r) \leqslant -\sigma_f$ implies $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$ for any points such that $(\boldsymbol{x}, \boldsymbol{y}) \in (X^* - \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{r-1}\}) \times Y$. Since also $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$ for any $(\boldsymbol{x}, \boldsymbol{y}) \in \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{r-1}\} \times Y$, we prove that $\Delta(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$ for any $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$, and the point $(\boldsymbol{x}^a, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

A similar reasoning proves the sufficiency of (3.29).

4. An algorithm for solving the problem BLP. The results obtained in Section 3 enable us to propose an algorithm for finding an optimal solution of the general problem BLP. It uses standard linear programming procedures, the method of maximizing a linear form on a convex hull of vertices of a convex polyhedral set (see [4]) and the method for ranking vertices of a convex polyhedral set in a sequence for which the values of a linear form do not increase, proposed in [7].

For clearity of the description we start with presenting algorithms solving problems (i′)-(iii′) which are tools for verifying conditions (i)-(iii) of Remark 4, respectively. We denote those algorithms by A(i), A(ii) and A(iii).

Let us assume that $(\boldsymbol{x}^a, \boldsymbol{y}^\beta)$ is a basic feasible solution of problem BLP, being an equilibrium point and satisfying conditions (3.12)-(3.15), and

(²) Compare with the proof of Theorem 1.

that vectors $\bar{z}_\xi$ and $\bar{u}_\eta$ are given with coordinates defined by the formulae

(4.1) $$\bar{z}_j = \max\{V_{\eta\{j\}}^{\mathrm{T}} y_\eta \mid y \in \bar{Y}_c\} \quad \text{for } j \in \xi,$$

(4.2) $$\bar{u}_i = \max\{V_{\{i\}\xi} x_\xi \mid x \in \bar{X}_c\} \quad \text{for } i \in \eta,$$

where $\bar{X}_c$ and $\bar{Y}_c$ are determined by (3.18) and (3.19), respectively.

The algorithm A(i)

Step 1. Find a basic optimal solution y^1 of the problem

$$\max\{\bar{u}_\eta^{\mathrm{T}} y_\eta \mid y \in \bar{Y}_c\}.$$

Step 2. Check whether $\bar{u}_\eta^{\mathrm{T}} y_\eta^1 \leqslant 0$. If so, condition (i) is satisfied. Otherwise go to Step 3.

Step 3. Check whether $\max\{(y_\eta^1)^{\mathrm{T}} V_{\eta\xi} x_\xi \mid x \in \bar{X}_c\} \leqslant 0$. If so, replace y^1 by the vertex $y^2 \in \bar{Y}_c^*$ which follows directly y^1 (3) with respect to nonincreasing values of the function $\bar{u}_\eta^{\mathrm{T}} y_\eta$ and return to Step 2. Otherwise condition (i) is not satisfied.

The algorithm A(ii)

Step 1. Find the value $\bar{\sigma}_f = \max\{\bar{u}_\eta^{\mathrm{T}} y_\eta \mid y \in Y_p\}$ (4) and a basic optimal solution x^1 of the problem $\max\{f(x) \mid x \in \bar{X}_c\}$, where $f(x)$ is defined by (3.22).

Step 2. Check whether $f(x^1) \leqslant -\bar{\sigma}_f$. If so, condition (ii) is satisfied. Otherwise go to Step 3.

Step 3. Check whether $\max\{y_\eta^{\mathrm{T}} V_{\eta\xi} x_\xi^1 \mid y \in Y\} \leqslant -f(x^1)$ (5). If so, replace x^1 by the vertex $x^2 \in \bar{X}_c^*$ which follows directly x^1 with respect to nonincreasing values of the function $f(x)$ and return to Step 2. Otherwise condition (ii) is not satisfied.

The algorithm A(iii) works as A(ii) if $\bar{\sigma}_f$, $f(x)$ and $\bar{X}_c^*$ are replaced by $\bar{\sigma}_g = \max\{\bar{z}_\xi^{\mathrm{T}} x_\xi \mid x \in X_p\}$, $g(y)$ (defined by (3.23)) and $\bar{Y}_c^*$, respectively.

The general algorithm for problem BLP

Step 1. Find a basic feasible solution (x^α, y^β) of problem BLP.

Step 2. Compute the vector p_ξ according to (3.3) and check whether (3.10) is satisfied.

(a) If so, go to Step 3.

(b) If not, then pick $j \in \xi$ such that $p_j = \max\{p_t > 0 \mid t \in \xi\}$ and check whether $j \in \xi^0$.

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(3) We can find the vertex $y^2 \in \bar{Y}_c^*$ by the method from [7].

(4) Analogously to formula (3.24).

(5) Since we use the algorithms A(ii) and A(iii) knowing that condition (i) is satisfied, we can take, due to Corollary 2, the set Y instead of Y_p.

(b$_1$) If so, the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

(b$_2$) If not, then compute ν_j according to (3.6), replace $\boldsymbol{x}^\alpha$ by $\boldsymbol{x}^{\alpha'}$ obtained from (3.8) and repeat Step 2.

Step 3. Compute the vector $\boldsymbol{q}_\eta$ according to (3.4) and check whether (3.11) is satisfied.

(a) If so, go to Step 4.

(b) If not, then pick $i \in \eta$ such that $q_i = \max\{q_t > 0 \mid t \in \eta\}$ and check whether $i \in \eta^0$.

(b$_1$) If so, the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

(b$_2$) If not, then compute μ_i according to (3.7), replace $\boldsymbol{y}^\beta$ by $\boldsymbol{y}^{\beta'}$ obtained from (3.9) and return to Step 2.

Step 4. Compute the matrix $V_{\eta\xi}$ according to (3.5) and check whether $V_{\eta\xi} \leqslant [\boldsymbol{0}]$.

(a) If so, $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

(b) If not, then select the sets $\xi^1 = \{j \in \xi \mid V_{\{i\}\{j\}} > 0\}$ and $\eta^1 = \{i \in \eta \mid V_{\{i\}\{j\}} > 0\}$ and check whether $\xi^1 \cap \xi^0 = \emptyset$.

(b$_1$) If so, go to Step 5.

(b$_2$) If not, then check whether (3.12) are satisfied for $j \in \xi^1 \cap \xi^0$.

(b$_2^1$) If so, go to Step 5.

(b$_2^2$) If not, then the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 5. Check whether $\eta^1 \cap \eta^0 = \emptyset$.

(a) If so, go to Step 6.

(b) If not, then check whether (3.13) are satisfied for $i \in \eta^1 \cap \eta^0$.

(b$_1$) If so, go to Step 6.

(b$_2$) If not, then the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 6. Check whether $\xi^1 - \xi^0 = \emptyset$.

(a) If so, the point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP ([6]).

(b) If not, then compute the values ν_j for $j \in \xi^1 - \xi^0$ according to (3.6) and check whether (3.14) are satisfied.

(b$_1$) If so, go to Step 7.

(b$_2$) If not, then pick $j \in \xi^1 - \xi^0$ and $\bar{\boldsymbol{y}} \in Y$ such that

$$N_j = (\boldsymbol{q}_\eta + \nu_j V_{\eta\{j\}})^{\mathrm{T}} \bar{\boldsymbol{y}}_\eta = \max\{(\boldsymbol{q}_\eta + \nu_j V_{\eta\{j\}})^{\mathrm{T}} \boldsymbol{y}_\eta \mid \boldsymbol{y} \in Y\} > -p_j \nu_j .$$

([6]) Condition (3.26) of Theorem 7 is satisfied.

If $N_j < +\infty$, replace $\boldsymbol{x}^{\alpha}$ by $\boldsymbol{x}^{\alpha'}$ obtained from (3.8) and $\boldsymbol{y}^{\beta}$ by $\bar{\boldsymbol{y}}$ and return to Step 2. Otherwise the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 7. Check whether $\eta^1 - \eta^0 = \emptyset$.

(a) If so, the point $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is an optimal solution of problem BLP (7).

(b) If not, then compute the values μ_i for $i \in \eta^1 - \eta^0$ according to (3.7) and check whether (3.15) are satisfied.

(b_1) If so, go to Step 8.

(b_2) If not, then pick $i \in \eta^1 - \eta^0$ and $\bar{\boldsymbol{x}} \in X$ such that

$$M_i = (\boldsymbol{p}_{\xi}^{\mathrm{T}} + \mu_i V_{\{i\}\xi})\bar{\boldsymbol{x}}_{\xi} = \max\{(\boldsymbol{p}_{\xi}^{\mathrm{T}} + \mu_i V_{\{i\}\xi})\boldsymbol{x}_{\xi} \mid \boldsymbol{x} \in X\} > -q_i\mu_i.$$

If $M_i < +\infty$, replace $\boldsymbol{x}^{\alpha}$ by $\bar{\boldsymbol{x}}$ and $\boldsymbol{y}^{\beta}$ by $\boldsymbol{y}^{\beta'}$ obtained from (3.9) and return to Step 2. Otherwise the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 8. Find the values z_j for $j \in \xi^1 - \xi^0$ according to (3.16) and check whether $z_j < +\infty$.

(a) If so, check whether (3.26) are satisfied for $j \in \xi^1 - \xi^0$.

(a_1) If so, the point $(\boldsymbol{x}^{\alpha}, \boldsymbol{y}^{\beta})$ is an optimal solution of problem BLP.

(a_2) If not, then find the values u_i for $i \in \eta^1 - \eta^0$ according to (3.17) and check whether $u_i < +\infty$.

(a_2^1) If so, go to Step 9.

(a_2^2) If not, then find the vector $\bar{\boldsymbol{u}}_{\eta}$ according to (4.2) and apply the algorithm A(ii). If condition (ii) is satisfied, replace u_i for $i \in \{i \in \eta^1 \mid u_i = +\infty\}$ by the values u_i^{p} defined by (3.21) and go to Step 9. Otherwise the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

(b) If not, then find the values u_i for $i \in \eta^1 - \eta^0$ according to (3.17) and check whether $u_i < +\infty$.

(b_1) If so, go to Step 10.

(b_2) If not, then find the vector $\bar{\boldsymbol{u}}_{\eta}$ according to (4.2) and apply the algorithm A(i) to check whether condition (i) is satisfied.

(b_2^1) If so, apply the algorithm A(ii). If condition (ii) is satisfied, replace u_i for $i \in \{i \in \eta^1 \mid u_i = +\infty\}$ by the values u_i^{p} defined by (3.21) and go to Step 10. Otherwise the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

(7) Condition (3.27) of Theorem 7 is satisfied.

(b_2^2) If not, then the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 9. Check whether (3.27) are satisfied for $i \in \eta^1 - \eta^0$.

(a) If so, the point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

(b) If not, go to Step 11.

Step 10. Check if (3.27) are satisfied for $i \in \eta^1 - \eta^0$.

(a) If so, the point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

(b) If not, then find the vector $\bar{\boldsymbol{z}}_\xi$ according to (4.1) and apply the algorithm A(iii) to check if condition (iii) is satisfied.

(b_1) If so, replace z_j for $j \in \{j \in \xi^1 \mid z_j = +\infty\}$ by the values z_j^p defined by (3.20) and check whether (3.26) are satisfied for $j \in \xi^1 - \xi^0$.

(b_1^1) If so, the point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

(b_1^2) If not, go to Step 11.

(b_2) If not, then the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Step 11. Find the value σ_f defined by (3.24) and take $\boldsymbol{x}^1 = \boldsymbol{x}^\alpha$ as the point maximizing the function $f(\boldsymbol{x})$ (defined by (3.22)) on the set X^* and go to Step 12.

Step 12. Find (8) the vertex $\boldsymbol{x}^2 \in X^*$ which follows directly $\boldsymbol{x}^1$ and check whether $f(\boldsymbol{x}^2) \leqslant -\sigma_f$.

(a) If so, the point $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of problem BLP.

(b) If not, then replace $\boldsymbol{x}^1$ by $\boldsymbol{x}^2$ and go to Step 13.

Step 13. Find an optimal solution $\bar{\boldsymbol{y}}$ of the problem $\max\{\Delta(\boldsymbol{x}^1, \boldsymbol{y}) \mid \boldsymbol{y} \in Y\}$ and check whether $\Delta(\boldsymbol{x}^1, \bar{\boldsymbol{y}}) \leqslant 0$.

(a) If so, return to Step 12.

(b) If not, then replace $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ by $(\boldsymbol{x}^1, \bar{\boldsymbol{y}})$ and return to Step 2 (9).

Remark 6. Since the number of vertices of the set $X \times Y$ is finite and the algorithm does not generate new vertices, the algorithm terminates in a finite number of iterations.

5. Numerical examples. Examples given in this section are relatively simple. They are only aimed to illustrate some paths of the algorithm in the case of an unbounded set of feasible solutions.

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(8) By the method from [7].

(9) If the algorithm goes from Step 13 back to Step 2, then all parts of the algorithm dealing with the boundedness of the function $F(\boldsymbol{x}, \boldsymbol{y})$ should be omitted. Let us notice then that the examination of the boundedness has been completed in Step 10.

Example 2. Maximize

$$F(\boldsymbol{x}, \boldsymbol{y}) = [3 \quad -1 \quad -1]\boldsymbol{x} + [2 \quad 1]\boldsymbol{y} + \boldsymbol{y}^{\mathrm{T}}\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}\boldsymbol{x}$$

subject to

$$\boldsymbol{x} \in X = \left\{\boldsymbol{x} \in R^3 \,\middle|\, \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix},\ \boldsymbol{x} \geqslant \boldsymbol{0}\right\},$$

$$\boldsymbol{y} \in Y = \{\boldsymbol{y} \in R^2 \mid [1 \quad 2]\boldsymbol{y} = [4],\ \boldsymbol{y} \geqslant \boldsymbol{0}\}.$$

Step 1. Taking $\alpha = \{2, 3\}$, $\beta = \{2\}$ and, consequently,

$$A_\alpha = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}, \quad B_\beta = [2], \quad \xi = \{1\}, \quad \eta = \{1\},$$

we get a basic feasible solution $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) = ([0 \quad 1 \quad 2]^{\mathrm{T}}, [0 \quad 2]^{\mathrm{T}})$.

Step 2. According to (3.3) we obtain

$$\boldsymbol{p}_\xi = [p_1]$$

$$= [3] + [2 \quad 0]\begin{bmatrix} 0 \\ 2 \end{bmatrix} - [-2 \quad -1]\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = [10] > [0].$$

Let us notice that condition (3.10) is not satisfied and $E_{\{1\}} = [-2 \; -1]^{\mathrm{T}} \leqslant \mathbf{0}$. Hence we conclude (by Proposition 1) that the function $F(\boldsymbol{x}, \boldsymbol{y})$ is unbounded from above on the set $X \times Y$.

Remark 7. Let us notice that any point $\boldsymbol{x} \in X$ can be expressed as follows:

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad x_1 \geqslant 0.$$

Hence the function $F(\boldsymbol{x}, \boldsymbol{y}^\beta) = 3x_1 + x_2 + 5x_3 + 2$ is of the form $F(\boldsymbol{x}, \boldsymbol{y}^\beta) = 10x_1 + 13$ for $x_1 \geqslant 0$ and

$$\lim_{x_1 \to +\infty} F(\boldsymbol{x}, \boldsymbol{y}^\beta) = +\infty.$$

Example 3. Maximize

$$F(\boldsymbol{x}, \boldsymbol{y}) = [3 \quad -1 \quad -1]\boldsymbol{x} + [2 \quad 1]\boldsymbol{y} + \boldsymbol{y}^{\mathrm{T}}\begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\boldsymbol{x}$$

on the unbounded set $X \times Y$ defined in Example 2.

Iteration 1.

Step 1. We get a basic feasible solution

$$(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) = ([0 \quad 1 \quad 2]^{\mathrm{T}}, [0 \quad 2]^{\mathrm{T}}),$$

where $\alpha = \{2, 3\}$, $\beta = \{2\}$, $\xi = \{1\}$ and $\eta = \{1\}$ as in Example 2.

Step 2. According to (3.3) we obtain

$$\boldsymbol{p}_\xi = [p_1]$$

$$= [3]+[2 \quad 0]\begin{bmatrix}0\\2\end{bmatrix}-[-2 \quad -1]\left(\begin{bmatrix}-1\\-1\end{bmatrix}+\begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\right) = 0.$$

Condition (3.10) is satisfied.

Step 3. According to (3.4) we obtain

$$\boldsymbol{q}_\eta = [q_1]$$

$$= [2]+[0 \quad 1 \quad 2]\begin{bmatrix}2\\-2\\1\end{bmatrix}-\begin{bmatrix}\frac{1}{2}\end{bmatrix}\left([1]+[0 \quad 1 \quad 2]\begin{bmatrix}0\\1\\-2\end{bmatrix}\right) = [3] > 0,$$

which does not satisfy condition (3.11). Then $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is not an equilibrium point. Since $\boldsymbol{H}_\eta = \boldsymbol{H}_{\{1\}} = [\frac{1}{2}] \nleqslant 0$, we apply (3.7) to compute $\mu_1 = 2 : \frac{1}{2} = 4$ and an adjacent basic feasible solution $\boldsymbol{y}^{\beta'} = [4 \quad 0]^{\mathrm{T}}$ with respect to the basis $\boldsymbol{B}_{\beta'} = [1]$. Now we take the point $([0 \quad 1 \quad 2]^{\mathrm{T}}, [4 \quad 0]^{\mathrm{T}})$ as $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$, where $\alpha = \{2, 3\}$, $\beta = \{1\}$, $\xi = \{1\}$, $\eta = \{2\}$.

Iteration 2.

Step 2. $\boldsymbol{p}_\xi = [p_1] = [-4] \leqslant 0$. Thus condition (3.10) is satisfied.

Step 3. $\boldsymbol{q}_\eta = [q_2] = [-6] \leqslant 0$ and condition (3.11) is satisfied. Then $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta) = ([0 \quad 1 \quad 2]^{\mathrm{T}}, [4 \quad 0]^{\mathrm{T}})$ is an equilibrium point.

Step 4. According to (3.5) we compute $V_{\eta\xi}$:

$$V_{\eta\xi} = V_{\{2\}\{1\}}$$

$$= [2]\left([-2 \quad 1]\begin{bmatrix}-2\\-1\end{bmatrix}-[2]\right)-[1 \quad -2]\begin{bmatrix}-2\\-1\end{bmatrix}+[0] = [2] > [0].$$

Now $\xi^1 = \{1\}$, $\eta^1 = \{2\}$ but

$$\boldsymbol{E}_{\{1\}} = \begin{bmatrix}-2\\-1\end{bmatrix} \leqslant 0,$$

so $\xi^0 = \{1\}$ and $\xi^1 \cap \xi^0 = \{1\} \neq \emptyset$. Since

$$\max\{V_{\eta\{1\}}^{\mathrm{T}} \boldsymbol{y}_\eta \mid \boldsymbol{y} \in Y\} = \max\{2y_2 \mid 0 \leqslant y_2 \leqslant 2\} = 4 \leqslant -p_1 = 4,$$

condition (3.12) is satisfied.

Step 5. $H_{\{2\}} = [2] \nleqslant 0$, so $\eta^0 = \emptyset$ and $\eta^1 \cap \eta^0 = \emptyset$. (It is understood that (3.13) is satisfied.)

Step 6. $\xi^1 - \xi^0 = \emptyset$, thus $(\boldsymbol{x}^\alpha, \boldsymbol{y}^\beta)$ is an optimal solution of the problem.

Therefore

$$\max\{F(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \in X \times Y\} = F([0 \quad 1 \quad 2]^{\mathrm{T}},\ [4 \quad 0]^{\mathrm{T}}) = 5.$$

Remark 8. Let us notice that the matrix corresponding to the optimal solution of the problem from Example 3 is of the form $V_{\eta\xi} = [2]$. It should be emphasized that in this case one of the sufficient conditions for optimality given in Theorem 4 is not satisfied.

Furthermore, according to (3.17), we have

$$u_2 = \max\{V_{\{2\}\xi}\boldsymbol{x}_\xi \mid \boldsymbol{x} \in X\} = \max\{2x_1 \mid x_1 \geqslant 0\} = +\infty.$$

This shows that the finite values defined by (3.16) and (3.17) may not exist even when the objective function of problem BLP is bounded from above on the set of feasible solutions (i.e. an optimal solution exists).

References

[1] M. Altman, *Bilinear programming*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 16 (1968), p. 741-746.

[2] A. V. Cabot and R. L. Francis, *Solving certain nonconvex quadratic minimization problems by ranking extreme points*, Operations Res. 18 (1970), p. 82-86.

[3] I. Czochralska, *The method of bilinear programming for nonconvex quadratic programming*, this fascicle, p. 515-525.

[4] — and W. Grabowski, *Maximization of a linear form over a convex hull of vertices of a convex polyhedral set*, Zastos. Mat. 18 (1983) (to appear).

[5] G. Gallo and A. Ülkücu, *Bilinear programming: an exact algorithm*, Math. Programming 12 (1977), p. 173-194.

[6] H. Konno, *A cutting plane algorithm for solving bilinear programs*, ibidem 11 (1976), p. 14-27.

[7] K. G. Murty, *Solving the fixed charge problems by ranking the extreme points*, Operations Res. 16 (1968), p. 268-279.

[8] E. N. Sokirjanskaja (Е. Н. Сокирянская), *Некоторые алгоритмы билинейного программирования*, Кибернетика 4 (1974), p. 106-112.

[9] H. Vaish and C. M. Shetty, *A cutting plane algorithm for the bilinear programming problem*, Naval Res. Logist. Quart. 24 (1977), p. 83-94.

[10] — *The bilinear programming problem*, ibidem 23 (1976), p. 303-309.

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PROGRAMOWANIE BILINIOWE

STRESZCZENIE

W pracy rozpatruje się ogólny problem programowania biliniowego określony przez (1.1)-(1.3), w którym zbiór rozwiązań dopuszczalnych jest wielościennym zbiorem wypukłym (niekoniecznie ograniczonym). Problem ten należy do klasy wieloekstremalnych zagadnień programowania matematycznego i charakteryzuje się tym, że jego maksima lokalne (a w związku z tym również rozwiązanie optymalne) znajdują się w wierzchołkach zbioru rozwiązań dopuszczalnych.

Po szczegółowym omówieniu własności ogólnego problemu programowania biliniowego, które zawierają m. in. warunki konieczne i dostateczne istnienia rozwiązania optymalnego, opisano algorytm rozwiązujący ten problem i zilustrowano go prostymi przykładami numerycznymi.