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S. LEWANOWICZ (Wrocław)

CONSTRUCTION OF THE LOWEST-ORDER RECURRENCE RELATION FOR THE JACOBI COEFFICIENTS

1. INTRODUCTION

A function f which is continuous in the interval $\langle -1, 1\rangle$ and satisfies the required conditions (see, e.g., [3], Vol. II, § 10.19, or [7], Vol. I, § 8.3) may be expanded into a uniformly convergent series of Jacobi polynomials in the form

$$f(x) = \sum_{k=0}^{\infty} a_k[f] P_k^{(\alpha,\beta)}(x) \quad (-1 \leqslant x \leqslant 1), \tag{1.1}$$

where $P_k^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ is the usual notation for the k-th Jacobi polynomial (cf. [3], Vol. II, § 10.8), and the coefficients $a_k[f]$ are given by

$$a_k[f] := \frac{(2k+\lambda)k!\,\Gamma(k+\lambda)}{2^{\lambda}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_k^{(\alpha,\beta)}(x) f(x)\,dx \tag{1.2}$$
$$(k = 0, 1, \ldots),$$

where

$$\lambda := \alpha+\beta+1. \tag{1.3}$$

Various techniques are available for the determination of the coefficients $a_k[f]$ (see, e.g., [7], Vol. II, Chapter IX). It is relatively easy to find the values of these coefficients if they satisfy a recurrence relation of the form

$$\sum_{j=0}^{r} \omega_j(k) a_{k+j}[f] = \omega(k), \tag{1.4}$$

where ω_j and ω are given functions of the variable k.

A simple and universal method of constructing the relation of the form (1.4) may be applied in the case where the function f satisfies the linear differential equation

$$\sum_{i=0}^{n} p_i f^{(i)} = q, \tag{1.5}$$

where $p_0, p_1, \ldots, p_n$ ($p_n \not\equiv 0$) are polynomials, and the coefficients $a_k[q]$ ($k = 0, 1, \ldots$) are known. Such a method was first proposed by Clenshaw [1] for the Chebyshev series of f, which is closely related to series (1.1) for $\alpha = \beta = -1/2$, and generalized by Elliott [2] to the case of the Gegenbauer series of f, which is in fact the Jacobi series (1.1) with $\alpha = \beta > -1$.

Paszkowski ([9], Section 13) gave some significant improvements of Clenshaw's method and raised the problem of constructing the recurrence relation for the Chebyshev coefficients which has the *lowest order* among all such relations following from (1.5) and from the basic difference and differential properties of the Chebyshev polynomials. (Relation (1.4) is said to be of *order* r if functions ω_0 and ω_r do not vanish identically.)

The complete solution to this problem, even in the more general case of the Gegenbauer series expansion of f, was given by the author [4]. The techniques developed in [4] were also successfully used in the construction of recurrence relations for the so-called modified moments [5] and for the coefficients of the Neumann-Gegenbauer expansion in Bessel functions of the first kind [6].

Robertson [10] wrote an ALTRAN program for constructing a recurrence relation for the Gegenbauer series coefficients, which implements the method given in [4].

Some examples (see, e.g., [8]) show that the case $\alpha \neq \beta$ can also be of practical interest. The possibility of construction of a recurrence relation, starting from equation (1.5) and using some basic properties of the Jacobi polynomials (see Section 2), is obvious. The purpose of this paper is to present an algorithmic description of the method leading to a recurrence relation for $a_k[f]$, which has the lowest possible order.

The method, called the *optimum method*, is expressed in terms of a certain type linear operator discussed in Section 3. Section 4 is devoted to the description of the optimum method. In Section 5 we give another method, called the *Paszkowski-type method*, which is not the optimum one but is much simpler than the first method.

2. BASIC PROPERTIES OF THE JACOBI COEFFICIENTS

We assume in the sequel that the parameters α and β are fixed, $\alpha \neq \beta$, $\alpha > -1$, $\beta > -1$.

Let us recall the recurrence formulas

$$(2.1)\qquad (2k+\lambda-1)_3 x P_k^{(\alpha,\beta)}(x) = 2(k+\alpha)(k+\beta)(2k+\lambda+1)P_{k-1}^{(\alpha,\beta)}(x)+ \\ +(\beta^2-\alpha^2)(2k+\lambda)P_k^{(\alpha,\beta)}(x)+2(k+1)(k+\lambda)(2k+\lambda-1)P_{k+1}^{(\alpha,\beta)}(x) \\ (k = 1, 2, \ldots),$$

$$\text{(2.2)}\qquad \frac{(2k+\lambda-1)_3}{2(k+\lambda)}(1-x^2)\frac{d}{dx}P_k^{(\alpha,\beta)}(x)$$
$$= (k+\alpha)(k+\beta)(2k+\lambda+1)P_{k-1}^{(\alpha,\beta)}(x)+(\alpha-\beta)k(2k+\lambda)P_k^{(\alpha,\beta)}(x)-$$
$$-(k)_2(2k+\lambda-1)P_{k+1}^{(\alpha,\beta)}(x) \qquad (k=1,2,\ldots)$$

([3], Vol. II, § 10.8), where we use the Pochhammer symbol

$$(a)_m = a(a+1)\ldots(a+m-1)\ (m=1,2,\ldots), \qquad (a)_0 = 1.$$

Using the above equations one can obtain the following relations for coefficients (1.2):

$$\text{(2.3)}\qquad (2k+\lambda-2)_2(2k+\lambda+1)_2 a_k[xf(x)]$$
$$= 2k(k+\lambda)(2k+\lambda+1)_2 a_{k-1}[f]+(\beta^2-\alpha^2)\big((2k+\lambda)^2-4\big)a_k[f]+$$
$$+2(k+\alpha+1)(k+\beta+1)(2k+\lambda-2)_2 a_{k+1}[f],$$

$$\text{(2.4)}\qquad \tfrac{1}{2}(k+\lambda)(2k+\lambda-2)_2(2k+\lambda+1)_2 a_k[f]$$
$$= (k+\lambda-1)_2(2k+\lambda+1)_2 a_{k-1}[f']+(\alpha-\beta)(k+\lambda)\big((2k+\lambda)^2-4\big)a_k[f]'-$$
$$-(k+\alpha+1)(k+\beta+1)(2k+\lambda-2)_2 a_{k+1}[f'].$$

A useful simplification is obtained by introducing the notation

$$\text{(2.5)}\qquad b_k[f] := \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\lambda)(2k+\lambda-1)_3}a_k[f].$$

We call $b_k[f]$ the *Jacobi coefficients* of the function f. Relations (2.3) and (2.4) imply

$$\text{(2.6)}\qquad (2k+\lambda-1)_3 b_k[xf(x)] = 2k(k+\alpha)(2k+\lambda-3)b_{k-1}[f]+$$
$$+(\beta^2-\alpha^2)(2k+\lambda)b_k[f]+2(k+\lambda)(k+\beta+1)(2k+\lambda+3)b_{k+1}[f],$$

$$\text{(2.7)}\qquad \tfrac{1}{2}(2k+\lambda-1)_3 b_k[f] = (k+\alpha)(2k+\lambda-3)b_{k-1}[f']+$$
$$+(\alpha-\beta)(2k+\lambda)b_k[f']-(k+\beta+1)(2k+\lambda+3)b_{k+1}[f'].$$

If we define $P_k^{(\alpha,\beta)} \equiv 0$ for $k = -1, -2, \ldots;$ then (2.1) and (2.2) are valid for all integer values of k. Consequently, we have

$$b_k[f] := a_k[f] := 0 \qquad \text{for } k = -1, -2, \ldots,$$

and (2.6) and (2.7) can be used for all integer values of k.

3. DIFFERENCE OPERATORS

3.1. Preliminaries. Algorithms given in Sections 4 and 5 are expressed in terms of a certain type linear operator. Let $\mathscr{S}$ denote the linear space of "doubly infinite" sequences of complex numbers with addition of sequences and scalar multiplication defined as usual. Obviously, $\mathscr{S}$ is the space of all complex-valued functions defined on the set of all integers. Let $\mathscr{S}_{\text{rat}}$ denote the set of all rational functions $s \in \mathscr{S}$.

Consider the set $\mathscr{T}$ of all linear operators mapping $\mathscr{S}$ into itself. If $T \in \mathscr{T}$ and $\{z_k\} \in \mathscr{S}$, we denote the k-th coordinate of the sequence $T\{z_k\} \in \mathscr{S}$ by Tz_k, so that $T\{z_k\} = \{Tz_k\}$. The *zero operator*, the *identity operator*, and the *m-th shift operator* in $\mathscr{T}$ are denoted by Θ, I and E^m, respectively. Then we have

$$Iz_k = z_k, \quad \Theta z_k = 0, \quad E^m z_k = z_{k+m} \tag{3.1}$$

for every $\{z_k\} \in \mathscr{S}$. Clearly, $E^0 = I$.

Let $\mathscr{L}$ be the set of all operators $L \in \mathscr{T}$ such that

$$L = \sum_{j=0}^{r} \lambda_j(k) E^{u+j}, \tag{3.2}$$

where $r \geqslant 0$ and u are integers, and $\lambda_0, \lambda_1, \ldots, \lambda_r \in \mathscr{S}_{\text{rat}}$. Every non-zero operator $L \in \mathscr{L}$ can be expressed in the form (3.2) with $\lambda_0 \not\equiv 0$ and $\lambda_r \not\equiv 0$. The number $r = r(L)$ is referred to as the *order of the operator* L, while $\lambda_0, \lambda_1, \ldots, \lambda_r$ are called the *coefficients* of L. The elements of the set $\mathscr{L}$ are known as *difference operators*.

Let $L \in \mathscr{L}$ be defined by (3.2) and let $M \in \mathscr{L}$ be such that

$$M = \sum_{j=0}^{t} \mu_j(k) E^{v+j}.$$

We define the product of L and M to be the operator

$$LM := \sum_{i=0}^{r} \lambda_i(k) \sum_{j=0}^{t} \mu_j(k+u+i) E^{u+v+i+j}.$$

It can be seen that under this definition of multiplication and the addition defined in a natural manner $\mathscr{L}$ forms a ring with identity I.

Let $L \in \mathscr{L}$ and $\omega \in \mathscr{S}$. The equation $Lz_k = \omega(k)$ is the *recurrence relation* for the sequence $\{z_k\} \in \mathscr{S}$. The *order of the recurrence relation* is the order of the difference operator L.

3.2. The operator X. Using the symbol of the operator X, defined by

$$\begin{aligned} X := {} & (2k+\lambda-1)_3^{-1}[2k(k+a)(2k+\lambda-3)E^{-1} + \\ & + (\beta^2 - a^2)(2k+\lambda)I + 2(k+\lambda)(k+\beta+1)(2k+\lambda+3)E], \end{aligned} \tag{3.3}$$

we can rewrite (2.6) in the following form:

$$Xb_k[f] = b_k[xf(x)]. \tag{3.4}$$

Let p be the polynomial defined by

$$p(x) := \sum_{j=0}^{d} c_j x^j.$$

The symbol $p(X)$ denotes the following difference operator:

$$p(X) := \sum_{j=0}^{d} c_j X^j.$$

Now, it is easy to generalize (3.4). Namely, we have

$$p(X)b_k[f] = b_k[p(x)f(x)]. \tag{3.5}$$

3.3. The operator D and related operators. The operator $D \in \mathscr{L}$ defined by

$$D := \delta_0(k)E^{-1} + \delta_1(k)I + \delta_2(k)E, \tag{3.6}$$

where

$$\begin{aligned} \delta_0(k) &:= (k+a)(2k+\lambda-3), \qquad \delta_1(k) := (a-\beta)(2k+\lambda), \\ \delta_2(k) &:= -(k+\beta+1)(2k+\lambda+3), \end{aligned} \tag{3.7}$$

plays a very important role in the sequel. Observe that (2.7) can be rewritten as [1]

$$Db_k[f'] = \tfrac{1}{2}(2k+\lambda-1)_3 b_k[f]. \tag{3.8}$$

Before generalizing this identity, let us introduce the difference operators B_i, S_{ij}, and P_i by means of the formulas

$$\begin{cases} B_0 := D, \\ B_1 := \dfrac{(k+a)(2k+\lambda+2)}{2k+\lambda-1}E^{-1} + (a-\beta)\dfrac{(2k+\lambda)^2-4}{(2k+\lambda)^2-1}I - \\ \qquad\qquad - \dfrac{(k+\beta+1)(2k+\lambda-2)}{2k+\lambda+1}E, \\ B_i := (2k+\lambda-1)_3^{-1}\big(\delta_0(k)(2k+\lambda+2i-3)_4E^{-1} + \\ \quad + \delta_1(k)(2k+\lambda-2i)_2(2k+\lambda+2i-1)_2 I + \delta_2(k)(2k+\lambda-2i)_4 E\big) \\ \qquad\qquad\qquad\qquad (i = 2, 3, \ldots), \end{cases} \tag{3.9}$$

---

[1] Notice that in the case $a = \beta$ operator (3.6) takes the form $2D^*[(k+a)_2 I]$, where $D^* := E^{-1} - E$. Introducing the notation $c_k[f] := 2(k+a)_2 b_k[f]$, we have the identity $D^* c_k[f'] = 2(k+a+1/2)c_k[f]$ which is much simpler than (3.8) (cf. [4]).

$$S_{ij} := \begin{cases} I & (i < j), \\ B_i B_{i-1} \dots B_j & (i \geqslant j \geqslant 0), \end{cases} \tag{3.10}$$

$$P_i := S_{i-1,0} \quad (i = 0, 1, \dots). \tag{3.11}$$

Let $\gamma_0, \gamma_1, \dots \in \mathscr{S}$ be polynomials given by

$$\gamma_0(k) := 1, \quad \gamma_1(k) := (2k+\lambda-1)_3, \tag{3.12}$$
$$\gamma_i(k) := (2k+\lambda-2i+2)_{4i-3} \quad (i = 2, 3, \dots).$$

It may be checked that

$$B_i\big(\gamma_i(k)I\big) = (2k+\lambda-1)_3^{-1}\gamma_{i+1}(k)D \quad (i = 0, 1, \dots),$$

which together with (3.8) implies

$$B_i\big(\gamma_i(k)b_k[f']\big) = \tfrac{1}{2}\gamma_{i+1}(k)b_k[f] \quad (i = 0, 1, \dots). \tag{3.13}$$

Lemma 3.1. *For any $i = 0, 1, \dots$ we have the identity*

$$P_i b_k[f^{(i)}] = 2^{-i}\gamma_i(k)b_k[f]. \tag{3.14}$$

Proof. We apply the method of induction on i. For $i = 0$ equation (3.14) holds trivially, and for $i = 1$ it has the form (3.8). Assuming that (3.14) holds for a certain i $(i \geqslant 1)$ and using the equality $P_{i+1} = B_i P_i$ (cf. (3.11)) and (3.13), we get

$$P_{i+1}b_k[f^{(i+1)}] = 2^{-i-1}\gamma_{i+1}(k)b_k[f].$$

Identity (3.14) is therefore true for any $i = 0, 1, \dots$

3.4. The set $\mathscr{A}^*(L)$. Given an operator $L \in \mathscr{L}$ we define the following sets of difference operators:

$$\mathscr{A}(L) := \big\{A \in \mathscr{L}\setminus\{\Theta\} \mid \underset{Q\in\mathscr{L}\setminus\{\Theta\}}{\exists} AL = QD\big\}, \tag{3.15}$$

$$\mathscr{A}^*(L) := \big\{A \in \mathscr{A}(L) \mid \underset{B\in\mathscr{A}(L)}{\forall} r(A) \leqslant r(B)\big\}, \tag{3.16}$$

where $r(P)$ denotes the order of the operator $P \in \mathscr{L}$, and D is defined by (3.6).

The following lemma can be derived from the results obtained in [4]:

Lemma 3.2. (i) *If $A \in \mathscr{A}(L)$ and $C \in \mathscr{L}\setminus\{\Theta\}$, then $CA \in \mathscr{A}(L)$.*

(ii) *If $A \in \mathscr{A}(L)$ and $B \in \mathscr{A}^*(L)$, then there exists an operator $C \in \mathscr{L}\setminus\{\Theta\}$ such that the equation $A = CB$ holds.*

(iii) *If $A, B \in \mathscr{A}^*(L)$, then there exist $\varrho \in \mathscr{S}_{\text{rat}}$ and an integer m such that $A = \varrho(k)E^m B$.*

The last part of the lemma means that all the operators belonging to the set $\mathscr{A}^*(L)$ are inessential modifications of any fixed operator from this set. In Lemma 3.4 (below) we construct an operator belonging to $\mathscr{A}^*(L)$.

First we show the following

LEMMA 3.3. *Let $L \in \mathscr{L}$ be a non-zero operator of order r given by*

$$L = \sum_{j=0}^{r} \lambda_j(k) E^{u+j}. \tag{3.17}$$

Define $\nu_0, \nu_1, \ldots, \nu_r \in \mathscr{S}_{\mathrm{rat}}$ recursively by

$$\nu_j(k) := \begin{cases} 0 & (j = r-1, r), \\ [\lambda_{j+2}(k) - \delta_1(k+j+u+2)\nu_{j+1}(k) - \\ \quad - \delta_0(k+j+u+3)\nu_{j+2}(k)]/\delta_2(k+j+u+1) & \\ & (j = r-2,\ r-3, \ldots, 0), \end{cases} \tag{3.18}$$

where δ_0, δ_1, and δ_2 are the coefficients of the operator D, given by (3.7). *Put*

$$N := \sum_{j=0}^{r} \nu_j(k) E^{u+j+1}. \tag{3.19}$$

Define $\sigma_0, \sigma_1 \in \mathscr{S}_{\mathrm{rat}}$ by

$$\begin{aligned} \sigma_0(k) &:= \lambda_0(k) - \delta_0(k+u+1)\nu_0(k), \\ \sigma_1(k) &:= \begin{cases} 0 & (r = 0), \\ \lambda_1(k) - \delta_1(k+u+1)\nu_0(k) - \delta_0(k+u+2)\nu_1(k) & \\ & (r = 1, 2, \ldots). \end{cases} \end{aligned} \tag{3.20}$$

Let

$$W := \sigma_0(k)E^u + \sigma_1(k)E^{u+1}. \tag{3.21}$$

Then

$$L = ND + W \tag{3.22}$$

and, moreover, the equation

$$\mathscr{A}^*(L) = \mathscr{A}^*(W) \tag{3.23}$$

holds.

Proof. Equation (3.22) can be easily checked by calculating the coefficients of the operator $ND+W$ and comparing them with the corresponding coefficients of the operator L.

We show that

$$\mathscr{A}(L) = \mathscr{A}(W), \tag{3.24}$$

which clearly implies (3.23).

1° If $A \in \mathscr{A}(L)$, then according to (3.15) there exists an operator $Q \in \mathscr{L} \setminus \{\Theta\}$ such that $AL = QD$, which together with (3.22) implies the relation $AW = (Q - AN)D$ meaning that $A \in \mathscr{A}(W)$.

2° Let $A \in \mathscr{A}(W)$. Applying the operator A to both sides of (3.22) and taking into account that there exists an operator $R \in \mathscr{L} \setminus \{\Theta\}$ such that $AW = RD$, we obtain the equation $AL = (AN + R)D$. Hence we get $A \in \mathscr{A}(L)$, which completes the proof.

For $\lambda, \mu \in \mathscr{S}_{\mathrm{rat}}$ let us define

$$\{\lambda, \mu\} := \frac{\gcd(\underline{\lambda}, \underline{\mu})}{\gcd(\bar{\lambda}, \bar{\mu})},$$

where $\underline{\lambda}$, $\underline{\mu}$ and $\bar{\lambda}$, $\bar{\mu}$ are the numerators and denominators of λ, μ, respectively.

LEMMA 3.4. *Let $L \in \mathscr{L}$ be defined by* (3.17) *and let $\sigma_0, \sigma_1 \in \mathscr{S}_{\mathrm{rat}}$ be functions given by* (3.20). *Let $A, R \in \mathscr{L}$ be defined by*

$$A := a_0(k)E^{-1} + a_1(k)I + a_2(k)E, \tag{3.25}$$

$$R := \varrho_0(k)I + \varrho_1(k)E, \tag{3.26}$$

where the coefficients $a_0, a_1, a_2, \varrho_0, \varrho_1 \in \mathscr{S}_{\mathrm{rat}}$ are determined in the following manner.

Case I. $\sigma_0 = \sigma_1 \equiv 0$. *Define*

$$a_0(k) := a_2(k) := 0, \qquad a_1(k) := 1, \tag{3.27}$$

$$\varrho_0(k) := \varrho_1(k) := 0. \tag{3.28}$$

Case II. $\sigma_0 \not\equiv 0$, $\sigma_1 \not\equiv 0$, $\sigma_1(k) = \tau(k+u)\sigma_0(k)$, *where $\tau \in \mathscr{S}_{\mathrm{rat}} \setminus \{0\}$ satisfies the difference equation*

$$\delta_0(k)\tau(k)\tau(k-1) - \delta_1(k)\tau(k) + \delta_2(k) = 0. \tag{3.29}$$

Define

$$a_0(k) := 0, \qquad a_i(k) := \omega(k)a_i^*(k) \ (i = 1, 2), \tag{3.30}$$

$$\varrho_0(k) := 0, \qquad \varrho_1(k) := \omega(k)\sigma_0(k)\sigma_1(k+1), \tag{3.31}$$

where

$$a_1^*(k) := \delta_0(k+u+1)\sigma_1(k+1), \qquad a_2^*(k) := \delta_2(k+u+1)\sigma_0(k), \tag{3.32}$$

$$\omega(k) := 1/\{a_1^*(k), a_2^*(k)\}. \tag{3.33}$$

Case III. $\sigma_0 \not\equiv 0$, $\sigma_1(k) = \tau(k+u)\sigma_0(k)$, *where* $\tau \in \mathscr{S}_{\mathrm{rat}}$ *does not satisfy* (3.29). *Define*

$$a_i(k) := \omega(k)\, a_i^*(k) \qquad (i = 0, 1, 2), \tag{3.34}$$

$$\varrho_j(k) := \omega(k)\varphi(k+u-j+1)\sigma_0(k-1)\sigma_0(k)\sigma_j(k+1) \qquad (j = 0, 1), \tag{3.35}$$

where

$$\begin{cases} a_0^*(k) := \delta_0(k+u)\varphi(k+u+1)\sigma_0(k)\sigma_0(k+1), \\ a_1^*(k) := \delta_0(k+u+1)\varphi(k+u)\sigma_0(k-1)\sigma_1(k+1) - \\ \qquad - \varphi(k+u+1)\sigma_0(k+1)[\delta_0(k+u)\sigma_1(k-1) - \\ \qquad - \delta_1(k+u)\sigma_0(k-1)], \\ a_2^*(k) := \delta_2(k+u+1)\varphi(k+u)\sigma_0(k)\sigma_0(k-1), \end{cases} \tag{3.36}$$

$$\omega(k) := 1/\big\{\{a_1^*(k),\, a_2^*(k)\},\, a_3^*(k)\big\}, \tag{3.37}$$

$$\varphi(k) := \delta_0(k)\tau(k)\tau(k-1) - \delta_1(k)\tau(k) + \delta_2(k). \tag{3.38}$$

Case IV. $\sigma_0 \equiv 0$, $\sigma_1 \not\equiv 0$. *Define*

$$a_i(k) := \omega(k)\,\delta_i(k+u+1)\sigma_1(k-1)\sigma_1(k)\sigma_1(k+1)/\sigma_1(k+i-1) \qquad (i = 0, 1, 2), \tag{3.39}$$

$$\varrho_0(k) := 0, \qquad \varrho_1(k) := \omega(k)\sigma_1(k-1)\sigma_1(k)\sigma_1(k+1), \tag{3.40}$$

where

$$\omega(k) := 1/\big\{\{\delta_0(k+u+1)\sigma_1(k),\, \delta_1(k+u+1)\sigma_1(k-1)\}\sigma_1(k+1),\; \delta_2(k+u+1)\sigma_1(k-1)\sigma_1(k)\big\}. \tag{3.41}$$

Here δ_0, δ_1, $\delta_2 \in \mathscr{S}_{\mathrm{rat}}$ *are the functions given by* (3.7).

Then in any of the cases I-IV *the operator* A *belongs to the set* $\mathscr{A}^*(L)$ *and we have*

$$AL = QD, \tag{3.42}$$

where $Q \in \mathscr{L}$,

$$Q := AN + RE^u, \tag{3.43}$$

$N \in \mathscr{L}$ *being the difference operator defined by* (3.19).

Proof. It suffices to show that in any of the cases I-IV the operator A, occurring in (3.25), belongs to the set $\mathscr{A}^*(W)$, where $W \in \mathscr{L}$ is given by (3.21), and that the equality

$$AW = RE^uD \tag{3.44}$$

holds. The thesis of the lemma follows then from (3.23) and (3.22).

For $\sigma_0 = \sigma_1 \equiv 0$ (case I) we have $W = \Theta$ and (3.44) holds for $A := I$ and $R := \Theta$. Hence (3.27) and (3.28) follow.

Let $\sigma_0 \not\equiv 0$ or $\sigma_1 \not\equiv 0$. Performing multiplications of operators in both sides of (3.44) and equating coefficients of the obtained operators, we get four equations:

$$
(3.45)\quad
\begin{aligned}
a_0(k)\sigma_0(k-1) &= \delta_0(k+u)\varrho_0(k),\\
a_0(k)\sigma_1(k-1)+a_1(k)\sigma_0(k) &= \delta_1(k+u)\varrho_0(k)+\delta_0(k+u+1)\varrho_1(k),\\
a_1(k)\sigma_1(k)+a_2(k)\sigma_0(k+1) &= \delta_1(k+u+1)\varrho_1(k)+\delta_2(k+u)\varrho_0(k),\\
a_2(k)\sigma_1(k+1) &= \delta_2(k+u+1)\varrho_1(k).
\end{aligned}
$$

Eliminating ϱ_0 and ϱ_1 we obtain the system

$$
(3.46)\quad
\begin{aligned}
&\delta_2(k+u+1)[\delta_0(k+u)\sigma_1(k-1)-\delta_1(k+u)\sigma_0(k-1)]a_0(k)+\\
&\quad+\delta_0(k+u)\delta_2(k+u+1)\sigma_0(k)a_1(k)-\\
&\quad-\delta_0(k+u)\delta_0(k+u+1)\sigma_1(k+1)a_2(k) = 0,\\
&\delta_2(k+u)\delta_2(k+u+1)\sigma_0(k-1)a_0(k)-\\
&-\delta_0(k+u)\delta_2(k+u+1)\sigma_1(k)a_1(k)+\delta_0(k+u)[\delta_1(k+u+1)\sigma_1(k+1)-\\
&\quad-\delta_2(k+u+1)\sigma_0(k+1)]a_2(k) = 0.
\end{aligned}
$$

Clearly, if any two of the functions a_0, a_1, a_2 vanish identically, then the third of them vanishes also identically. This means that the set $\mathscr{A}(W)$ *does not contain operators of the zero order.*

Assume that $\sigma_0 \not\equiv 0$. Define $\tau \in \mathscr{S}_{\mathrm{rat}}$ by $\tau(k) := \sigma_1(k-u)/\sigma_0(k-u)$. Then equations (3.46) take the form

$$
(3.47)\quad
\begin{aligned}
&\delta_2(k+u+1)\sigma_0(k-1)[\tau(k+u-1)\delta_0(k+u)-\delta_1(k+u)]a_0(k)+\\
&\quad+\delta_0(k+u)\delta_2(k+u+1)\sigma_0(k)a_1(k)-\\
&\quad-\delta_0(k+u)\delta_0(k+u+1)\tau(k+u+1)\sigma_0(k+1)a_2(k) = 0,\\
&\delta_2(k+u)\delta_2(k+u+1)\sigma_0(k-1)a_0(k)-\\
&\quad-\delta_0(k+u)\delta_2(k+u+1)\tau(k+u)\sigma_0(k)a_1(k)+\\
&+\delta_0(k+u)\sigma_0(k+1)[\delta_1(k+u+1)\tau(k+u+1)-\delta_2(k+u+1)]a_2(k) = 0.
\end{aligned}
$$

Put

$$\varphi(k) := \delta_0(k)\tau(k)\tau(k-1)-\delta_1(k)\tau(k)+\delta_2(k).$$

If $\varphi \equiv 0$, i.e., τ is a solution of (3.29) (case II), then (3.47) is reduced to the single equation

$$
\begin{aligned}
&\delta_2(k+u)\delta_2(k+u+1)\sigma_0(k-1)a_0(k)-\\
&\quad-\delta_0(k+u)\delta_2(k+u+1)\tau(k+u)\sigma_0(k)a_1(k)+\\
&\quad+\delta_0(k+u)\delta_0(k+u+1)\tau(k+u)\tau(k+u+1)\sigma_0(k+1)a_2(k) = 0.
\end{aligned}
$$

The operator (3.25), belonging to the set $\mathscr{A}(W)$, is therefore of the *first order*, e.g., for

$$a_0(k) := 0, \quad a_1(k) := \delta_0(k+u+1)\sigma_1(k+1),$$
$$a_2(k) := \delta_2(k+u+1)\sigma_0(k).$$

In this case, the first and the last equations of (3.45) imply

$$\varrho_0(k) := 0, \quad \varrho_1(k) := \sigma_0(k)\sigma_1(k+1).$$

Hence (3.30) and (3.31) hold, where the factor $\omega(k)$, defined by (3.33), was introduced in order to simplify the form of the operators A and R, which by Lemma 3.1 does not change the fact that $A \in \mathscr{A}^*(W)$.

If $\varphi \not\equiv 0$ (case III), then from (3.47) we get

$$\delta_2(k+u+1)\sigma_0(k-1)\varphi(k+u)a_0(k) -$$
$$- \delta_0(k+u)\sigma_0(k+1)\varphi(k+u+1)a_2(k) = 0.$$

Therefore, $r(A) \geqslant 2$ for every $A \in \mathscr{A}(W)$. The *second-order* operator (3.25) from the set $\mathscr{A}(W)$ is obtained, e.g., for $a_i := a_i^*$ $(i = 0, 1, 2)$, a_i^* being given by (3.36). From the first and the fourth equations of (3.45) we obtain

$$\varrho_0(k) := \varphi(k+u+1)\sigma_0(k-1)\sigma_0(k)\sigma_0(k+1),$$
$$\varrho_1(k) := \varphi(k+u)\sigma_0(k-1)\sigma_0(k)\sigma_1(k+1).$$

This proves (3.34) and (3.35), where the factor $\omega(k)$ defined by (3.37) has only a simplifying character.

Assume now that case IV occurs, i.e., $\sigma_0 \equiv 0$, $\sigma_1 \not\equiv 0$. Then system (3.46) takes the form

$$\delta_2(k+u+1)\sigma_1(k-1)a_0(k) - \delta_0(k+u+1)\sigma_1(k+1)a_2(k) = 0,$$
$$\delta_2(k+u+1)\sigma_1(k)a_1(k) - \delta_1(k+u+1)\sigma_1(k+1)a_2(k) = 0.$$

Thus it is clear that if $a_0, a_1, a_2 \in \mathscr{S}_{\mathrm{rat}}$ is a non-trivial solution of that system, then $a_j \not\equiv 0$ for $j = 0, 1, 2$, which means that $r(A) \geqslant 2$ for every $A \in \mathscr{A}(W)$. One of the possible solutions is given by

$$a_i(k) := \delta_i(k+u+1)\sigma_1(k-1)\sigma_1(k)\sigma_1(k+1)/\sigma_1(k+i-1) \quad (i = 0, 1, 2).$$

From (3.45) we obtain

$$\varrho_0(k) := 0, \quad \varrho_1(k) := \sigma_1(k-1)\sigma_1(k)\sigma_1(k+1).$$

Introducing the factor $\omega(k)$ defined by (3.41) we get (3.39) and (3.40).

COROLLARY 3.1. *For every $L \in \mathscr{L}$ and for every $A \in \mathscr{A}^*(L)$ we have* $0 \leqslant r(A) \leqslant 2$.

Remark 3.1. It can be checked that the functions $\tau_1, \tau_2 \in \mathscr{S}_{\text{rat}}$ defined by

$$(3.48)\qquad \tau_1(k) := \frac{2k+\lambda+3}{2k+\lambda-1} = -\frac{k+\alpha+1}{k+\beta+1}\,\frac{\delta_2(k)}{\delta_0(k+1)},$$

$$(3.49)\qquad \tau_2(k) := \frac{\delta_2(k)}{\delta_0(k+1)}$$

satisfy (3.29).

LEMMA 3.5. *Let p be a polynomial and let q and r, $r(x) = cx+d$, be the quotient and the remainder, respectively, from the division of p by x^2-1. The difference operator $A \in \mathscr{L}$ defined by*

$$(3.50)\qquad A := \begin{cases} I & \text{for } c = d = 0,\\ I+\tau_2(k)E & \text{for } c = d \neq 0,\\ I+\tau_1(k)E & \text{for } c = -d \neq 0,\\ D & \text{for } |c| \neq |d| \end{cases}$$

belongs in any case to the set $\mathscr{A}^*\big(p(X)\big)$. Moreover, the equation

$$(3.51)\qquad Ap(X) = QD$$

holds, where $Q \in \mathscr{L}$,

$$(3.52)\qquad Q := [Aq(X)C+S](2k+\lambda-1)_3^{-1}I,$$

C, $S \in \mathscr{L}$ being given by

$$(3.53)\qquad C := 4(2k+\lambda-1)_3^{-1}[(k-1)_2\,\delta_0(k)E^{-1}-k(k+\lambda)\,\delta_1(k)I+ \\ +(k+\lambda)_2\,\delta_2(k)E],$$

$$(3.54)\qquad S := \begin{cases} \Theta & \text{for } c = d = 0,\\ 2c[kI-(k+\lambda+1)\tau_2(k)E] & \text{for } c = d \neq 0,\\ 2c[kI-(k+\lambda+1)\tau_1(k)E] & \text{for } c = -d \neq 0,\\ 2c(k-1)\,\delta_0(k)E^{-1}+ & \\ +[d(2k+\lambda-1)_3-c(\lambda+1)\,\delta_1(k)]I+ & \\ +2c(k+\lambda+1)\,\delta_2(k)E & \text{for } |c| \neq |d|, \end{cases}$$

where $\delta_0, \delta_1, \delta_2 \in \mathscr{S}_{\text{rat}}$ are the functions defined in (3.7).

Proof. The equation $p(x) = q(x)(x^2-1)+cx+d$ implies

$$p(X) = q(X)(X^2-I)+cX+dI.$$

It can be verified that the identity $X^2-I = CD$ holds for C defined by (3.53). Hence we have

$$(3.55)\qquad p(X) = q(X)CD+cX+dI.$$

Now, it is clear that $\mathscr{A}^*(p(X)) = \mathscr{A}^*(cX+dI)$. Applying an operator $A \in \mathscr{A}^*(cX+dI)$ to both sides of (3.55), we obtain (3.51), where $Q \in \mathscr{L}$ is given by (3.52) and $S \in \mathscr{L}$ is such that $A(cX+dI) = SD$.

If $c = d = 0$, then, obviously, $I \in \mathscr{A}^*(p(X))$, which means that (3.51) holds for $Q := q(X)C$.

Assume that $|c|+|d| \neq 0$. Applying Lemma 3.3 to the operator

(3.56) $$L := cX+dI$$

we obtain (3.22) for

$$N := -\frac{2(k+\lambda)c}{(2k+\lambda-1)_3}I, \qquad W := \sigma_0(k)E^{-1}+\sigma_1(k)I,$$

$$\sigma_0(k) := \frac{2c(k+a)(2k+\lambda-3)}{(2k+\lambda)^2-1}, \qquad \sigma_1(k) := \frac{c(a-\beta)}{2k+\lambda-1}+d.$$

We consider three cases.

1° If $c = d \neq 0$, then $\sigma_1(k) = \tau_1(k-1)\sigma_0(k)$, τ_1 being the function defined by (3.48). The application of Lemma 3.4, where case II occurs for $\tau := \tau_1$, gives the difference operators $A^+ \in \mathscr{A}^*(L)$, $R^+ \in \mathscr{L}$,

$$A^+ := (k+a+1)I - \frac{(k+\beta+1)(2k+\lambda+3)}{2k+\lambda-1}E, \qquad R^+ := \frac{2c(k+a+1)}{(2k+\lambda)^2-1}E,$$

satisfying the equality $A^+L = S^+D$, where $S^+ := A^+N+R^+E^{-1}$. Clearly, $A := (k+a+1)^{-1}A^+$ is in $\mathscr{A}^*(L)$ and $AL = SD$ for $S := (k+a+1)^{-1}S^+$ taking the form as in the second part of (3.54).

2° If $c = -d \neq 0$, then $\sigma_1(k) = \tau_2(k-1)\sigma_0(k)$, τ_2 being the function defined by (3.49). By Lemma 3.4 (case II, $\tau := \tau_2$, $\omega := 1/a_1^*$) the difference operators A and R, defined as

$$A := I+\tau_1(k)E, \qquad R := \frac{2c}{(2k+\lambda)^2-1}E,$$

are such that $A \in \mathscr{A}^*(L)$ and $AL = SD$, where $S := AN+RE^{-1}$. It may be seen that S takes the form as in the third part of (3.54).

3° If $|c| \neq |d|$, then applying Lemma 3.4 to (3.56) we get case III ($c \neq 0$) or case IV ($c = 0$). In both cases we have $r(A) \geqslant 2$ for all A in $\mathscr{A}(L)$. It can be checked by straightforward calculations that D belongs to the set $\mathscr{A}^*(W)$ $(= \mathscr{A}^*(L))$ and that

(3.57) $$DL = SD,$$

where

$$S := DN+RE^{-1}, \qquad R := \frac{2c(k+a)}{2k+\lambda-1}I+\sigma_1(k+1)E.$$

It can be checked that S has coefficients given in the fourth part of (3.54).

Remark 3.2. It can be verified that the identities

$$(3.58)\qquad \left(I-\frac{k+\beta+1}{k+a+1}E\right)\left(\delta_0(k)E^{-1}\right)\left(I+\tau_1(k)E\right) = D,$$

$$(I+E)\left(\delta_0(k)E^{-1}\right)\left(I+\tau_2(k)E\right) = D$$

hold. Thus, one can deduce, in virtue of Lemmas 3.2 and 3.5, that $D \in \mathscr{A}\left(p(X)\right)$ for an arbitrary polynomial p. (Clearly, this result can also be obtained by letting the operator D act on both sides of (3.55) and using (3.57).)

4. OPTIMUM METHOD

4.1. Fundamental system of recurrence relations. The k-th Jacobi coefficients of both sides of equation (1.5) are equal, which implies

$$\sum_{i=0}^{n} b_k[p_i f^{(i)}] = b_k[q]$$

or, by (3.5),

$$(4.1)\qquad \sum_{i=0}^{n} p_i(X)b_k[f^{(i)}] = b_k[q].$$

The equation obtained and the relations

$$(4.2)\qquad Db_k[f^{(j)}] = 2^{-1}(2k+\lambda-1)_3 b_k[f^{(j-1)}] \quad (j = 1, 2, \ldots, n),$$

following from (3.8), form a *system* of $n+1$ recurrence relations for sequences $\{b_k[f^{(i)}]\}$ $(i = 0, 1, \ldots, n)$.

We show (see Theorem 4.1 below) that $\{b_k[f']\}, \{b_k[f'']\}, \ldots, \{b_k[f^{(n)}]\}$ can be eliminated from this system, which implies a *single* recurrence relation for Jacobi coefficients $\{b_k[f]\}$ of the function f.

4.2. The class $\mathscr{P}(\Lambda_0, \Lambda_1, \ldots, \Lambda_s)$.

Definition 4.1. Given the difference operators $\Lambda_0, \Lambda_1, \ldots, \Lambda_s \in \mathscr{L}$, we define the class $\mathscr{P}(\Lambda_0, \Lambda_1, \ldots, \Lambda_s)$ of pairs $\langle P, L\rangle$ such that P, L are in $\mathscr{L}\setminus\{\Theta\}$ and that the identity

$$(4.3)\qquad P\sum_{j=0}^{s} \Lambda_j b_k[f^{(j)}] = Lb_k[f]$$

holds.

LEMMA 4.1. *If* $\langle P, L\rangle \in \mathscr{P}(\Lambda_0, \Lambda_1, \ldots, \Lambda_s)$, *then*

$$PV = L\big(\gamma_s^{-1}(k)P_s\big), \tag{4.4}$$

where

$$V := \sum_{j=0}^{s} 2^{-j}\Lambda_j\big(\gamma_{s-j}^{-1}(k)P_{s-j}\big),$$

and the notation used is that of (3.11) *and* (3.12).

Proof. Making use of Lemma 3.1, we transform (4.3) into

$$P\sum_{j=0}^{s} 2^{s-j}\Lambda_j\big(\gamma_{s-j}^{-1}(k)P_{s-j}\big)b_k[f^{(s)}] = 2^s L\big(\gamma_s^{-1}(k)P_s\big)b_k[f^{(s)}],$$

which implies (4.4).

LEMMA 4.2. *If* $\langle P, L\rangle \in \mathscr{P}\big(p_0(X), p_1(X), \ldots, p_n(X)\big)$, *where* $p_0, p_1, \ldots$ $\ldots, p_n$ *are polynomials, then*

$$r(L) = r(P) + 2\max_{0\leqslant j\leqslant n,\, p_j\neq 0}(d_j - j), \tag{4.5}$$

d_j *being the degree of the polynomial* p_j.

Proof. Applying Lemma 3.2 we obtain

$$PV = L\big(\gamma_n^{-1}(k)P_n\big),$$

where

$$V := \sum_{j=0}^{n} 2^{-j}p_j(X)\big(\gamma_{n-j}^{-1}(k)P_{n-j}\big).$$

It can be seen (cf. [4]) that the equation

$$r(P) + r(V) = r(L) + r(P_n)$$

holds. Now, it is sufficient to observe that

$$r(P_n) = 2n \quad \text{and} \quad r(V) = 2n + 2\max_{0\leqslant j\leqslant n,\, p_j\neq 0}(d_j - j).$$

The last formula can be proved in the way analogous to the one used in the proof of Lemma 2.11 in [4].

4.3. Recurrence relation for Jacobi coefficients $b_k[f]$.

Definition 4.2. Let the operators $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)} \in \mathscr{L}$ be defined by

$$L_0^{(j)} := p_j(X) \quad (j = 0, 1, \ldots, n), \tag{4.6}$$

and let $\pi_0 \in \mathscr{S}$,

$$\pi_0(k) := b_k[q], \tag{4.7}$$

$p_0, p_1, \ldots, p_n$ and q being the coefficients and the right-hand side of the differential equation (1.5), respectively. For any $m := 1, 2, \ldots, n$ we define the difference operators $L_m^{(0)}, L_m^{(1)}, \ldots, L_m^{(n-m)} \in \mathscr{L}$ and the function $\pi_m \in \mathscr{S}$ by the recurrence forms

$$L_m^{(i)} := A_{m-1} L_{m-1}^{(i)} \quad (i = 0, 1, \ldots, n-m-1), \tag{4.8}$$

$$L_m^{(n-m)} := A_{m-1} L_{m-1}^{(n-m)} + 2^{-1} Q_{m-1} [(2k+\lambda-1)_3 I], \tag{4.9}$$

$$\pi_m(k) := A_{m-1} \pi_{m-1}(k), \tag{4.10}$$

where

$$A_{m-1} := A, \quad Q_{m-1} := Q, \tag{4.11}$$

and A, $Q \in \mathscr{L}$ are the difference operators formed in Lemma 3.4, when applied to the operator $L := L_{m-1}^{(n-m+1)}$.

LEMMA 4.3. *Let $\Pi_n := A_{n-1} A_{n-2} \ldots A_0$. Then*

(i) $\langle \Pi_n, L_n^{(0)} \rangle \in \mathscr{P}(L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)})$;

(ii) *for every pair $\langle P, L \rangle \in \mathscr{P}(L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)})$ there exists an operator $\Phi \in \mathscr{L} \setminus \{\Theta\}$ such that $P = \Phi \Pi_n$ and $L = \Phi L_n^{(0)}$.*

The proof is based on the argument analogous to the one used in [4] in the proofs of Lemmas 2.13 and 2.14.

The main result of this paper is contained in the following

THEOREM 4.1. *Let f be a function satisfying* (1.5) *and such that its n-th derivative can be expanded into a uniformly convergent Jacobi series. Then we have the recurrence relation*

$$L_n^{(0)} b_k[f] = \pi_n(k), \tag{4.12}$$

where $L_n^{(0)} \in \mathscr{L} \setminus \{\Theta\}$ and $\pi_n \in \mathscr{S}$ are an operator and a function, respectively, formed in the manner given in Definition 4.2. The order r_{I} of this relation is expressed by

$$r_{\mathrm{I}} := \sum_{j=0}^{n-1} r(A_j) + 2 \max_{0 \leqslant i \leqslant n, p_i \neq 0} (d_i - i), \tag{4.13}$$

d_i being the degree of the polynomial p_i.

Among the recurrence relations for the Jacobi coefficients $\{b_k[f]\}$ of the function f, which can be obtained by virtue of the differential equation (1.5) *and relations* (2.1), (2.2), *the recurrence relation* (4.12) *is that of the lowest order.*

Proof. From (1.5) and (2.1), (2.2) we have obtained the system of recurrence relations (4.1), (4.2). The process of eliminating the terms of sequences $\{b_k[f^{(i)}]\}$ $(i = 1, 2, \ldots, n)$ from that system, leading to a single relation of the form

$$L b_k[f] = \omega(k) \quad (L \in \mathscr{L} \setminus \{\Theta\}, \omega \in \mathscr{S})$$

is equivalent to the action, on both sides of (4.1), of the operator $P \in \mathscr{L} \setminus \{\Theta\}$ such that $\langle P, L\rangle \in \mathscr{P}(L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)})$. By Lemmas 4.2 and 4.3 the operator L has the *lowest order* in case of $P := \Pi_n$. We then have $L = L_n^{(0)}$ and $\omega(k) = \pi_n(k) = \Pi_n b_k[q]$, whence (4.12) follows. The expression given in (4.13) for the order of $L_n^{(0)}$ is a consequence of (4.5) and (4.11).

Using (4.13) one cannot predict the order of the recurrence relation (4.12) before forming the operator $L_n^{(0)}$, as the operators $A_0, A_1, \ldots, A_{n-1}$ are constructed recursively. By Corollary 3.1 we obtain immediately the following

COROLLARY 4.1. *We have*

$$2 \max_{0\leqslant i\leqslant n, p_i\neq 0} (d_i - i) \leqslant r_{\mathrm{I}} \leqslant 2n+2 \max_{0\leqslant i\leqslant n, p_i\neq 0} (d_i - i).$$

If f satisfies the *first-order* differential equation of the form (1.5), we can give more details than in the general case. Namely, we have the following

THEOREM 4.2. *Let f satisfy the first-order differential equation*

$$p_1 f' + p_0 f = q, \tag{4.14}$$

p_0, p_1 being polynomials. Assume that f' can be expanded into a uniformly convergent Jacobi series. The Jacobi coefficients of f satisfy the recurrence relation

$$\big(Ap_0(X) + \tfrac{1}{2}Q[(2k+\lambda-1)_3 I]\big)b_k[f] = Ab_k[q], \tag{4.15}$$

where A, $Q \in \mathscr{L}$ are the operators formed in Lemma 3.5 when applied to the polynomial $p := p_1$. The order of this recurrence relation is equal to $2\max(d_0, d_1 - 1) + 2 - z$ for $p_0 \not\equiv 0$ or to $2d_1 - z$ for $p_0 \equiv 0$, where z is 0, 1 or 2 according as none, one or two numbers from the set $\{-1, 1\}$ satisfy the equation $p_1(x) = 0$.

Proof. Equation (4.14) is a special case of (1.5) for $n = 1$. By Theorem 4.1 equation (4.12) holds and it is easy to see that it takes the form (4.15) for the difference operators $A \in \mathscr{A}^*\big(p_1(X)\big)$ and $Q \in \mathscr{L}$ satisfying the equality $Ap_1(X) = QD$. The formulas for A and Q can be obtained by the application of Lemma 3.5. The expression for the order of equation (4.15) follows readily from (4.13).

Example 4.1. The function

$$f(x) = (1-x)^{\varrho} \tag{4.16}$$

satisfies the first-order differential equation

$$(x-1)f' - \varrho f = 0 \quad (-1 < x < 1). \tag{4.17}$$

It is known ([3], Section 10.20) that under the assumption that $-\varrho < \min(a+1, a/2+3/4)$ the function (4.16) can be expanded into the Jacobi series (1.1), and

$$b_k[f] = \frac{2^\varrho \Gamma(a+\varrho+1)(-\varrho)_k}{[(2k+\lambda)^2-1]\Gamma(k+\lambda+\varrho+1)}.$$

Applying Theorem 4.2 to (4.17) we obtain

$$p_1(X) = X-I, \quad p_0(X) = -\varrho I, \quad b_k[q] \equiv 0,$$

$$A = I+\tau_1(k)E, \quad Q = \big(kI-(k+\lambda+1)\tau_1(k)E\big)\left(\frac{2}{(2k+\lambda-1)_3}I\right)$$

and, finally,

$$(4.18) \qquad (k-\varrho)b_k[f]-(k+\lambda+\varrho+1)\tau_1(k)b_{k+1}[f] = 0$$

or, in a slightly modified form,

$$(k-\varrho)(2k+\lambda-1)b_k[f]-(k+\lambda+\varrho+1)(2k+\lambda+3)b_{k+1}[f] = 0.$$

5. PASZKOWSKI-TYPE METHOD

In this section we describe the second method of constructing a recurrence relation for Jacobi coefficients of a function satisfying the differential equation (1.5). The method is based on the idea proposed by Paszkowski ([9], Section 13) in connection with recurrence relations for the Chebyshev coefficients, and applied by the author in constructing the recurrence relations for the Gegenbauer coefficients [4], the modified moments [5], and the coefficients of the Neumann-Gegenbauer expansion in Bessel functions of the first kind [6].

The method, called the *Paszkowski-type method*, is much simpler and easier to use than the optimum method given in Section 4, but in some cases (i.e., for some differential equations) it leads to a recurrence relation of order greater than the order of the relation obtained by the optimum method. In other words, the present method is, in general, not the optimum one in the sense of Theorem 4.1.

It can be shown that (1.5) is equivalent to

$$(5.1) \qquad \sum_{i=0}^{n} (q_i f)^{(i)} = q,$$

where

$$(5.2) \qquad q_i := \sum_{j=i}^{n} (-1)^{j-i}\binom{j}{i} p_j^{(j-i)} \quad (i = 0, 1, \ldots, n)$$

(see, e.g., [9], p. 232).

THEOREM 5.1. *If the function f satisfies the assumptions of Theorem 4.1, then the recurrence relation*

$$Lb_k[f] = \omega(k) \tag{5.3}$$

holds, where $L \in \mathscr{L}$, $\omega \in \mathscr{S}$,

$$L := \sum_{i=0}^{n} 2^{-i} S_{n-1,i}\big(\gamma_i(k) q_i(X)\big), \tag{5.4}$$

$$\omega(k) := P_n b_k[q], \tag{5.5}$$

the notation being that of (5.2) *and* (3.10)-(3.12).

The order of relation (5.3) *is equal to*

$$r_{\mathrm{II}} := 2n + 2 \max_{0 \leqslant j \leqslant n, q_j \neq 0} (e_j - j), \tag{5.6}$$

where e_j is the degree of the polynomial q_j.

Proof. Equation (5.1) implies the identity

$$\sum_{i=0}^{n} b_k[(q_i f)^{(i)}] = b_k[q].$$

Let the operator P_n act on both sides of this identity and take into account that $P_n = S_{n-1,j} P_j$ for $j = 0, 1, \ldots, n$. We get

$$\sum_{i=0}^{n} S_{n-1,i} P_i b_k[(q_i f)^{(i)}] = P_n b_k[q],$$

which, by Lemma 3.1, implies

$$\sum_{i=0}^{n} 2^{-i} S_{n-1,i}\big(\gamma_i(k) b_k[q_i f]\big) = P_n b_k[q].$$

Using the notation of (3.5), we can transform this equation into the form (5.3), where $L \in \mathscr{L}$ and $\omega \in \mathscr{S}$ are given by (5.4) and (5.5), respectively.

It follows from (4.13) that

$$r_{\mathrm{II}} = r(L) = \max_{0 \leqslant j \leqslant n, q_j \neq 0} \big[r(S_{n-1,j}) + r\big(q_j(X)\big)\big].$$

It is not difficult to see that $r\big(q_j(X)\big) = 2e_j$, e_j being the degree of the polynomial q_j, and that $r(S_{n-1,j}) = 2(n-j)$. Hence we obtain (5.6).

Remark 5.1. The operators $S_{n-1,j}$ ($j = 0, 1, \ldots, n$) and P_n, occurring in Theorem 5.1, can be formed by using the formulas $S_{n-1,n} := I$, $S_{n-1,j} := S_{n-1,j+1} B_j$ for $j = n-1, n-2, \ldots, 0$, and $P_n := S_{n-1,0}$ (cf. (3.10) and (3.11)).

Remark 5.2. From (5.2) it can be easily deduced that the number r_{II} given by (5.6) can be also defined as

$$r_{II} := 2n + \max_{0 \leqslant j \leqslant n, p_j \neq 0} (d_j - j),$$

d_j being the degree of the polynomial p_j. Therefore, by Corollary 4.1, the order of relation (5.3) is not lower than the order of relation (4.12), i.e., $r_I \leqslant r_{II}$. Examples 4.1 and 5.1 show that the sharp inequality may occur.

Example 5.1. Using Theorem 5.1 we construct the recurrence relation (5.3) for the Jacobi coefficients $b_k[f]$ of the function f given by (4.16), satisfying the differential equation (4.17) which is equivalent to

$$[(x-1)f]' - (\varrho+1)f = 0,$$

i.e., to equation (5.1) for $n = 1$, $q_0(x) \equiv -\varrho-1$, $q_1(x) = x-1$, $q \equiv 0$. Since $S_{01} = I$, $S_{00} = P_1 = D$, $\gamma_0(k) \equiv 1$, and $\gamma_1'(k) = \frac{1}{2}(2k+\lambda-1)_3$, formulas (5.4) and (5.5) define $L \in \mathscr{L}$, $\omega \in \mathscr{S}$ as

$$\begin{aligned} L := & -(\varrho+1)D + \tfrac{1}{2}(2k+\lambda-1)(X-I) \\ = & (k-\varrho-1)(k+a)(2k+\lambda-3)E^{-1} - \\ & -\tfrac{1}{2}(2k+\lambda)[(2k+\lambda)^2 + (a-\beta)(\lambda+2\varrho+1)-1]I + \\ & +(k+\lambda+\varrho+1)(k+\beta+1)(2k+\lambda+3)E, \end{aligned}$$

$$\omega(k) := Db_k[q] \equiv 0.$$

By Theorem 5.1 the second-order recurrence relation

$$Lb_k[f] = 0 \tag{5.7}$$

holds with the above-calculated $L \in \mathscr{L}$. Remember that the optimum method yielded the first-order relation (4.18). Relation (5.7) can be obtained by letting the operator

$$\left(I - \frac{k+\beta+1}{k+a+1} E\right)\left((k+a)(2k+\lambda-3)E^{-1}\right)$$

act on both sides of (4.18). (It is not difficult to guess the form of the above operator if one knows identity (3.58).)

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INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
51-151 WROCŁAW

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S. LEWANOWICZ (Wrocław)

KONSTRUKCJA ZWIĄZKU REKURENCYJNEGO NAJNIŻSZEGO RZĘDU DLA WSPÓŁCZYNNIKÓW JACOBIEGO

STRESZCZENIE

Funkcję f, określoną w przedziale $\langle -1, 1\rangle$ i spełniającą odpowiednie warunki, można rozwinąć w jednostajnie zbieżny w tym przedziale *szereg Jacobiego* (1.1). Wartości współczynników (1.2) tego szeregu można znaleźć stosunkowo łatwo, jeśli spełniają one związek rekurencyjny postaci (1.4).

Wśród metod konstrukcji związków typu (1.4) dla współczynników $\{a_k[f]\}$ danej funkcji f prostotą i uniwersalnością wyróżnia się metoda stosowana wówczas, gdy f spełnia równanie różniczkowe (1.5), w którym $p_0, p_1, \ldots, p_n$ ($p_n \not\equiv 0$) są wielomianami, a współczynniki $\{a_k[g]\}$ są dane.

Przypadek rozwinięć (1.1) dla $\alpha = \beta$ został szczegółowo zbadany w [4], gdzie opisano metodę prowadzącą od równania różniczkowego (1.5) do związku rekurencyjnego typu (1.4), najniższego możliwego rzędu.

W tej pracy opisujemy analogiczną metodę dla $\alpha \neq \beta$. Metoda ta jest sformułowana w języku pewnych operatorów liniowych omówionych w § 3.

W § 4 opisano *optymalną metodę* konstrukcji związku (1.4) najniższego możliwego rzędu na podstawie równania (1.5).

Druga metoda, opisana w § 5, wykorzystuje pomysły zastosowane przez Paszkowskiego ([9], § 13) w konstrukcji związków rekurencyjnych dla współczynników *szeregu Czebyszewa* (ściśle związanego z szeregiem (1.1) dla $\alpha = \beta = -1/2$) funkcji f. Metoda ta nie jest, ogólnie biorąc, optymalna.
