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Z. CYLKOWSKI (Wrocław)

CHEBYSHEV SERIES EXPANSIONS OF THE FUNCTIONS $J_\nu(kx)/(kx)^\nu$ AND $I_\nu(kx)/(kx)^\nu$

Applying the Laplace transformation, Elliott and Szekeres [1] have proved that

$$(1)\qquad \frac{J_1(kx)}{x} = k\sum_{m=0}^{\infty}{}' (-1)^m\left[J_m^2\left(\frac{k}{2}\right) - J_{m-1}\left(\frac{k}{2}\right)J_{m+1}\left(\frac{k}{2}\right)\right] T_{2m}(x),$$

where J_m denotes the Bessel function of first kind, $T_n(x) = \cos(n \arccos x)$ — the n-th Chebyshev polynom, k any arbitrary real constant; the sign$'$ at Σ denotes that the summation component for $m = 0$ has to be halved. This note contains a completely elementary proof of the series expansion of the functions mentioned in the title, thus also a proof of formula (1).

It is known that

$$(2)\qquad \left(\frac{y}{2}\right)^{\mu-\nu} J_\nu(y) = \sum_{j=0}^{\infty} \frac{(\mu+2j)\Gamma(\mu+j)\Gamma(\nu+1-\mu)}{j!\,\Gamma(\nu+1-\mu-j)\Gamma(\nu+j+1)} J_{\mu+2j}(y)$$

([2], vol. II, p. 99, formula (2)). For $\mu = 0$ formula (2) allows an expression of the quotient $J_\nu(kx)/(kx)^\nu$ (for ν integer, nonnegative) by a linear combination of Bessel functions with even indices, as follows

$$\frac{J_\nu(kx)}{(kx)^\nu} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{j=0}^{\infty}{}' \frac{\nu^{(j)}}{(\nu+j)^{(j)}} J_{2j}(kx),$$

where

$$a^{(j)} = \begin{cases} 1 & (j = 0), \\ a(a-1)\ldots(a-j+1) & (j = 1, 2, \ldots) \end{cases}$$

(more precisely, it is necessary to calculate the limit of (2) for $\mu \to 0$ and to observe for $j = 0$ that $\lim\limits_{\mu\to 0}\mu\Gamma(\mu) = 1$). From this and from the identity

$$e^{\pm\nu\pi i/2} I_\nu(z) = J_\nu(ze^{\pm\pi i/2})$$

([4], p. 160) one obtains that

$$\frac{I_\nu(kx)}{(kx)^\nu} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{j=0}^{\infty}{}' \frac{(-1)^j \nu^{(j)}}{(\nu+j)^{(j)}} I_{2j}(kx).$$

An application of

$$J_{2j}(kx) = 2 \sum_{m=0}^{\infty}{}' J_{j-m}\left(\frac{k}{2}\right) J_{j+m}\left(\frac{k}{2}\right) T_{2m}(x) \tag{3}$$

([3], p. 24) and of

$$I_{2j}(kx) = 2 \sum_{m=0}^{\infty}{}' I_{j-m}\left(\frac{k}{2}\right) I_{j+m}\left(\frac{k}{2}\right) T_{2m}(x) \tag{4}$$

gives the announced expansions:

$$\frac{J_\nu(kx)}{(kx)^\nu} = \frac{1}{2^{\nu-2}\Gamma(\nu+1)} \sum_{m=0}^{\infty}{}' T_{2m}(x) \sum_{j=0}^{\infty}{}' \frac{\nu^{(j)}}{(\nu+j)^{(j)}} J_{j-m}\left(\frac{k}{2}\right) J_{j+m}\left(\frac{k}{2}\right), \tag{5}$$

$$\frac{I_\nu(kx)}{(kx)^\nu} = \frac{1}{2^{\nu-2}\Gamma(\nu+1)} \sum_{m=0}^{\infty}{}' T_{2m}(x) \sum_{j=0}^{\infty}{}' \frac{(-1)^j \nu^{(j)}}{(\nu+j)^{(j)}} I_{j-m}\left(\frac{k}{2}\right) I_{j+m}\left(\frac{k}{2}\right). \tag{6}$$

If ν is a natural number then it suffices to take j up to ν in the inner summation of both formulae. For $\nu = 1$ one obtains formula (1) from (5). The identity (2) allows also a Chebyshev series expansion of the functions

$$\frac{J_\nu(kx)}{(kx)^\lambda}, \quad \frac{I_\nu(kx)}{(kx)^\lambda} \quad (\nu-\lambda = 1, 2, \ldots). \tag{7}$$

However, formulae (5) and (6) have a greater value since they give approximation polynoms of $J_\nu(kx)$ and $I_\nu(kx)$ with a small relative error.

References

[1] D. Elliott and G. Szekeres, *Some estimates of the coefficients in the Chebyshev series expansion of a function,* Math. of Comput. 19 (1965), pp. 25-32.

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[3] G. N. Lance, *Numerical methods for high speed computers,* London 1960.

[4] N. W. McLachlan, *Bessel functions for engineers* (*Funkcje Bessela dla inżynierów*), Polish edition, Warszawa 1964.

DEPARTMENT OF NUMERICAL METHODS
WROCŁAW UNIVERSITY

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Z. CYLKOWSKI (Wrocław)

SZEREGI CZEBYSZEWA DLA FUNKCJI $J_\nu(kx)/(kx)^\nu$ I $I_\nu(kx)/(kx)^\nu$

STRESZCZENIE

Nota zawiera elementarny dowód rozwinięć (5) i (6) funkcji wymienionych w tytule względem wielomianów Czebyszewa, korzystający tylko z tożsamości (2)-(4). Podobnie można rozwinąć w szeregi Czebyszewa funkcje (7).

З. ЦИЛЬКОВСКИ (Вроцлав)

РЯДЫ ЧЕБЫШЕВА ДЛЯ ФУНКЦИЙ $J_\nu(kx)/(kx)^\nu$ И $I_\nu(kx)/(kx)^\nu$

РЕЗЮМЕ

Автор предлагает элементарный вывод разложений (5) и (6) в ряд относительно полиномов Чебышева функций указанных в заглавии статьи. Для вывода здесь употребляются только тождества (2)-(4). Таким же образом получаются разложения функций (7).
