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SOME ESTIMATIONS OF SYSTEM RELIABILITY WITH POSITIVELY DEPENDENT SURVIVAL TIMES OF CRITICAL PATHS

1. Introduction. Let us consider the system of some elements with specified n minimal critical paths. Let A_j ($j \in J = \{1, 2, \ldots, n\}$) denote the random event that all elements of the j-th path are failed, and let B denote the random event that the system is failed. The minimal critical paths may be considered in the system as subsystems being connected in series, namely

$$B = \bigcup_{j \in J} A_j.$$

By the *reliability* P of the system we understand the probability of complementation of the event B, so that

$$P = \Pr(B^c) = \Pr\big(\bigcap_{j \in J} A_j^c\big).$$

In common practical situations the random events A_j, $j \in J$, are dependent, and then for the computation of the probability P a probability distribution on the unions $A_{i_1} A_{i_2} \ldots A_{i_k}$ for every $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, tedious in estimation however, is required. Some partial information on this distribution allows us to estimate the probability P. In this note we consider the estimation of P of second degree, i.e., using the probabilities $P_j = \Pr(A_j)$, $P_{ij} = \Pr(A_i A_j)$, $i, j \in J$, and the additional condition of positive dependence of events. We say that random events A_j, $j \in J$, are *positively en bloc dependent* if for every $J_1 \subset J$ and $J_2 \subset J$ the inequality

$$\Pr\big(\bigcap_{j \in J_1 \cup J_2} A_j\big) \geqslant \Pr\big(\bigcap_{j \in J_1} A_j\big) \Pr\big(\bigcap_{j \in J_2} A_j\big)$$

holds.

From among estimations of second degree we recall Hunter's result [2] distinguished by simplicity. Consider any tree on J, i.e., a connected subgraph with $n-1$ nodes. Let

$$\tau = \big\{(\tau(j), j) \colon \tau(j) < j,\ j = 2, 3, \ldots, n\big\}$$

be the arcs of this tree. Hunter showed that

$$\Pr(B) \leqslant \sum_{j\in J} \Pr(A_j) - \max_{\tau} \sum_{j=2}^{n} \Pr(A_{\tau(j)} A_j). \tag{1}$$

In the class of estimations of first degree with positive dependence condition of events $A_1^c, A_2^c, \ldots, A_n^c$, Esary et al. [1] showed easily that

$$P = \Pr(B^c) \geqslant \prod_{j\in J} \Pr(A_j^c) = \prod_{j\in J} (1-P_j). \tag{2}$$

2. Some estimation of second degree. Let μ be a $2m$-element permutation of the set J with repetition of its elements,

$$\mu = (i_1, j_1, i_2, j_2, \ldots, i_m, j_m),$$

where $2m \geqslant n$. Hence

$$B^c = \bigcap_{k=1}^{m} A_{i_k}^c A_{j_k}^c,$$

and so for positively en bloc dependent events $A_1^c, A_2^c, \ldots, A_n^c$ we get the inequality

$$\Pr(B^c) \geqslant \prod_{k=1}^{m} Q_{i_k j_k}, \tag{3}$$

where $Q_{ij} = \Pr(A_i^c A_j^c) = 1 - P_i - P_j + P_{ij}$.

This implies the estimation

$$P \geqslant 1 - \max_{\mu} \prod_{k=1}^{m} Q_{i_k j_k}. \tag{4}$$

The problem of optimum estimation of P in the class of estimations (3) is reduced to the optimum choice of pairs, strictly, to the optimum division of the set J into one- or two-element subsets. It is easy to see that the class of admissible permutations with repetitions μ may be limited to the class of permutations without repetition if n is even and then $m = n/2$, or to the class of permutations with one permutation if n is odd and then $m = (n+1)/2$.

The optimization problem may also be formulated as a linear programming problem. Write for some μ

$$x_{ij} = \begin{cases} 1 & \text{for } i = i_k, j = j_k \text{ or } i = j_k, j = i_k, k = 1, 2, \ldots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Using the symmetry of matrices (Q_{ij}) and (x_{ij}) we get

$$\prod_{k=1}^{m} Q_{i_k j_k} = \prod_{1\leqslant i\leqslant j\leqslant n} Q_{ij}^{x_{ij}} = \Big[\prod_{i,j\in J} Q_{ij}^{x_{ij}(1+\delta_{ij})}\Big]^{1/2} = \exp\Big[-\frac{1}{2}\sum_{i,j\in J} w_{ij} x_{ij}\Big],$$

where δ_{ij} is Kronecker's delta, and $w_{ij} = -(1+\delta_{ij}) \log Q_{ij}$.

The linear programming problem is as follows: given the symmetric matrix (w_{ij}), find the $(0, 1)$-valued symmetric matrix (x_{ij}) such that

$$\sum_{i\in J} x_{ij} \geqslant 1, \quad \sum_{j\in J} x_{ij} \geqslant 1, \quad x_{ij} = x_{ji},$$

$$\sum_{i,j\in J} w_{ij} x_{ij} = \min.$$

This is the so-called *non-bivariate matching problem.* It is solved by Edmonds, and descriptions of computer programs are also given (cf. [3]).

3. Comparison of estimations. Estimation (4) is comparable with estimation (1). The first one is sharper if the random events A_j, $j \in J$, are almost independent, and the second one is sharper if the events have a large common part. We show this by examples.

Let A_j, $j \in J$, be independent events with the same probability p. From (4) we obtain the exact value

$$\Pr(B) = 1-(1-p)^n,$$

and from (1) we get the estimation

$$\Pr(B) \leqslant np-(n-1)p^2.$$

Let $A_j = A$ for $j \in J$, $n = 2m$, and let $\Pr(A) = p$. From (4) we get the estimation

$$\Pr(B) \leqslant 1-(1-p)^m = p\big(1+(1-p)+ \ldots +(1-p)^{m-1}\big),$$

and from (1) we obtain the exact value

$$\Pr(B) = 2mp-(2m-1)p = p.$$

References

[1] J. D. Esary, F. Proschan and D. W. Walkup, *Association of random variables, with applications*, Ann. Math. Statist. 38 (1967), p. 1466-1474.

[2] D. Hunter, *An upper bound for the probability of a union,* J. Appl. Prob. 13 (1976), p. 597-603.

[3] E. L. Lawler, *Combinatorial optimization: networks and matroids,* New York 1976.

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PEWNE OSZACOWANIE NIEZAWODNOŚCI SYSTEMU PRZY DODATNIO ZALEŻNYCH CZASACH PRZEŻYCIA ŚCIEŻEK KRYTYCZNYCH

STRESZCZENIE

Niezawodność systemu definiuje się jako prawdopodobieństwo sumy pewnych zdarzeń losowych. Przy założeniu, że zdarzenia te są dodatnio en bloc zależne, w pracy znaleziono oszacowanie niezawodności, w którym skorzystano z macierzy prawdopodobieństw iloczynów par zdarzeń. Algorytm optymalnego szacowania sprowadza się do problemu optymalnego kojarzenia par. Oszacowanie porównano ze znanym oszacowaniem Huntera; okazało się przy tym, że żadnego z nich nie można generalnie wyróżnić.