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DISCRETE AND NONDISCRETE DUELS

1. Introduction. The nondiscrete duels are a generalization of the classical discrete duels. A model of the discrete duel can be described in the following way. Each of two players taking part in the game has some fixed number of "bullets" to be fired in the time interval $\langle 0, 1 \rangle$. For $i = 1, 2$, we associate with player i a nondecreasing continuous function $P_i(t)$ from $\langle 0, 1 \rangle$ onto $\langle 0, 1 \rangle$, called an *accuracy function*. We interpret it as the probability of hitting the opponent by the i -th player at the time t when he fires a bullet at this moment. If one of the players hits the other, the game is over. The payoff for the first player amounts to $+1$ or -1 when he hits the opponent or the opponent hits him, respectively. Otherwise, his payoff is equal to 0 . The above-described model is considered as a zero-sum game.

A duel is *noisy* if each player knows when the other fires his bullets and a duel is *silent* if neither of players knows when the other fires. One can also consider a mixed *silent-noisy duel*. The first two kinds of duels were solved by Restrepo [6] and by Kimeldorf and Fox [1]. The silent-noisy duels in their generality are still unsolved. The strongest result in this field was given by Radzik and Orłowski in [4] and [5].

A model of nondiscrete duel differs from the above-described one in the fact that player i ($i = 1, 2$) instead of some number of indivisible bullets has some amount of "ammunition" which may be arbitrarily distributed in the time interval $\langle 0, 1 \rangle$.

In this game, the accuracy function $P_i(t)$ ($i = 1, 2$) expresses the probability of hitting the opponent by the i -th player when he fires at moment t the amount of ammunition equal to 1 . All the remaining assumptions are common for the two models mentioned above. The nondiscrete duels in the most complete form were solved by Lang and Kimeldorf in [2] and [3].

It follows from the description of the models of discrete and nondiscrete duels that they are very closely related. The aim of this paper is a careful examination of the connection between these two kinds of

games of timing. Among other things, the formula for the payoff function in a nondiscrete duel is derived in a different way from that used before.

2. Notation. In the paper we use extended real numbers, i.e., real numbers with the symbols $\pm\infty$. In this set we define

$$\begin{aligned}\infty \cdot 0 = 0 \cdot \infty = 0, \quad \infty + x = x + \infty = \infty \text{ for } x > -\infty, \\ 1/\infty = 0, \quad \log 0 = -\infty, \quad \exp(-\infty) = 0, \quad 0^0 = 1.\end{aligned}$$

We denote by $\bar{x}_n, \bar{y}_n, \bar{z}_n, \bar{t}_n$ vectors the components of which form a nondecreasing sequence of numbers from $\langle 0, 1 \rangle$.

The letters μ and η with various indices are reserved to denote measures. The symbols $\mu_{\bar{x}_n}, \eta_{\bar{y}_n}$ depending on measures μ, η and on vectors \bar{x}_n, \bar{y}_n , respectively, denote the measures which place at each point x_i of the vector $\bar{x}_n = (x_1, \dots, x_n)$ the mass $\mu\langle 0, 1 \rangle/n$ or at each point y_i of $\bar{y}_n = (y_1, \dots, y_n)$ the mass $\eta\langle 0, 1 \rangle/n$.

3. Construction of the payoff function. Let the accuracy function $P_i(t)$ ($i = 1, 2$) in a nondiscrete silent duel be a continuous and non-decreasing function from $\langle 0, 1 \rangle$ onto $\langle 0, 1 \rangle$. Now we ask with what probability $P_i^{(a)}(t)$ player i hits the opponent when firing the ammunition of amount a at the time t . In order to answer this question we make two natural assumptions:

$$(I) \quad P_i^{(a)}(t) \leq P_i^{(\beta)}(t) \text{ for } 0 \leq a \leq \beta, \quad 0 \leq t \leq 1;$$

$$(II) \quad 1 - P_i^{(a+\beta)}(t) = (1 - P_i^{(a)}(t))(1 - P_i^{(\beta)}(t)) \text{ for } a \geq 0, \beta \geq 0, \quad 0 \leq t \leq 1.$$

The first condition means that the probability of hitting the opponent in the time t is the greater the more ammunition the player i fires.

The second condition states that the event "player i firing at moment t ammunition of amount $a + \beta$ does not hit the other" has the same probability as the event "player i firing at moment t two cartridges mutually independently, of masses a and β , does not hit the opponent".

From (I) and (II) we obtain

$$(1) \quad P_i^{(a)}(t) = 1 - [1 - P_i(t)]^a \quad \text{for } a \geq 0, \quad 0 \leq t \leq 1.$$

Let μ_i be a strategy for player i , i.e., a measure on $\langle 0, 1 \rangle$ such that $\mu_i\langle 0, 1 \rangle = M_i$, where M_i is the amount of ammunition of player i . Let $Q_i^{\mu_i}(t)$ be the probability of the event that player i hits the other in the interval $\langle 0, t \rangle$ applying the strategy μ_i . In order to find $Q_i^{\mu_i}(t)$ we make the following three assumptions:

(III) If $\mu_i = \mu(t_1, a_1, \dots, t_n, a_n)$, where $0 \leq t_1 \leq \dots \leq t_n \leq 1$, is a measure which places at the points t_1, \dots, t_n the masses a_1, \dots, a_n , respectively, then

$$(*) \quad 1 - Q_i^{\mu_i}(t) = \prod_{k(t_k \leq t)} [1 - P_i^{(a_k)}(t)].$$

(IV) If μ_i and μ'_i are two strategies for player i such that $\mu_i\langle 0, t_0 \rangle = \mu'_i\langle 0, t_0 \rangle$ for some t_0 , $0 \leq t_0 \leq 1$, and $\mu_i\langle 0, s \rangle \leq \mu'_i\langle 0, s \rangle$ for any $s \in \langle 0, t_0 \rangle$, then $Q_i^{\mu_i}(t_0) \leq Q_i^{\mu'_i}(t_0)$.

(V) For any strategy μ_i such that $\mu_i\langle t_i^*, t \rangle = 0$, where t_i^* is the point defined as $\min\{t: P_i(t) = 1\}$, we have $Q_i^{\mu_i}(t) = Q_i^{\mu_i}(t_i^* -)$, $t_i^* \leq t \leq 1$.

Condition (III) means that the event "player i applying the strategy $\mu(t_1, a_1, \dots, t_n, a_n)$ does not hit his opponent in the time interval $\langle 0, t \rangle$ " has the same probability as the event "player i firing, at the moments t_1, \dots, t_n , n cartridges mutually independently, of masses a_1, \dots, a_n , respectively, does not hit the opponent".

Condition (IV) says that the more we displace ammunition forward in time, the greater is the efficiency of strategy (according with the fact that accuracy functions are nondecreasing).

THEOREM 1. *Conditions (I)-(V) imply*

$$Q_i^{\mu_i}(t) = 1 - \exp \left\{ \int_{[0,t]} \ln[1 - P_i(t)] d\mu_i \right\}, \quad 0 \leq t \leq 1.$$

Proof. Fix a strategy μ_i . It follows from the definition of $Q_i^{\mu_i}(t)$ and from conditions (II) and (III) that $Q_i^{\mu_i}(0) = P_i^{\mu_i(0)}(0) = 0$. Hence the equality is valid at $t = 0$.

Let $t_i^* \leq t \leq 1$. If $\mu_i\langle t_i^*, t \rangle \neq 0$, then by condition (III) and formula (*) we get $Q_i^{\mu_i}(t) = 1$, and we easily verify that the theorem holds in this case. If $\mu_i\langle t_i^*, t \rangle = 0$, then by condition (V) we have $Q_i^{\mu_i}(t) = Q_i^{\mu_i}(t_i^* -)$, and the theorem is also true. Therefore, to complete the proof it suffices to consider the case $0 < t < t_i^*$.

Let $0 = x_0 < x_1 < \dots < x_n < x_{n+1} = t < t_i^*$, where x_1, \dots, x_n are arbitrarily fixed numbers. Denote by μ'_i a measure which at the points x_1, x_2, \dots, x_{n+1} places the masses a_1, a_2, \dots, a_{n+1} , where $a_1 = \mu_i\langle 0, x_1 \rangle$, $a_k = \mu_i\langle x_{k-1}, x_k \rangle$, $k = 2, 3, \dots, n+1$. Then, putting $z_k = -\ln[1 - P_i(x_k)]$, we infer from the assumptions (IV), (III) and from (*) that

$$\begin{aligned} Q_i^{\mu_i}(t) &\leq Q_i^{\mu'_i}(t) = 1 - \prod_{k=1}^{n+1} [1 - P(x_k)]^{a_k} \\ &= 1 - \exp \left\{ - \sum_{k=1}^{n+1} (z_k \mu_i\{t: z_{k-1} < -\ln[1 - P_i(t)] \leq z_k\}) \right\} \\ &\rightarrow 1 - \exp \left\{ \int_{[0,t]} \ln[1 - P_i(t)] d\mu_i \right\} \quad \text{as } \max_{0 \leq k \leq n} (x_{k+1} - x_k) \rightarrow 0. \end{aligned}$$

$Q_i^{\mu_i}(t) = 1$ for $t \geq t_i^*$, which gives the thesis of the theorem. Therefore, the theorem has been proved.

In the next part of the paper we will omit μ_i in the expression $Q_i^{\mu_i}(t)$ when it does not lead to misunderstanding. Put $A_i(t) = -\ln[1 - P_i(t)]$. This formula establishes a one-to-one correspondence between the set of continuous nondecreasing functions from $\langle 0, 1 \rangle$ onto $\langle 0, 1 \rangle$ and the set of continuous nondecreasing functions from $\langle 0, 1 \rangle$ onto $\langle 0, \infty \rangle$.

The payoff function takes the form (see [3])

$$(2) \quad K(\mu_1, \mu_2) = \int_{\langle 0,1 \rangle} \bar{Q}_2 dQ_1 - \int_{\langle 0,1 \rangle} \bar{Q}_1 dQ_2,$$

where

$$Q_i(t) = 1 - \exp\left\{-\int_{\langle 0,t \rangle} A_i(t) d\mu_i\right\}, \quad \bar{Q}_i(t) = 1 - Q_i(t), \quad i = 1, 2.$$

Thus we can relate the nondiscrete silent duel to the functions $A_1(t)$ and $A_2(t)$ which are continuous, nondecreasing, from $\langle 0, 1 \rangle$ onto $\langle 0, \infty \rangle$.

In what follows we mean by $K_{m \times n}(\mu, \eta)$ or $K_{m \times n}(\bar{x}_m, \bar{y}_n)$ the payoff functions in discrete silent duels in which the players have m and n bullets and behave according to the strategies μ, η or \bar{x}_m, \bar{y}_n (to fire their bullets at moments denoted by \bar{x}_m, \bar{y}_n), respectively.

Example 1 (reduction to discrete case). Let us consider the nondiscrete silent duel with parameters $P_i(t)$ and M_i ($i = 1, 2$). Let μ and η be fixed strategies for the players. Then, using formula (2) we can easily show that

$$K(\mu_{\bar{x}_m}, \eta_{\bar{y}_n}) = K_{m \times n}(\bar{x}_m, \bar{y}_n),$$

where the right-hand side is the payoff in a discrete silent duel with the following accuracy functions:

$$Q_1(t) = 1 - [1 - P_1(t)]^{M_1/m}, \quad Q_2(t) = 1 - [1 - P_2(t)]^{M_2/n}.$$

4. An example of approximation of a nondiscrete duel by a discrete one. In this section we give a theorem which has been proved by Lang and Kimeldorf in [2]. We prove it once more but in another way, by approximation.

THEOREM 2. *The nondiscrete silent duel with $P_1(t) = P_2(t) = t$ and $M_1 = M_2 = 1$ has the value $v = 0$ and the optimal strategies for players are measures with the common distribution*

$$(3) \quad F(t) = 1 + \frac{1}{2 \ln(1-t)}, \quad a < t < 1,$$

where $a = 1 - \exp(-\frac{1}{2})$.

We need some lemmas to prove this theorem.

Let us consider the discrete silent duel of type $n \times n$ (each player has n bullets) with accuracy functions $\tilde{P}_1(t) = \tilde{P}_2(t) = Q_n(t)$, where $Q_n(t)$ is an absolutely continuous and strictly increasing function from $\langle 0, 1 \rangle$ onto $\langle 0, 1 \rangle$.

It is known (see [6]) that this game has the value 0 and the optimal strategies have the common form

$$\mu_n^*(\bar{t}_n) = \prod_{i=1}^n \mu_{ni}^*(t_i),$$

where μ_{ni}^* is a measure in $(a_{ni}, a_{n,i+1})$ with density

$$d\mu_{ni}^*(t_i) = k_{ni} \frac{Q_n'(t_i)}{Q_n^3(t_i)} dt_i,$$

where

$$a_{n,n+1} = 1, \quad a_{ni} = Q_n^{-1} \left(\frac{1}{2(n-i)+3} \right), \quad k_{ni} = \frac{1}{4(n-i+1)} \\ (i = 1, \dots, n).$$

We interpret $\mu_{ni}^*(t_i)$ as the distribution of the moment at which the player fires his i -th bullet.

Since the value of this game is 0, we have

$$(4) \quad K_{n \times n}(\mu_n^*, \eta) \geq 0$$

for every strategy η of the second player.

LEMMA 1. If $Q_n(t) = 1 - (1-t)^{1/n}$, then

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} (a_{n,i+1} - a_{ni}) = 0.$$

Proof. We use the following inequality:

$$a^n - b^n \leq n(a-b)a^{n-1} \quad \text{for } 0 \leq b \leq a.$$

Since $Q_n^{-1}(t) = 1 - (1-t)^n$ for $0 \leq t \leq 1$, we have

$$\begin{aligned} \max_{1 \leq i \leq n} (a_{n,i+1} - a_{ni}) &= \max_{1 \leq i \leq n} \left[Q_n^{-1} \left(\frac{1}{2i-1} \right) - Q_n^{-1} \left(\frac{1}{2i+1} \right) \right] \\ &= \max_{1 \leq i \leq n} \left[\left(\frac{2i}{2i+1} \right)^n - \left(\frac{2i-2}{2i-1} \right)^n \right] \leq \max_{1 \leq i \leq n} \left[n \left(\frac{2i}{2i+1} - \frac{2i-2}{2i-1} \right) \left(\frac{2i}{2i+1} \right)^{n-1} \right] \\ &= \max_{1 \leq i \leq n} \left[\frac{2n}{(2i+1)(2i-1)} \left(\frac{2i}{2i+1} \right)^{n-1} \right]. \end{aligned}$$

With the help of differential calculus it is easy to show that

$$\lim_{n \rightarrow \infty} \max_{1 \leq x \leq n} \left\{ \frac{2n}{(2x+1)(2x-1)} \left(\frac{2x}{2x+1} \right)^{n-1} \right\} = 0,$$

and this completes the proof.

Let η be a certain strategy for the second player in the nondiscrete duel with $P_1(t) = P_2(t) = t$ and $M_1 = M_2 = 1$. Thus $\eta \langle 0, 1 \rangle = 1$. Associate with the measure η a measure $\eta_{\bar{s}_n}$, where the vector $\bar{s}_n(\eta) = (s_1, \dots, s_n)$ satisfies the condition

$$s_j = \min\{x: 0 \leq x \leq 1, \eta \langle 0, x \rangle \geq j/n\} \quad \text{for any } j = 1, \dots, n.$$

Further, we define a vector $\bar{s}'_n(\eta) = (s'_1, \dots, s'_n)$ by

$$s'_i = \begin{cases} a_{nn} & \text{if } s_i = 1, \\ s_i & \text{if } s_i < a_{n1}, \\ a_{nj} & \text{if } a_{nj} \leq s_j < a_{n,j+1} \text{ for some } j \leq n \quad (i = 1, \dots, n). \end{cases}$$

Let $\mathcal{A}_n(\eta)$ denote the class of all vectors $\bar{x}_n = (x_1, \dots, x_n)$ satisfying the following two conditions:

- (i) if $s_k < a_{n1}$, then $x_k = s_k$;
- (ii) if $a_{nj} \leq s_k < a_{n,j+1}$ for some j , then $a_{nj} \leq x_k < a_{n,j+1}$ ($k = 1, \dots, n$).

Thus, for any strategy η for the second player in the nondiscrete duel, the vector $\bar{s}'_n(\eta)$ and the class $\mathcal{A}_n(\eta)$ are constructed.

We put $\bar{a}_n = (a_{n2}, a_{n3}, \dots, a_{n,n+1})$.

LEMMA 2. For any strategy η we have

$$K_{n \times n}(\mu_n^*, \bar{s}'_n(\eta) +) \leq K_{n \times n}(\bar{a}_n, \bar{s}'_n(\eta) +).$$

The lemma follows easily from the definition of $K_{n \times n}$ (see [6]) and from the fact that the accuracy function $Q_n(t)$ is increasing.

Similarly, by the definition of $\mathcal{A}_n(\eta)$ we obtain

LEMMA 3. For any $\varepsilon > 0$ and for any strategy η there exists a vector $\bar{y}_n \in \mathcal{A}_n(\eta)$ such that

$$K_{n \times n}(\bar{a}_n, \bar{s}'_n(\eta) +) \leq K_{n \times n}(\bar{a}_n, \bar{y}_n) + \varepsilon.$$

LEMMA 4. If $\bar{y}_n \in \mathcal{A}_n(\eta)$ ($n = 1, 2, \dots$), then the sequence $\{\eta_{\bar{y}_n}\}$ converges weakly to η .

The lemma follows easily from the definition of $\mathcal{A}_n(\eta)$.

LEMMA 5. If $\mu \langle 0, 1 \rangle = 1$, then the sequence $\{\mu_{\bar{a}_n}\}$ converges weakly to the measure μ^* with distribution (3) on $(a, 1)$.

This lemma is presented in [2] as Theorem 2.

LEMMA 6. We have

$$\int_0^1 -\ln(1-t) d\mu^* = \infty.$$

This result follows immediately by direct computation.

Proof of Theorem 2. Fix in an arbitrary way a strategy η for the second player in the nondiscrete duel. By (4) and Lemmas 2 and 3, for any $\varepsilon > 0$ we obtain

$$\begin{aligned} 0 &\leq K_{n \times n}(\mu_n^*, \bar{s}_n(\eta) +) \leq K_{n \times n}(\bar{a}_n, \bar{s}'_n(\eta) +) \\ &\leq K_{n \times n}(\bar{a}_n, \bar{y}_n) + \varepsilon \quad \text{for a certain } \bar{y}_n \in \mathcal{A}_n(\eta). \end{aligned}$$

Hence (see Example 1) $K(\mu_{\bar{a}_n}^*, \eta_{\bar{y}_n}) \geq -\varepsilon$ for $\bar{y}_n \in \mathcal{A}_n(\eta)$.

Now, using Theorem 2.1 from [3] (this theorem is valid for strategies on $\langle 0, 1 \rangle$ if the intervals of integration in (i), (ii), and (2.7) are regarded there as open) and Lemmas 4-6, because of the free choice of ε we get $K(\mu^*, \eta) \geq 0$, which completes the proof of Theorem 2.

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