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SOME REMARKS ON THE ENUMERATION OF ROOTED TREES

0. Summary. A solution is given to a general problem of finding the number of different rooted trees when a specification $(n_1, n_2, \ldots, n_s)$ is given stating that the tree has n_i knots of kind i $(i = 1, 2, \ldots, s)$, where

$$\sum_{i=1}^{s} n_i = n.$$

1. Introduction. The enumeration of different rooted trees having a given number of knots was first discussed in 1857 by Cayley [3] for the case where all the knots were treated as being of the same kind and, later, for the case where they were regarded as being different (labelled). The introduction of "figurate series" and "enumerating series" by Pólya (see [5] and [6]) led to a simplification of Cayley's arguments. Riordan, in his textbook [7], discussed with the aid of Pólya's Fundamental Theorem of Enumeration Theory the case of rooted trees having m labelled knots, the remainder $p-m$ being identical. However, this is not a particularly interesting case and in applications it is often necessary to find the answer to a more general problem of finding the number of different rooted trees which have knots of s different kinds when a specification $(n_1, n_2, \ldots, n_s)$ is given stating that the tree has n_i knots of kind i $(i = 1, 2, \ldots, s)$ so that the total number of trees is equal to $\sum_{i=1}^{n} n_i = n$. For instance, for some family trees, it can be necessary to enumerate the trees according to the number of males and females or according to the number of persons in various survival age-groups; similarly, when rooted trees are used in other branches of science (e.g. chemistry), the general problem can be again needed. By a (perhaps somewhat tedious) routine application of Pólya's ideas it is possible to solve this problem giving as well a numerical answer to it. A similar problem for the two-terminal series-parallel networks was discussed by the author in [4].

Following Pólya (see [5] and [6]) let us define as a *tree* a connected linear graph built of p points and $p-1$ lines. It contains no closed paths and no slings. If a certain unique point of the tree in which just one line ends is distinguished, we have a *rooted tree*: this point is called the *root* of the tree, the line starting from the root — its *trunk*, and any point different from the root is called a *knot*. The trunk is bounded by the root on one side and on another side by a knot which we call *principal*. In our drawings we indicate a root by an arrow and each knot by a small circle.

Fig. 1

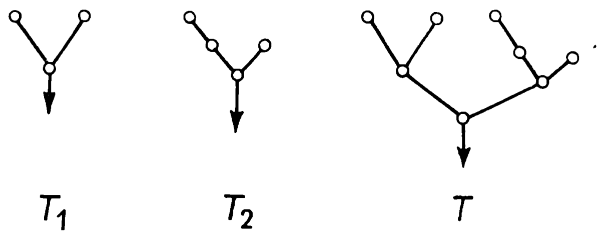

Fig. 2

The simplest rooted tree T_0 consists of a root, a trunk and just one (principal) knot (Fig. 1). To build any rooted tree different from the simplest one we take a number of rooted trees and treat the roots of these trees as one principal knot of the new tree. Thus it can be said that the trunks of those chosen trees are inserted into the upper point (the knot) of the trunk of the new tree, and the trees treated so form the so-called *main branches* of the new tree. For example, the two trees T_1 and T_2 form the two main branches of the tree T (Fig. 2).

If $T_1, T_2, \ldots, T_s$ are s trees ($s \geqslant 1$) having $p_1, p_2, \ldots, p_s$, respectively, knots, then we can combine them into one tree T having as the principal knot the combined roots of these trees and as the trunk an added simple tree T_0. We write this as

$$T = T_0(T_1, T_2, \ldots, T_s) \tag{1}$$

and, clearly, T has $1+p_1+p_2+\ldots+p_s$ knots. In our example T_1 has 3 knots, T_2 has 4 knots and T has $1+3+4 = 8$ knots. It should be stressed that the order in which the given set of rooted trees is combined to build a new rooted tree is regarded as of no importance so that two trees having the same main branches are treated as identical: if $(r_1, r_2, \ldots, r_s)$ is a permutation of numbers $(1, 2, \ldots, s)$, then

$$T_0(T_1 T_2 \ldots T_s) = T_0(T_{r_1} T_{r_2} \ldots T_{r_s}). \tag{2}$$

For instance (see Fig. 3), if the knots are marked by small letters $a, b, \ldots$, then according to (2) the trees T_A and T_B shown there should be regarded as identical. Indeed, their root p and their principal knot a are the same and it is easy to verify that their main branches are identical although the order in which they are combined to build T_A and T_B is different. The fact that the appearance of some main branches (like e.g. those containig b, c, d) is slightly different in T_A and T_B is of no importance since according to (2) they again have as their main branches the identical trees (simple trees (b, c) and (b, d)) combined only in different order.

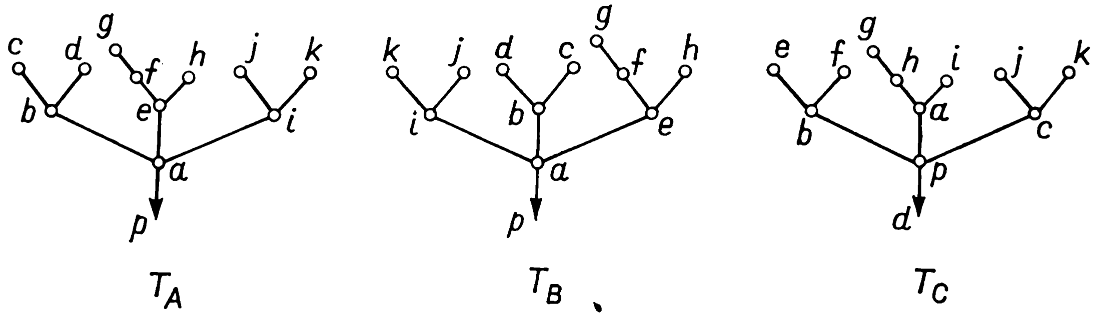

Fig. 3

Let us assume now that there exists a one-to-one transformation φ of a rooted tree T into a rooted tree T' which maps the elements of T into the elements of T' so that the basic relations between the mapped elements in T' are the same as between the corresponding elements of T, i.e. the root of T is mapped into the root of T', the trunk of T into the trunk of T', the line connecting knots p and q of T into the line connecting the transformed p' and q' of T' etc. Then we say that the rooted trees T and T' are *equivalent*; the two non-equivalent rooted trees will be called *essentially different*. For example, let us assume that the root and the knots of the tree T_A (Fig. 3) were transformed by a function φ in such a way that

x	p	a	b	c	d	e	f	g	h	i	j	k
$\varphi(x)$	d	p	b	e	f	a	h	g	i	c	j	k

It is easy to verify that the root p of T was transformed into the root d of T', the principal knot a into the principal knot p, the trunk (p, a) into the trunk (d, p), the line connecting (i, j) into the line connecting (c, j) etc. Thus the trees T_A and T_C are equivalent (1); however, being not identical they are different, although not essentially different.

(1) In [6], in a footnote on p. 695, Pólya expressed the opinion that „it may be sufficient and in some respect even advantageous if at a first reading the reader takes the definition intuitively and supplements it by examples". A more elaborate definition of rooted trees and of their equivalence can be found in [2], p. 181-191.

Among the s trees $T_1, T_2, \ldots, T_s$ in (1) some can be equivalent and we denote them by the same letter W_j; if there are r_j of them, we say that the multiplicity of the main branch W_j is r_j and we write $W_j^{r_j}$ for the repetition $W_j W_j \ldots W_j$ of these r_j equivalent trees. Then we can rewrite (1) as

$$T = T_0(W_1^{r_1} W_2^{r_2} \ldots W_w^{r_w}), \tag{3}$$

where w is the number of essentially different rooted trees among the trees shown in the brackets of (1). Obviously, the trees T and T' are equivalent if and only if their main branches are equivalent and occur with the same multiplicity.

2. Pólya's proof of Cayley's recurrence generating series. Let us denote by $U_{1k}, U_{2k}, \ldots, U_{t_k k}$ all the essentially different rooted trees having exactly k knots, and let t_k be their number. The simplest tree T_0, as shown in Fig. 1, will be denoted by U_{11} and, obviously, $t_1 = 1$. There is only one rooted tree having two knots so that $t_2 = 1$ as well. In order to get some information about the numbers t_k let us follow Pólya and construct the appropriate "figurate series". To do this let us define by U_{00} an empty main branch, i.e. let us agree that if U_{00} appears anywhere between the brackets in a formula like (3), then it will be ignored as a non-existing main branch or, in other words, it will be treated in this "product" like a unity. Then the "figurate series" can be written as:

$$\begin{aligned} \sum_{k=1}^{\infty}\Big\{\sum_{i=1}^{t_k} U_{ik}\Big\} y^k = {} & U_{11}y(U_{00} + U_{11}y + U_{11}^2y^2 + U_{11}^3y^3 + \ldots) \times \\ & \times (U_{00} + U_{12}y^2 + U_{12}^2y^4 + U_{12}^3y^6 + \ldots) \times \\ & \times (U_{00} + U_{13}y^3 + U_{13}^2y^6 + U_{13}^3y^9 + \ldots) \ldots \times \\ & \times \prod_{i=1}^{t_m} (U_{00} + U_{im}y^m + U_{im}^2y^{2m} + U_{im}^3y^{3m} + \ldots) \ldots \end{aligned} \tag{4}$$

Indeed, if a rooted tree T is represented by formula (3), where W_j $(j = 1, 2, \ldots, w)$ has p_j knots, then T has $k = 1 + r_1p_1 + r_2p_2 + \ldots + r_wp_w$ knots and it appears as one of the t_k terms on the left-hand side of (4) multiplied by y^k. The powers of the appropriate W_j's multiplied by $y^{r_jp_j}$ can be found in one of the appropriate lines of the right-hand side of (4), i.e. in one of the t_m lines for $m = p_j$ at the r_j-th place; taking the "product" of U_{11} and of these expressions and multiplying them by U_{00} from all other lines, we obtain the same tree T on the right-hand side of (4) as the coefficient of $1 + \sum_{j=1}^{w} r_jp_j = k$ power of y, and this completes the proof of the formula.

Putting $U_{ik} = 1$ ($k = 0, 1, \ldots; i = 0$ if $k = 0$; $i = 1, \ldots, t_k$ if $k > 0$), we obtain the "enumerating series" leading to

$$\sum_{k=1}^{\infty} t_k y^k = y(1-y)^{-t_1}(1-y^2)^{-t_2}(1-y^3)^{-t_3}\ldots \tag{5}$$

which is Cayley's remarkable result (cf., for instance, [6], formula (8)).

The comparison of coefficients of y^k does not give us the values of t_k-numbers but only a recurrence formula for calculating them, and that is why in the title of this section the term "recurrence generating series" was used. Taking logarithms on both sides of (5), we get

$$\log\Big[\sum_{k=1}^{\infty} t_k y^k\Big] = \log y - \sum_{k=1}^{\infty} t_k \log(1-y^k)$$
$$= \log y + \sum_{k=1}^{\infty} t_k (y^k + y^{2k}/2 + y^{3k}/3 + \ldots).$$

Differentiating with respect to y and multiplying by y, we obtain

$$\sum_{k=1}^{\infty} k t_k y^k = \Big[\sum_{k=1}^{\infty} t_k y^k\Big]\Big[1 + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} k t_k y^{kj}\Big].$$

The double sum can be written as $\sum_{m=1}^{\infty} y^m \big(\sum_{d/m} d t_d\big)$, and if we write

$$e_m = \sum_{d/m} d t_d, \tag{6}$$

we get

$$\sum_{k=1}^{\infty} k t_k y^k = \Big[\sum_{k=1}^{\infty} t_k y^k\Big]\Big[1 + \sum_{m=1}^{\infty} e_m y^m\Big]. \tag{7}$$

Comparing the coefficients of y_k, we arrive at the following recurrence formula:

$$\begin{aligned} t_1 &= 1, \\ (k-1)t_k &= t_{k-1}e_1 + t_{k-1}e_2 + t_{k-3}e_3 + \ldots + t_1 e_{k-1} \qquad (k > 1). \end{aligned} \tag{8}$$

The evaluation of t_k-numbers can be easily achieved by the use of Table 1 where the line for $k = 1$ has unities in cols. (2) and (3). The numbers in col. (4) are obtained as sums of cross-products of numbers in cols. (2) and (3) of previous lines. By dividing the results in col. (4) of line k by $k-1$ we obtain t_k. To obtain e_k we find the divisors of k, multiply the numbers in cols. (1) and (2) for those divisors and add the results. The numbers in col. (2) agree with the published data (cf., for instance, [7], Table 4 on p. 138).

TABLE 1

k	t_k	e_k	$\sum_{i=1}^{k-1} e_i t_{k-i}$
(1)	(2)	(3)	(4)
1	1	1	—
2	1	3	1
3	2	7	4
4	4	19	12
5	9	46	36
6	20	129	100
7	48	337	288
8	115	939	805
9	286	2 581	2 288
10	719	7 238	6 471
11	1 842	20 263	18 420
12	4 766	57 337	52 426
13	12 486	162 319	149 832
14	32 973	461 961	428 649
15	87 811	1 317 217	1 229 354
16	235 381	3 767 035	3 530 715
17	634 847	10 792 400	10 157 552
18	1 721 159	30 983 565	29 259 703
19	4 688 676	89 084 845	84 396 168
20	12 826 228	256 531 814	243 698 332

3. Permutation group associated with a structure, its cycle index and Pólya's Fundamental Theorem of Enumeration Theory. To each rooted tree we can find a group $\mathfrak{G}$ of permutations which leave this tree unchanged; we call this group a *permutation group associated with this tree.* Thus, for instance, if we have a tree T as that drawn in Fig. 4 with 7 knots marked by the numbers *1-7*, then the following 12 permutations, written in cyclic form, leave the tree unchanged:

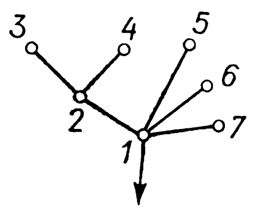

Fig. 4

$$\begin{aligned}
\varrho_1 &= (1)(2)(3)(4)(5)(6)(7), & \varrho_7 &= (1)(2)(34)(5)(6)(7),\\
\varrho_2 &= (1)(2)(3)(4)(5)(67), & \varrho_8 &= (1)(2)(34)(5)(67),\\
\varrho_3 &= (1)(2)(3)(4)(57)(6), & \varrho_9 &= (1)(2)(34)(57)(6),\\
\varrho_4 &= (1)(2)(3)(4)(56)(7), & \varrho_{10} &= (1)(2)(34)(56)(7),\\
\varrho_5 &= (1)(2)(3)(4)(567), & \varrho_{11} &= (1)(2)(34)(567),\\
\varrho_6 &= (1)(2)(3)(4)(576), & \varrho_{12} &= (1)(2)(34)(576).
\end{aligned}$$

If a permutation splits into b_1 cycles of length 1, b_2 cycles of length 2 etc., we say that it is *of the type* $(b_1, b_2, \ldots, b_n)$, where some b's can be zero and where $b_1+2b_2+3b_3+\ldots+nb_n = n$, n being the degree of the group, i.e. the number of permuted elements. For each permutation of the type $(b_1, b_2, \ldots, b_n)$ we form a monomial $f_1^{b_1}f_2^{b_2}\ldots f_n^{b_n}$ and we define as the *cycle index of the group* $\mathfrak{G}$ the polynomial

$$P(\mathfrak{G}; f_1, f_2, \ldots, f_n) = g^{-1}\sum_{\mathfrak{G}} f_1^{b_1}f_2^{b_2}\ldots f_n^{b_n}, \tag{9}$$

where g is the order of the group $\mathfrak{G}$, the number of permutations in this group. Thus, for instance, the cycle index of the permutation group associated with the tree of Fig. 4 is equal to

$$P(\mathfrak{G}; f_1, f_2, \ldots, f_7) = (f_1^7+4f_1^5f_2+2f_1^4f_3+3f_1^3f_2^2+2f_1^2f_2f_3)/12.$$

Instead of speaking of the "cycle index of the permutation group associated with a given tree" to save space we shall call it the "cycle index of the tree" and write it shortly as $P(T; f)$.

Let us now put in (9) the substitution

$$f_i = x_1^i+x_2^i+\ldots+x_n^i. \tag{10}$$

According to Pólya's Fundamental Theorem of Enumeration Theory [5] (discussed e.g. in [1]), the number of different structures with specification $(n_1, n_2, \ldots, n_r)$ generated by structure T is the coefficient of the term $x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r}$ in the expansion of the function $I(\mathfrak{G}; x_1, x_2, \ldots, x_n)$ obtained from (9) by substitution (10).

Let us now assume that in our example we can have various kinds of knots denoted by letters $a, b, \ldots$ If all the knots are of the same kind marked all by letter a, i.e. with specification $(7, 0, \ldots, 0)$, then the substitution (10) yields as the coefficient of x_1^7 unity since $(1+4+2+3+2)/12 = 1$ which means that there is only one tree of this kind. If all the knots are different, we have the specification $(1, 1, \ldots, 1)$ and the coefficient of $x_1x_2\ldots x_7$ is $7!/12 = 420$. Indeed, there are 7 ways in which one of the seven letters can be used to mark the principal knot, 6 ways in which the knot denoted by *2* on Fig. 4 can be then given a letter and there are 10 ways in which from the remaining five letters two can be chosen to mark the knots indicated by *3* and *4* on Fig. 4 and three to mark the remaining knots, total number of ways being $7\cdot 6\cdot 10 = 420$.

4. Number of rooted trees with a given knot specification. Our problem will be solved if we find the cycle indices of all rooted trees U_{ik} appearing on the left-hand side of formula (4). If $P(U_{ik}; f) = P(U_{ik}; f_1, f_2, \ldots, f_k)$ is the corresponding cycle index, we write a polynomial

$$\psi_k(f_1, f_2, \ldots, f_k) = \sum_{i=1}^{t_k} P(U_{ik}; f_1, f_2, \ldots, f_k). \tag{11}$$

Substituting (10) into this polynomial and calculating the coefficient of the term $x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r}$, we obtain the number of different rooted trees having a knot specification $(n_1, n_2, \ldots, n_r)$, where $\sum_{i=1}^{r} n_i = k$.

Let us assume that the rooted tree T was defined by formula (3), i.e. was obtained by juxtaposition of smaller non-equivalent trees W_i each with p_i knots which formed the main branches of the tree T. The multiplicities of these branches were r_i. The problem is to find the cycle index $P(T; f_1, f_2, \ldots, f_k)$ of the tree T, where $k = 1+\sum_{i=1}^{w} r_i p_i$ if the cycle indices $P(W_i; f_1, f_2, \ldots, f_{p_i})$ of branches are known.

Notice first that the trees W_i $(i = 1, 2, \ldots, w)$ are non-equivalent so that the group of permutations leaving T unchanged must be the direct product of permutation groups which leave the "powers" $W_1^{r_1}$, $W_2^{r_2}$, $\ldots$, $W_w^{r_w}$ unchanged; consequently, the cycle index of the direct product being the product of cycle indices of factors (cf. [1], [5] or [7]), we have

$$(12)\qquad P(T; f) = f_1 P(W_1^{r_1}; f) P(W_2^{r_2}; f) \ldots P(W_w^{r_w}; f).$$

To express the cycle indices $P(W_i^{r_i}; f)$ by the cycle indices of branches W_i let us notice that the permutations leaving the product $W_i W_i \ldots W_i$ of r_i, branches unchanged are those of group $\mathfrak{G}_i$ which leaves W_i unchanged and also those of the symmetric group γ_{r_i} of degree r_i which change the order of W_i in the product; the number of such permutations is $r_i! g_i^{r_i}$, where g_i is the order of the permutation group $\mathfrak{G}_i$. Following Pólya's definition in [5] (also discussed e.g. in [1]), this means that all these permutations form a "Kranz" group $\gamma_{r_i}[\mathfrak{G}_i]$, i.e. the "Kranz" of group $\mathfrak{G}_i$ over the symmetric group γ_{r_i} of degree r_i ([2]). The cycle index of such a group can be expressed very elegantly in terms of the cycle indices of γ_{r_i} and $\mathfrak{G}_i$. Indeed, it is equal to the cycle index of the symmetric group γ_{r_i} of degree r_i denoted by $Z_{r_i}(h_1, h_2, \ldots, h_{r_i})$ in which each h_j is replaced by

$$h_j = P(W_i; f_j, f_{2j}, \ldots, f_{p_i j}) \qquad (j = 1, 2, \ldots, r_i).$$

Thus we get

$$(13)\qquad P(W_i^{r_i}; f_1, f_2, \ldots, f_{r_i p_i}) = Z_{r_i}[P(W_i; f_1, f_2, \ldots, f_{p_i}), P(W_i; f_2, f_4, \ldots, f_{2p_i}), \ldots, \ldots, P(W_i; f_{r_i}, f_{2r_i}, \ldots, f_{p_i r_i})].$$

([2]) A generalization of Pólya's concept of the "Kranz" group can be found in de Bruijn's paper [2], where the author defines a more general concept of the crown. However, our results can be obtained without this generalization by using simply Pólya's ideas.

To save space let us write $Z_{r_i}[P(W_i;f)]$ for the right-hand side of (13), and then we get from (12)

$$(14)\qquad P(T;f) = f_1 Z_{r_1}[P(W_1;f)] Z_{r_2}[P(W_2;f)] \dots Z_{r_w}[P(W_w;f)],$$

which allows us to find the cycle index of a rooted tree from the knowledge of cycle indices of the main branches and of multiplication of these branches.

Reverting to Pólya's "figurate series" given by formula (4) we can rewrite it as

$$(4')\qquad \sum_{k=1}^{\infty}\Big\{\sum_{i=1}^{t_k} U_{ik}\Big\} y^k = U_{11} y \prod_{k=1}^{\infty}\prod_{i=1}^{t_k}(1 + U_{ik}y^k + U_{ik}^2 y^{2k} + U_{ik}^3 y^{3k} + \dots).$$

Replacing in this formula each U_{ik}^r by its cycle index $Z_r[P(U_{ik};f)]$, we verify that, in view of (14), the following formula for cycle indices associated with these trees, similar to formula (4), is valid:

$$(15)\qquad \sum_{k=1}^{\infty}\Big\{\sum_{i=1}^{t_k} P(U_{ik};f)\Big\} y^k = f_1 y \prod_{k=1}^{\infty}\prod_{i=1}^{t_k}\{1 + Z_1[P(U_{ik};f)]y^k + $$
$$+ Z_2[P(U_{ik};f)]y^{2k} + Z_3[P(U_{ik};f)]y^{3k} + \dots\}.$$

It can be found in any textbook of combinatorial mathematics (cf., for instance, [1]) that the generating function of $Z_n(h_1, h_2, \dots, h_n)$ (of cycle indices of symmetric groups) is given by the formula

$$(16)\qquad 1 + Z_1(h_1)y + Z_2(h_1, h_2)y^2 + Z_3(h_1, h_2, h_3)y^3 + \dots$$
$$= \exp\{h_1 y + h_2 y^2/2 + h^3 y^3/3 + \dots\}.$$

This allows us to replace the expression in brackets { } on the right-hand side of (15) by

$$\exp\{P(U_{ik};f)y^k + P(U_{ik};f^{*2})y^{2k}/2 + P(U_{ik};f^{*3})y^{3k}/3 + \dots\},$$

where $P(U_{ik};f^{*r})$ is written for $P(U_{ik};f_r, f_{2r}, \dots, f_{kr})$.

Using this remark and the definition of polynomials ψ_k by formula (11) we can rewrite (15) as

$$(17)\qquad \sum_{k=1}^{\infty}\psi_k(f)y^k = f_1 y \prod_{k=1}^{\infty}\exp\{\psi_k(f)y^k + \psi_k(f^{*2})y^{2k}/2 + \psi_k(f^{*3})y^{3k}/3 + \dots\}.$$

Formula (17) is really a recurrence generating function for ψ_k-polynomials similar to Cayley's formula (5). In order to obtain a suitable recurrence formula, we apply a method similar to that used in the treatment of formula (5). Taking logarithms on both sides of (17), we get

$$(18)\quad \log\Big\{\sum_{k=1}^{\infty}\psi_k(f)y_k\Big\} = \log(f_1y)+\sum_{k=1}^{\infty}[\psi_k(f)y^k+\psi_k(f^{*2})y^{2k}/2+\dots].$$

The derivatives of both sides taken with respect to y multiplied by y yield

$$(19)\quad \sum_{k=1}^{\infty}k\psi_k(f)y^k = \Big[\sum_{k=1}^{\infty}\psi_k(f)y^k\Big]\Big[1+\sum_{k=1}^{\infty}k[\psi_k(f)y^k+\psi_k(f^{*2})y^{2k}+\dots]\Big]$$
$$=\Big[\sum_{k=1}^{\infty}\psi_k(f)y^k\Big]\Big\{1+\sum_{k=1}^{\infty}\sum_{r=1}^{\infty}k\psi_k(f^{*r})y^{rk}\Big\}.$$

The double sum can be written as $\sum_{m=1}^{\infty} y^m(\sum_{d/m} d\psi_d(f^{*m/d})$. Writing

$$e_m = \sum_{d/m} d\psi_d(f^{*m/d}),$$

we get (19) in the form

$$(20)\quad \sum_{k=1}^{\infty}k\psi_k(f)y^k = \Big[\sum_{k=1}^{\infty}\psi_k(f)y^k\Big]\Big[1+\sum_{m=1}^{\infty}y^m e_m\Big].$$

The comparison of coefficients yields then

$$(21)\quad (k-1)\psi_k(f) = \psi_{k-1}e_1+\psi_{k-2}e_2+\dots+\psi_1e_{k-1}.$$

The evaluation of ψ_k-polynomials can be easily achieved by the use of Table 2 where various operations are similar to those used in calculations of entrants to Table 1.

TABLE 2

k	$\psi_k(f)$	e_k	$\sum_{i=1}^{k-1}\psi_{k-i}e_i$
(1)	(2)	(3)	(4)
1	f_1	f_1	—
2	f_1^2	$2f_1^2+f_2$	f_1^2
3	$(3f_1^3+f_1f_2)/2$	$(9f_1^3+3f_1f_2)/2+f_3$	$3f_1^3+f_1f_2$
4	$(f_1f_3+3f_1^2f_2+8f_1^4)/3$	$(4f_1f_3+12f_1^2f_2+32f_1^4)/3+$ $+2f_2^2+f_4$	$f_1f_3+3f_1^2f_2+8f_1^4$
5	$(125f_1^5+54f_1^3f_2+$ $+6f_1^2f_3+15f_1f_2^2+$ $+6f_1f_4)/24$	$5(125f_1^5+54f_1^3f_2+16f_1^2f_3+$ $+15f_1f_2^2+6f_1f_4)/24+f_5$	$(125f_1^5+54f_1^3f_2+$ $+16f_1^2f_3+15f_1f_2^2+$ $+6f_1f_4)/6$
6	$(1296f_1^6+640f_1^4f_2+$ $+180f_1^3f_3+180f_1^2f_2^2+$ $+60f_1^2f_4+24f_1f_5+$ $+20f_1f_2f_3)/120$	$6(1296f_1^6+640f_1^4f_2+$ $+180f_1^3f_3+180f_1^2f_2^2+$ $+60f_1^2f_4+24f_1f_5+$ $+20f_1f_2f_3)/120+$ $+(9f_2^2+3f_2f_4)/2+2f_3^2+f_6$	$(1296f_1^6+640f_1^4f_2+$ $+180f_1^3f_3+180f_1^2f_2^2+$ $+60f_1^2f_4+24f_1f_5+$ $+20f_1f_2f_3)/24$
7		...	

Knowing the ψ_k-polynomials, we have our problem solved. If we ask for the number t_k of rooted trees with all n knots identical, we have to find the coefficient of x_1^n when the substitution (10) is made into the polynomial $\psi_n(f)$. It is then clear that $t_n = \psi_n(1, 1, \dots, 1)$ and we obtain the numbers discussed in Section 2. If we ask for the numbers s_n of rooted trees with all the n knots different, we have to find the coefficient of $x_1 x_2 \dots x_n$ which is equal to $s_n = n!\psi_n(1, 0, \dots, 0)$. Formula (17) for $f = (1, 0, \dots, 0)$ yields

$$\sum_{k=1}^{\infty} s_k y^k/k! = y \exp\Big\{\sum_{k=1}^{\infty} s_k y^k/k!\Big\}. \tag{22}$$

If $R(y)$ is substituted for the exponential generating function of s_k-numbers, then (22) becomes

$$R(y) = y \exp R(y), \tag{23}$$

the result due to Pólya (cf. [5] or [7]). It is easy to verify from Table 2 that $s_n = n^{n-1}$.

Various other specifications lead to the results shown in Table 3; they are obtained by finding the appropriate coefficents when substitution (10) is made in the appropriate ψ_n-polynomials.

TABLE 3

No. of knots	Specification	No. of different rooted trees	No. of knots	Specification	No. of different rooted trees
1	(1)	*1*	5	(3, 1, 1)	*119*
2	(2)	*1*		(2, 2, 1)	171
	(1, 1)	*2*		(2, 1, 1, 1)	*326*
3	(3)	*2*		(1, 1, 1, 1, 1)	*625*
	(2, 1)	*5*	6	(6)	*20*
	(1, 1, 1)	*9*		(5, 1)	*95*
4	(4)	*4*		(4, 2)	209
	(3, 1)	*13*		(3, 3)	268
	(2, 2)	18		(4, 1, 1)	*401*
	(2, 1, 1)	*34*		(3, 2, 1)	744
	(1, 1, 1, 1)	*64*		(2, 2, 2)	1077
5	(5)	*9*		(3, 1, 1, 1)	*1433*
	(4, 1)	*35*		(2, 2, 1, 1)	2078
	(2, 3)	63		(2, 1, 1, 1, 1)	*4016*
				(1, 1, 1, 1, 1, 1)	*7776*

The numbers in italics agree with the published data (cf, [7]. Table 3 on p. 134).

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KILKA UWAG O OBLICZANIU DENDRYTÓW Z WYRÓŻNIONYM WĘZŁEM

STRESZCZENIE

W pracy rozwiązuje się ogólne zagadnienie znalezienia liczby różnych dendrytów z wyróżnionym węzłem (*rooted tree*), gdy podana jest specyfikacja $(n_1, n_2, \ldots, n_s)$, mówiąca, że dendryt ma n_i węzłów i-tego rodzaju $(i = 1, 2, \ldots, s)$, gdzie $n_1+n_2+\ldots+n_s = n$.