ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVII, 4 (1983)

R. M. PHATARFOD (Clayton, Victoria)

ON INFINITE DAMS WITH MARKOV DEPENDENT INPUTS

0. The known expression for time-dependent probabilities of the contents of an infinite dam given certain initial conditions is in the form of a complicated double generating function. This makes the problem of numerical evaluation of the probabilities extremely difficult. An explicit expression for these probabilities is obtained by using combinatorial methods, and it is shown how a matrix operation facilitates their numerical evaluation.

1. Introduction. In this paper we consider an infinite dam whose inputs $X_n = 0, 1, \ldots, N \leqslant \infty$ during consecutive intervals $(n, n+1)$ $(n = -1, 0, 1, \ldots)$ form an ergodic Markov chain and whose release at the end of the time interval $(n, n+1)$ is unity. Defining Z_n as the dam contents at time n, we have the recurrence relation

(1) $$Z_{n+1} = Z_n + X_n - \min\{1, Z_n + X_n\} \quad (n = 0, 1, \ldots).$$

Since $\{Z_n, X_n\}$ and $\{Z_n, X_{n-1}\}$ are bivariate Markov chains, it is possible to study the stochastic behaviour of the process $\{Z_n\}$ by extending the methods employed for the case where $\{X_n\}$ are independent and identically distributed random variables. It is known that the results for the Markov inputs case are similar in form, though not in the degree of complexity, to those for the independent inputs case. For example, for the independent inputs case, by letting $\Pr\{X = i\} = p_i$ $(i = 0, 1, \ldots)$ and

$$P(\theta) = \sum_{i=0}^{\infty} p_i \theta^i \quad (|\theta| \leqslant 1),$$

the stationary distribution $\pi_u = \Pr\{Z = u\}$ $(u = 0, 1, \ldots)$ when $\mu = \mathrm{E}(X) < 1$ is given by

$$\pi(\theta) = \sum_{u=0}^{\infty} \pi_u \theta^u = (1-\mu)(1-\theta)[P(\theta)-\theta]^{-1} \quad (|\theta| \leqslant 1).$$

For the Markov inputs case with transition probability matrix $\boldsymbol{P} = (p_{ij})$, where $p_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\}$ and $\boldsymbol{P}(\theta) = (p_{ij}\theta^i)$, the limiting distribution

$$\pi_{ui} = \lim_{n\to\infty} \Pr\{Z_n = u, X_n = i\}$$

when $\mu = \mathrm{E}(X) < 1$ is given by (see [5])

$$\boldsymbol{\pi}(\theta) = \sum_{u=1}^{\infty} \boldsymbol{\pi}_u \theta^u = (1-\mu)(1-\theta)\boldsymbol{p}_0[\boldsymbol{P}(\theta)-\theta\boldsymbol{I}]^{-1},$$

where $\boldsymbol{\pi}_u = (\pi_{u0}, \pi_{u1}, \ldots, \pi_{uN})$ and $\boldsymbol{p}_0 = (p_{00}, p_{01}, \ldots, p_{0N})$.

Let us now consider the time-dependent probabilities; first the probability of the time to the first emptiness of the dam.

For independent inputs, defining T_u as the time of the first emptiness of the dam with initial contents u, i.e., $T_u = \min\{n \mid Z_n = 0, Z_0 = u\}$, and letting

$$g(n \mid u) = \Pr\{T_u = n\} \quad \text{and} \quad G(s \mid u) = \sum_{n\geqslant u} g(n \mid u)s^n,$$

it is known that

$$G(s \mid u) = G^u(s \mid 1) = \xi^u(s),$$

where $\xi(s)$ is the unique solution of $s = P(s)$ with the properties $0 < \xi(s) \leqslant 1$ for $0 < s \leqslant 1$. For the Markov inputs case, defining $T_i(u)$ as the time to the first emptiness with initial contents $Z_0 = u$ and initial input $X_{-1} = i$, and letting

$$g(n \mid u, i) = \Pr\{T_i(u) = n\} \quad (n \geqslant u)$$

and

$$G(s \mid u, i) = \sum_{n=u}^{\infty} g(n \mid u, i)s^n,$$

it is known (see [1] and [2]) that

$$\begin{aligned} G(s \mid u, i) &= G(s \mid 1, i)G(s \mid 1, 0)^{u-1} \quad (1 \leqslant i \leqslant N), \\ G(s \mid u, 0) &= G(s \mid 1, 0)^u. \end{aligned} \tag{2}$$

Explicit expressions for the stationary distribution and the probabilities of the time to the first emptiness have been obtained for the cases

(i) $N = 3$ (see [4] and [6]),

(ii) $N \leqslant \infty$ but $\boldsymbol{P}$ such that

$$\sum_j p_{ij}s^j = B(s)A^i(s)$$

(see [8]).

Our interest here is in the derivation of the time-dependent probabilities

(3) $$P[v, j \mid u, i, n] = \Pr\{Z_n = v, X_{n-1} = j \mid Z_0 = u, X_{-1} = i\}.$$

It is known that for the independent inputs case, letting

$$P_{uv}^{(n)} = \Pr\{Z_n = v \mid Z_0 = u\}$$

and

$$\Phi(\theta, s \mid u) = \sum_{n=0}^{\infty} \sum_{v=0}^{\infty} P_{uv}^{(n)} \theta^v s^n \quad (|\theta| \leqslant 1, \ |s| < 1),$$

we have

(4) $$\Phi(\theta, s \mid u) = [\theta - sP(\theta)]^{-1} \left(\theta^{u+1} - \frac{(1-\theta)\, \xi^{u+1}(s)}{1-\xi(s)} \right).$$

This is not a terribly convenient form for a numerical evaluation of the probabilities $P_{uv}^{(n)}$. However, either by expanding the right-hand side of (4) in powers of $sP(\theta)/\theta$ or by using combinatorial methods (see [7]) we can obtain

(5) $$\Pr\{Z_n \leqslant v \mid Z_0 = u\} = P_{n+v-u}^{(n)} - p_0 \sum_{m=2}^{n-u} P_{u0}^{(n-m)} p_{m+j}^{(m-1)},$$

where $\{p_i^{(n)}\}$ is the n-th convolution of $\{p_i\}$ and $P_j^{(n)} = \sum_{i \leqslant j} p_i^{(n)}$. The above expression involves $P_{u0}^{(n)}$ for which an explicit expression can be obtained as

$$P_{u0}^{(n)} = (p_0)^{-1} \sum_{j=u+1}^{n+1} \frac{j}{n+1}\, p_{n+1-j}^{(n+1)}.$$

An analogue of (4) for the case of Markov inputs does exist [1]; however, it is extremely complicated, and hence a numerical evaluation of the probabilities in (3) is very difficult.

In this paper we give an explicit expression of the form (5) suitable for a numerical evaluation, in those cases where $G_i(s) = G(s \mid 1, i)$ $(i = 0, 1, \ldots, N)$ are determinable, e.g., for the cases mentioned above. The method uses combinatorial arguments similar to those used in deriving (5).

2. A preliminary lemma. Let $S_0 = 0$, $S_n = X_0 + X_1 + \ldots + X_{n-1}$ $(n \geqslant 1)$. Then for $u \geqslant 1$ we have

$$\begin{aligned} g(n \mid u, i) &= \Pr\{T_i(u) = n\} \\ &= \Pr\{u + S_n - n = 0, \ u + S_r - r > 0, \ r = 1, 2, \ldots, n-1 \mid X_{-1} = i\}. \end{aligned}$$

LEMMA. *For $u \geqslant 1$ we have*

(6) $$\Pr\{u+S_r-r>0\ (r=1,2,\ldots,n-1),\ u+S_n-n=v,$$

$$X_{n-1}=k \mid X_{-1}=i\} = h^{(n)}_{n+v-u}(i,k) - \sum_{m=1}^{n-1} g(n-m \mid u,i)\, h^{(m)}_{m+v}(0,k),$$

where

$$h^{(n)}_j(i,k) = \Pr\{S_n=j,\, X_{n-1}=k \mid X_{-1}=i\}.$$

Proof. We have

$$\begin{aligned} h^{(n)}_{n+v-u}(i,k) &= \Pr\{S_n=n+v-u,\, X_{n-1}=k \mid X_{-1}=i\} \\ &= \Pr\{T_i(u)\geqslant n,\ u+S_n-n=v,\ X_{n-1}=k \mid X_{-1}=i\} + \\ &\quad + \Pr\{T_i(u)<n,\ u+S_n-n=v,\ X_{n-1}=k \mid X_{-1}=i\}. \end{aligned}$$

The first term on the right-hand side of the last expression is the required probability. The second term can be written as

$$\sum_{m=1}^{n-1} \Pr\{T_i(u)=m\}\Pr\{u+S_n-n=v,\ X_{n-1}=k \mid T_i(u)=m,\ X_{n-1}=i\},$$

and since $T_i(u)=m$ implies $X_{m-1}=0$, this reduces to

$$\sum_{m=1}^{n-1} g(m \mid u,i)\, h^{(n-m)}_{n+v-m}(0,k) = \sum_{m=1}^{n-1} g(n-m \mid u,i)\, h^{(m)}_{v+m}(0,k),$$

thus completing the proof.

3. Derivation of time-dependent probabilities. Equation (1) can be written as

$$Z_n = \max\{0,\, Z_{n-1}+X_{n-1}-1\}.$$

If $Z_0=u$, we obtain

$$Z_n = \max\{S_n-S_{n-r}-r\ (r=0,1,\ldots,n-1),\ u+S_n-n\}.$$

Hence

(7) $$\Pr\{Z_n\leqslant v,\ X_{n-1}=k \mid Z_0=u,\, X_{-1}=i\}$$

$$= \Pr\{S_n-S_{n-r}-r\leqslant v\ (r=0,1,\ldots,n-1),\ u+S_n-n\leqslant v,$$

$$X_{n-1}=k \mid X_{-1}=i\} = \sum_{j=u+1}^{\infty} \Pr\{S_n-S_{n-r}-r\leqslant v\ (r=0,1,\ldots,n-1),$$

$$S_n-n=v-j+1,\ X_{n-1}=k \mid X_{-1}=i\} = \sum_{j=u+1}^{\infty} \Pr\{S_{n-r}-(n-r)$$

$$\geqslant -j+1 \ (r = 0, 1, \ldots, n-1), \ S_n - n = v-j+1,$$

$$X_{n-1} = k \mid X_{-1} = i\} = \sum_{j=u+1}^{\infty} \Pr\{j+S_{n-r}-(n-r) > 0$$

$$(r = 0, 1, \ldots, n-1), \ j+S_n-n = v+1, \ X_{n-1} = k \mid X_{-1} = i\}$$

$$= \sum_{j=u+1}^{\infty} \Pr\{j+S_r - r > 0 \ (r = 1, 2, \ldots, n), \ j+S_n-n = v+1, \ X_{n-1} = k \mid X_{-1} = i\},$$

which, by (6), reduces to

$$\sum_{j=u+1}^{\infty} \Big[h^{(n)}_{n+v+1-j}(i, k) - \sum_{m=1}^{n-j} g(n-m \mid j, i) h^{(m)}_{m+v+1}(0, k)\Big]. \tag{8}$$

Putting now $H_j^{(n)}(i, k) = \sum_{l=0}^{j} h_l^{(n)}(i, k)$, we reduce (8) to

$$H^{(n)}_{n+v-u}(i, k) - \sum_{m=1}^{n-u-1} h^{(m)}_{m+v+1}(0, k) \sum_{j=u+1}^{n-m} g(n-m \mid j, i). \tag{9}$$

This form is the Markov analogue of (5) and the $g(n \mid j, i)$ may be obtained from (2) when $G_i(s)$ $(i = 0, 1, \ldots, N)$ are known. Alternatively, we can express the last summation in (9) in terms of the probabilities of emptiness not necessarily for the first time as in (5). We have

$$g(n+1 \mid j, i) = \Pr\{j+S_r - r > 0 \ (r = 1, 2, \ldots, n),$$

$$j+S_{n+1}-(n+1) = 0 \mid X_{-1} = i\} = \Pr\{j+S_r-r > 0 \ (r = 1, 2, \ldots, n-1),$$

$$j+S_n-n = 1, \ j+S_{n+1}-(n+1) = 0 \mid X_{-1} = i\}.$$

Since $j+S_n-n = 1$ and $j+S_r-r > 0$ $(r = 1, 2, \ldots, n-1)$ implies $X_{n-1} = 0$ or 1, the above expression reduces to

$$p_{00}\Pr\{j+S_r-r > 0 \ (r = 1, 2, \ldots, n-1), \ j+S_n-n = 1,$$

$$X_{n-1} = 0 \mid X_{-1} = i\} + p_{10}\Pr\{j+S_r-r > 0 \ (r = 1, 2, \ldots, n-1),$$

$$j+S_n-n = 1, \ X_{n-1} = 1 \mid X_{-1} = i\}.$$

Using (7) for $v = 0$ and $k = 0, 1$, we therefore obtain

$$\sum_{l=0}^{1} p_{l0}\Pr\{Z_n = 0, \ X_{n-1} = l \mid Z_0 = u, \ X_{-1} = i\} = \sum_{j=u+1}^{\infty} g(n+1 \mid j, i)$$

$$= \sum_{r=0}^{n-u} g(n+1 \mid n+1-r, i).$$

Substituting this in (9) we finally obtain

$$\Pr\{Z_n \leqslant v,\ X_{n-1} = k \mid Z_0 = u,\ X_{-1} = i\}$$
$$= H^{(n)}_{n+v-u}(i, k) - \sum_{m=2}^{n-u} \Big\{\sum_{l=0}^{1} p_{l0} P[0, l \mid u, i, n-m]\Big\} \times h^{(m-1)}_{m+v}(0, k).$$

The quantities $P[0, l \mid u, i, n-m]$ for $l = 0, 1$ can be obtained from their generating functions. It is known (see, e.g., [1]) that

$$P(0, 0 \mid s) = \sum_n P[0, 0 \mid u, i, n] s^n = \frac{s^{-1}(1-sp_{11}) G_0^u(s) G_i(s)}{\{p_{00} + s(p_{10}p_{01} - p_{00}p_{11})\}\{1 - G_0(s)\}}$$

and

$$P(0, 1 \mid s) = \sum_n P[0, 1 \mid u, i, n] s^n = \frac{p_{01} s P(0, 0 \mid s)}{1 - sp_{11}}$$

and these can be determined for the cases of Markov inputs mentioned above.

The derivation of $h_j^{(n)}(i, k)$ is facilitated by the following matrix operation due to Conover [3].

Consider the $[(p+1)\times(q+1)]$-matrix $\boldsymbol{A} = (a_{ij})$ and the $[(q+1)\times (r+1)]$-matrix $\boldsymbol{B} = (b_{jk})$ $(0 \leqslant i \leqslant p,\ 0 \leqslant j \leqslant q,\ 0 \leqslant k \leqslant r)$. Ordinary matrix multiplication of $\boldsymbol{A}$ by $\boldsymbol{B}$ results in the $[(p+1)\times(r+1)]$-matrix $\boldsymbol{C} = (c_{ij})$, say. The operation $\boldsymbol{A} * \boldsymbol{B}$ ($\boldsymbol{A}$ shift-multiplied by $\boldsymbol{B}$) results in the $[(p+1+r)\times(r+1)]$-matrix $\boldsymbol{D} = (d_{ij})$, where

$$d_{ij} = \begin{cases} c_{i-j,j} & \text{if } 0 < i-j \leqslant p+1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{bmatrix},$$

then

$$\boldsymbol{A} * \boldsymbol{B} = \begin{bmatrix} \sum a_{0a} b_{a0} & 0 \\ \sum a_{1a} b_{a0} & \sum a_{0a} b_{a1} \\ \sum a_{2a} b_{a0} & \sum a_{1a} b_{a1} \\ 0 & \sum a_{2a} b_{a1} \end{bmatrix},$$

i.e. the matrix $\boldsymbol{AB}$ with its i-th column shifted downwards i places and the empty positions replaced by zeros.

Let $\delta_j^{(m)} = \Pr\{S_m = j\}$ and let $\Delta_m = (\delta_j^{(m)},\ j = 0, 1, \ldots, mN)$ be the $[(mN+1)\times 1]$-vector defining the probability distribution of S_m. Then Δ_1 represents the distribution of X_0, namely $(p_{i0}, p_{i1}, \ldots, p_{iN})^T$. Denoting the $(N+1)\times(N+1)$ identity matrix by $\boldsymbol{I}$, and putting

$$\boldsymbol{Q}_1 = \Delta_1^T * \boldsymbol{I}, \quad \boldsymbol{Q}_t = \boldsymbol{Q}_{t-1} * \boldsymbol{P} \ (t = 1, 2, \ldots),$$

it is easy to prove by induction that the element in the $(j+1)$-st row and $(k+1)$-st column of $\boldsymbol{Q}_n$ is equal to $\Pr\{S_n = j,\ X_{n-1} = k \mid X_{-1} = i\}$, i.e. $h_j^{(n)}(i, k)$.

References

[1] M. S. Ali Khan and J. Gani, *Infinite dams with inputs forming a Markov chain*, J. Appl. Probability 5 (1968), p. 72-83.

[2] P. J. Brockwell and J. Gani, *A population process with Markovian progenies*, J. Math. Anal. Appl. 32 (1970), p. 264-273.

[3] W. J. Conover, *The distribution of* $\sum_0^m f(Y_t)$, *where* $(Y_0, Y_1, \ldots)$ *is a realization of a non-homogeneous finite-state Markov chain*, Biometrika 52 (1965), p. 277-279.

[4] J. Gani, *Recent advances in storage and flooding theory*, Advances in Appl. Probability 1 (1969), p. 90-110.

[5] S. Odoom and E. H. Lloyd, *A note on the equilibrium distribution of levels in a semi-infinite reservoir subject to Markovian inputs and unit withdrawals*, J. Appl. Probability 2 (1965), 215-222.

[6] G. G. S. Pegram, *Factors affecting draft from a Lloyd reservoir*, Water Resour. Res. 10 (1974), p. 63-66.

[7] N. U. Prabhu, *Queues and inventories*, J. Wiley, New York 1964.

[8] P. Tin and R. M. Phatarfod, *Some exact results for dams with Markovian inputs*, J. Appl. Probability 13 (1976), p. 329-337.

DEPT. OF MATHEMATICS
MONASH UNIVERSITY
CLAYTON, VICTORIA 3168
AUSTRALIA

Received on 15. 12. 1979

R. M. PHATARFOD (Clayton, Victoria)

O NIESKOŃCZONYCH TAMACH
Z WEJŚCIEM ZALEŻNYM W SPOSÓB MARKOWOWSKI

STRESZCZENIE

Znane wyrażenie na zależne od czasu prawdopodobieństwa zawartości nieskończonej tamy przy danych warunkach początkowych ma postać skomplikowanej podwójnej funkcji generującej. Tym samym numeryczne obliczenie tych prawdopodobieństw jest niezmiernie trudne. W pracy podano, przy użyciu metod kombinatorycznych, jawne wzory na te prawdopodobieństwa i pokazano, jak pewien operator macierzowy ułatwia ich numeryczne wyznaczenie.