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A MAJORANT PRINCIPLE FOR MULTIPOINT ITERATIONS WITHOUT MEMORY

The paper deals with multipoint iterations without memory for the solution of the non-linear scalar equation

$$\varphi(x) = 0. \tag{1}$$

The function φ has a suitable number of derivatives.

For equation (1) we construct a sequence of approximations $\{x^m\}$ by the formula

$$x^{m+1} = \Phi(\varphi)(x^m). \tag{2}$$

In the case of one-point methods the function Φ can be defined as a zero of a polynomial $p_n(x)$, interpolating φ at the point x with information $\varphi(x), \ldots, \varphi^{(n)}(x)$. The order of this method is equal to $n+1$. Such a method cannot be used in practice, since the zero of a polynomial of degree greater than 2 can be found only approximatively. This disadvantage can be avoided by linearization of the interpolating Taylor polynomial. A class of methods is obtained in [2], p. 270, including among others König's and Schröder's methods.

The same problem appears in the case of multipoint iterations. In paper [4], Kung and Traub apply an inverse interpolation. Their algorithm is the following:

Given x^m, construct $x_0 = x^m$, $x_1 = x_0 + \beta\varphi(x_0)$ for a non-zero β, x_k being a zero of the polynomial interpolating $\psi(y) = \varphi^{-1}(x)$ at distinct points $y_i = \varphi(x_i)$, $i = 0(1)k-1$, $k = 2(1)n+1$, $x^{m+1} = x_{n+1}$, and having the order of convergence 2^n.

Woźniakowski [5] presents a more general method using also the derivatives of φ. If at the point x_i, $i = 0(1)n$, we use the information $\varphi(x_i), \ldots, \varphi^{(l_i)}(x_i)$, l_i – non-negative integer, then the order of convergence μ is equal to

$$\mu = (l_0+1)\prod_{i=1}^{n}(l_i+2). \tag{3}$$

In this paper we present a method having its source in the linearization of a Newton interpolating formula. It is of the following form:

Given x^m, put $x_0 = x^m$, $x_1 = x_0 + \beta\varphi(x_0)$ for a non-zero β, and

(4)
$$x_{k+1} = x_k - \frac{\varphi_k}{\varphi_{k,k-1} + \varphi_{k,k-2}(x_k - x_{k-1}) + \ldots + \varphi_{k,0}(x_k - x_{k-1})\ldots(x_2 - x_1)},$$
$$k = 1(1)n,\ x^{m+1} = x_{n+1},$$

where $\varphi_{k,l}$ denotes the difference quotient of φ at the points $x_k, x_{k-1}, \ldots, x_l$.

The method of majorant, used here, has been introduced first by Kantorovich [3] for the operator case of the Newton method, and also by Altman [1] for methods of higher order.

Definition. $f(s) = 0$ is a *majorant equation* for equation (1) if there exist points x^0, s^0, s^* such that

(i) $f(s^*) = 0$;
(ii) $|\varphi(x^0)| \leqslant f(s^0)$;
(iii) $|\varphi'(x^0)| \geqslant -f'(s^0)$;
(iv) $|\varphi^{(k)}(x^0)| \leqslant f^{(k)}(s^0)$, $k = 2(1)n$;
(v) $|\varphi^{(n+1)}(x)| \leqslant f^{(n+1)}(s)$ for $|x - x^0| \leqslant s - s^0 \leqslant s^* - s^0$.

THEOREM 1. *Assume that $f(s^*) = 0$, s^* being the closest, lying on the right of s^0, zero of f, and*

$$f(s^0) > 0,\quad f'(s^0) < 0,\quad f''(s^0) \geqslant 0,\quad \ldots,$$
$$f^{(n)}(s^0) \geqslant 0,\quad f^{(n+1)}(s) \geqslant 0 \quad \textit{for } s^0 \leqslant s \leqslant s^*.$$

Then the sequence of iterations $\{s^m\}$ derived from the formulas

$$s_0 = s^m,\quad s_1 = s_0 + \beta f_0,\quad \textit{where } 0 < \beta < -1/f'(s^0),$$

(5)
$$s_{k+1} = s_k - \frac{f_k}{f_{k,k-1} + f_{k,k-2}(s_k - s_{k-1}) + \ldots + f_{k,0}(s_k - s_{k-1})\ldots(s_2 - s_1)},$$
$$k = 1(1)n,\ s^{m+1} = s_{n+1},$$

is monotonic and tends to s^*. The order of convergence of this method is equal to 2^n.

Proof. Observe that $f^{(k)}(s) \geqslant 0$, $k = 2(1)n$, $f'(s) < 0$, $f(s) > 0$ for $s^0 \leqslant s < s^*$. Put $s_0 = s^0$; then, since $0 < \beta < -1/f'(s^0)$, we have $s_1 > s_0$ and $s_1 \leqslant s^*$. Suppose that $s_0 < s_1 < \ldots < s_k < s^*$. Since

(6)
$$f(u_0, \ldots, u_k) = \int\limits_{\substack{t_j \geqslant 0 \\ \Sigma t_j = 1}} f^{(k)}(t_0 u_0 + \ldots + t_k u_k)\,dt_1 \ldots dt_k,$$

we get

$$(7)\qquad \begin{aligned} &f(u_0, u_1) < 0, \\ &f(u_0, \ldots, u_k) \geqslant 0, \quad k = 2(1)n+1,\ u_0, \ldots, u_k \in (s^0, s^*). \end{aligned}$$

We have

$$f(s^*) = 0 = f_k + f_{k,k-1}(s^*-s_k) + \ldots + f_{k,0}(s^*-s_k)\ldots(s^*-s_1) + \\ +(s^*-s_k)\ldots(s^*-s_0)f(s_k, \ldots, s_0, s^*).$$

This formula and (7) give

$$f_{k,k-1} + f_{k,k-2}(s_k - s_{k-1}) + \ldots + f_{k,0}(s_k - s_{k-1})\ldots(s_2 - s_1) < 0.$$

The inequality $f(s_k) > 0$ implies $s_{k+1} > s_k$ and

$$s_{k+1} - s_k = -\frac{f_k}{f_{k,k-1} + f_{k,k-2}(s_k - s_{k-1}) + \ldots + f_{k,0}(s_k - s_{k-1})\ldots(s_2 - s_1)}$$
$$< -\frac{f_k}{f_{k,k-1} + f_{k,k-2}(s^* - s_{k-1}) + \ldots + f_{k,0}(s^* - s_{k-1})\ldots(s^* - s_1)} \leqslant s^* - s_k.$$

By induction we have $s_0 < s_1 \leqslant \ldots \leqslant s_{n+1} \leqslant s^*$, i.e., $s^0 < s^1 \leqslant s^*$. Since this method is stationary and without memory, we get

$$s^0 < s^1 \leqslant \ldots \leqslant s^m \leqslant \ldots \leqslant s^*.$$

The fact that s^m tends to s^* is obvious.

The second part of the theorem will be also proved by induction. From the Newton interpolation formula we obtain

$$f_2 = f_1 + f_{1,0}(s_2 - s_1) + f_{2,0}(s_2 - s_1)(s_2 - s_0) = O[(s_2 - s_1)(s_2 - s_0)] \\ = O[(s^* - s_0)^2].$$

Since $f_2/(s^* - s_2) = f'(\theta) \neq 0$, $\theta \in (s_2, s^*)$, we get

$$s^* - s_2 = O[(s^* - s_0)^2].$$

Suppose that

$$(8)\qquad s^* - s_l = O[(s^* - s_0)^{2^{l-1}}] \quad \text{for } l = 1(1)k.$$

Compare the formulas

$$f(s) = f_k + f_{k,k-1}(s - s_k) + f_{k,k-2}(s - s_k)(s - s_{k-1}) + \ldots + \\ + f_{k,0}(s - s_k)\ldots(s - s_1) + (s - s_k)\ldots(s - s_0)f(s_k, \ldots, s_0, s)$$

and

$$0 = f_k + f_{k,k-1}(s_{k+1} - s_k) + f_{k,k-2}(s_{k+1} - s_k)(s_k - s_{k-1}) + \ldots + \\ + f_{k,0}(s_{k+1} - s_k)\ldots(s_2 - s_1).$$

By (8) we have

$$(s_{k+1}-s_k)(s_{k+1}-s_{k-1})\dots(s_{k+1}-s_p) = (s_{k+1}-s_k)(s_k-s_{k-1})\dots(s_{p+1}-s_p)+ \\ +O[(s^*-s_0)^{2^k}], \quad p = 1(1)k-1.$$

Hence

$$f_{k+1} = f_{k,k-2}[(s_{k+1}-s_k)(s_{k+1}-s_{k-1})-(s_{k+1}-s_k)(s_k-s_{k-1})]+\dots+ \\ +f_{k,0}[(s_{k+1}-s_k)\dots(s_{k+1}-s_1)-(s_{k+1}-s_k)\dots(s_2-s_1)]+ \\ +f_{k+1,0}(s_{k+1}-s_k)\dots(s_{k+1}-s_0) = O[(s^*-s_0)^{2^k}].$$

Analogously as before we get

$$s^*-s_{k+1} = O[(s^*-s_0)^{2^k}].$$

We have shown that

$$s^*-s^{m+1} = O[(s^*-s^m)^{2^n}],$$

and this completes the proof.

LEMMA. *Let* $s_{i+1}-s_i \geqslant |x_{i+1}-x_i|$, $i = 0(1)k-1$, *and*

$$f^{(k)}(s) \geqslant |\varphi^{(k)}(x)| \quad \textit{for } |x-x_0| \leqslant s-s_0 \leqslant s^*-s_0.$$

Then

$$|\varphi(x_0, \dots, x_k)| \leqslant f(s_0, \dots, s_k).$$

The lemma follows immediately from formula (6).

THEOREM 2. *If* $f(s) = 0$ *is a majorant equation for equation* (1), *then*

(i) *equation* (1) *has the root* x^*;

(ii) *if* $0 < |\beta| < -1/f'(s^0)$, *then the iteration process* (4) *is convergent to* x^*;

(iii) *the inequality* $|x^m-x^*| \leqslant s^*-s^m$ *holds for all* m.

Proof. Let assumptions (i)-(v) of the Definition be fulfilled for points x^m, s^m, $m \geqslant 0$. Then

$$|x_1-x_0| = |\beta|\,|\varphi(x_0)| \leqslant |\beta| f(x_0) = s_1-s_0.$$

Assume that $s_{i+1}-s_i \geqslant |x_{i+1}-x_i|$, $i = 0(1)k-1$. By the Lemma we have

$$(9) \qquad |x_{k+1}-x_k| = \left|\frac{\varphi_k}{\varphi_{k,k-1}+\dots+\varphi_{k,0}(x_k-x_{k-1})\dots(x_2-x_1)}\right| \\ \leqslant \frac{-f_k}{f_{k,k-1}+\dots+f_{k,0}(s_k-s_{k-1})\dots(s_2-s_1)} = s_{k+1}-s_k.$$

Applying induction we obtain

$$|x^{m+1}-x^m| \leqslant s^{m+1}-s^m.$$

Now, we show that assumptions (ii)-(iv) are satisfied if x^m and s^m are replaced by x^{m+1} and s^{m+1}, respectively. Assume that

$$|\varphi^{(k+1)}(x)| \leqslant f^{(k+1)}(s), \qquad |x-x^0| \leqslant s-s^0 \leqslant s^*-s^0 \quad \text{for } 2 \leqslant k \leqslant n.$$

For $k = n$ this inequality holds due to (v). We have

$$\begin{aligned} |\varphi^{(k)}(x)| &= \Big|\varphi^{(k)}(x^0)+(x-x^0)\int_0^1 \varphi^{(k+1)}\big(x^0+t(x-x^0)\big)(1-t)\,dt\Big| \\ &\leqslant f^{(k)}(s^0)+(s-s^0)\int_0^1 f^{(k+1)}\big(s^0+t(s-s^0)\big)(1-t)\,dt \\ &= f^{(k)}(s). \end{aligned}$$

Since $|x^{m+1}-x^0| \leqslant s^{m+1}-s^0 \leqslant s^*-s^0$, we get by induction

$$|\varphi^{(k)}(x^{m+1})| \leqslant f^{(k)}(s^{m+1}) \quad \text{for } k = 2(1)n.$$

We consider the case $k = 1$ separately:

$$\begin{aligned} |\varphi'(x^{m+1})| &= \Big|\varphi'(x^0)+\int_0^1 \varphi''\big(tx^{m+1}+(1-t)x^0\big)(1-t)\,dt\cdot(x^{m+1}-x^0)\Big| \\ &\geqslant |\varphi'(x^0)|-|x^{m+1}-x^0|\int_0^1 \big|\varphi''\big(tx^{m+1}+(1-t)x^0\big)(1-t)\big|\,dt \\ &\geqslant -f'(s^0)-(s^{m+1}-s^0)\int_0^1 f''\big(ts^{m+1}+(1-t)s^0\big)(1-t)\,dt \\ &= -f'(s^{m+1}). \end{aligned}$$

The proof of (ii) is a little more complicated. Let $|\varphi(x_l)| \leqslant f(s_l)$, $l = 0(1)k$. Then, using the Lemma and formula (9), we have

$$\begin{aligned} |\varphi(x_{l+1})| &= |\varphi(x_k)+\varphi_{k,k-1}(x_{k+1}-x_k)+\ldots+\varphi_{k+1,0}(x_{k+1}-x_k)\ldots(x_{k+1}-x_0)| \\ &\leqslant |\varphi(x_k)+\varphi_{k,k-1}(x_{k+1}-x_k)+\ldots+\varphi_{k,0}(x_{k+1}-x_k)\ldots(x_2-x_1)+ \\ &\quad +\varphi_{k+1,0}(x_{k+1}-x_k)\ldots(x_{k+1}-x_0)|+\sum_{j=2}^{n}|\varphi_{k,k-j}(x_{k+1}-x_k)[(x_{k+1}-x_{k-1})\ldots \\ &\quad \ldots(x_{k+1}-x_{k-j+1})-(x_k-x_{k-1})\ldots(x_{k-j+2}-x_{k-j+1})]| \\ &\leqslant f_{k+1,0}(s_{k+1}-s_k)\ldots(s_{k+1}-s_0)+\sum_{j=2}^{n}f_{k,k-j}(s_{k+1}-s_k)[(s_{k+1}-s_{k-1})\ldots \\ &\quad \ldots(s_{k+1}-s_{k-j+1})-(s_k-s_{k-1})\ldots(s_{k-j+2}-s_{k-j+1})] \\ &= f(s_{k+1}). \end{aligned}$$

Hence $|\varphi(x^{m+1})| \leqslant f(s^{m+1})$.

Applying induction once more we obtain $|x^{m+k}-x^m| \leqslant s^{m+k}-s^m$ for every $m \geqslant 0$, $k \geqslant 0$.

The sequence $\{x^m\}$ is Cauchy (because $\{s^m\}$ is convergent). Then the limit

$$x^* = \lim_{m\to\infty} x^m$$

exists.

Letting k tend to infinity in the last inequality we prove (iii) of the theorem.

Remark. There exist methods using also the derivatives of φ, constructed in a similar manner as (4). Our two theorems remain valid with the order of convergence defined by (3). Similar results hold also for one-point methods with memory.

Construction of the majorant equation. Let an initial approximation be chosen. Put

$$b_k = \frac{1}{k!}\varphi^{(k)}(x^0), \qquad k = 0(1)n,$$

and set

$$\tilde{f}(s) = b_n s^n + \ldots + b_2 s^2 - b_1 s + b_0.$$

The equation $\tilde{f}(s) = 0$ may have no positive roots or two. If it has no roots, then we must choose a new initial point x^0. In the second case, choose the greater one, say s^* (or, better, the point $s^* > 0$ where $\tilde{f}$ attains its minimum), and put

$$b_{n+1} = \frac{1}{(n+1)!}\max_{|x-x^0|\leqslant s^*}|\varphi^{(n+1)}(x)|.$$

If the equation $f(s) = 0$, where $f(s) = \tilde{f}(s) + b_{n+1}s^{n+1}$, has positive roots, then it is a majorant equation for (1).

Example. Let

$$\varphi(x) = \ln(2-x) - \sin\left(x+\frac{\pi}{6}\right).$$

Choose $x^0 = -0.6$, $n = 2$. We have

$$\varphi(x^0) = 1.032, \quad \varphi'(x^0) = -1.382, \quad \varphi''(x^0) = -0.224,$$

$$|\varphi'''(x)| < 0.9 \quad \text{for } x \in [-1.7, 0.5].$$

Put

$$f(s) = 1.04 - 1.38s + 0.12s^2 + 0.15s^3.$$

Then $f(0) > 0$ and $f(1) < 0$, so there exists a positive root s^* of the equation $f(s) = 0$, $s^* < 1.1$. Hence, putting $s^0 = 0$ we get the majorant equation.

It can be checked that, for $n = 1$, $x^0 = -0.6$ is not a good initial approximation in the sense of the Definition.

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ZASADA MAJORANT DLA METOD ITERACYJNYCH WIELOPUNKTOWYCH BEZ PAMIĘCI

STRESZCZENIE

W pracy opisana jest wielopunktowa metoda iteracyjna dla rozwiązywania równań skalarnych. Jej idea wywodzi się z linearyzacji wzoru interpolacyjnego Newtona. W dowodzie zbieżności zastosowano uogólnioną zasadę majoranty rzeczywistej pochodzącej od Kantorowicza [3].

Dla równania skalarnego (1) konstruujemy ciąg iteracji $\{x^m\}$ w następujący sposób:

Mając dany punkt x^m, ustalamy $x_0 = x^m$, $x_1 = x_0 + \beta\varphi(x_0)$ $(\beta \neq 0)$ oraz określamy x_{k+1} za pomocą wzoru (4), gdzie $\varphi_{k,l}$ oznacza iloraz różnicowy φ w punktach $x_k, x_{k-1}, \ldots, x_l$.

Definicja. $f(s)$ jest *majorantą* dla równania (1), jeżeli istnieją takie punkty x^0, s^0, s^*, że

(i) $f(s^*) = 0$;

(ii) $|\varphi(x^0)| \leqslant f(s^0)$;

(iii) $|\varphi'(x^0)| \geqslant -f'(s^0)$;

(iv) $|\varphi^{(k)}(x^0)| \leqslant f^{(k)}(s^0)$, $k = 2(1)n$;

(v) $|\varphi^{(n+1)}(x)| \leqslant f^{(n+1)}(s)$ dla $|x-x^0| \leqslant s-s^0 \leqslant s^*-s^0$.

TWIERDZENIE 1. *Załóżmy, że* $f(s^*) = 0$, *gdzie* s^* *jest najbliższym, leżącym na prawo od* s^0, *zerem funkcji* f, *oraz że*

$$f(s^0) > 0, \quad f'(s^0) < 0, \quad f''(s^0) \geqslant 0, \quad \ldots, \quad f^{(n)}(s^0) \geqslant 0,$$
$$f^{(n+1)}(s) \geqslant 0 \quad dla\ s^0 \leqslant s \leqslant s^*.$$

Wtedy ciąg iteracji $\{s^m\}$, *określony wzorami* (5), *jest monotoniczny i zbieżny do* s^*. *Wykładnik zbieżności tego ciągu wynosi* 2^n.

TWIERDZENIE 2. *Jeżeli* $f(s) = 0$ *jest majorantą dla równania* (1), *to*

(i) *równanie* (1) *ma pierwiastek* x^*;

(ii) *ciąg iteracji określony wzorami* (4) *jest zbieżny do* x^*;

(iii) *dla wszystkich* m *zachodzi nierówność* $|x^m - x^*| \leqslant s^* - s^m$