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ON THE ROTATION AND TRANSLATION OF AN ISOTROPIC CONDUCTING INCOMPRESSIBLE MEDIUM IN PLANE MOTION

1. Introduction. Dean [1] investigated theoretically Taylor's problem on the two-dimensional motion of an infinite cylinder in a rotating homogeneous incompressible fluid by comparing two fluid motions: a general two-dimensional motion of the cylinder in a fluid and a second motion derived from the first by the superposition, on the whole system, of a constant angular velocity Ω. It is noticed that the second motion is dynamically possible whenever the first is and the motion of the cylinder relative to the liquid is not altered by superposition. It is further observed that the stress system in the second flow differs from that in the first only by a scalar stress. Noll [3] and Oldroyd [4] noticed that the result holds for liquids having visco-elastic properties. Later, the present author [5] recorded the validity of Taylor's result for a general isotropic conducting medium in the presence of a constant axial magnetic field. Recently, Huilgol [2] examined the Taylor problem with the superposition of a translation, in addition, to constant angular velocity.

The aim of the present note is to investigate this problem for a general isotropic incompressible conducting medium in the presence of a constant axial magnetic field. The essential assumptions are the following: the medium is weakly conducting (which always happens when the cylinder is of infinite conductivity), non-magnetic, free of charges, isotropic and incompressible, the motion is two-dimensional, the applied magnetic field $\boldsymbol{H}$ is perpendicular to the plane of motion of the cylinder and the perturbation in it due to interaction with the fluid velocity is negligibly small.

2. Basic equations. The two-dimensional motion of such a liquid is governed by the equations of magneto-hydrodynamics, written in the simplified form as

$$\varrho(D\boldsymbol{q}/Dt) = \mathrm{Div}\, S - \sigma\mu_e^2 H^2 \boldsymbol{q} \tag{1}$$

together with the continuity equation

$$(2) \qquad \operatorname{div} \boldsymbol{q} = 0,$$

where $\boldsymbol{q} = (u, v)$ is the fluid velocity, ϱ the density, σ the electric conductivity, μ_e the magnetic permeability of the liquid, and H the intensity of the applied magnetic field. Also, S denotes the stress tensor which, for an incompressible fluid in motion, can be expressed as $S = -pI + T$, where p is the isotropic pressure, and T the deviatoric stress tensor. Body forces are neglected here, but any conservative system of body forces can be freely accommodated by a simple adjustment in the pressure contingent.

The equation of continuity (2) is identically satisfied by introducing the stream function ψ, satisfying the equations $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$.

We compare two plane motions: (i) a motion of a cylinder in the liquid in an inertial frame OXY and (ii) the motion of cylinder in a second frame, denoted by 2, which is obtained from (i) by a three-dimensional translation $\boldsymbol{V}(t)$ and by a superposed rigid rotation $\Omega(t)$, normal to the plane of motion. These two motions are designated by the subscripts 1 and 2, respectively.

If (u_1, v_1) and (a_1, b_1) be the components of the velocity and acceleration, respectively, in the first motion, then the velocity (u_2, v_2) and acceleration (a_2, b_2) in the second motion can be expressed in terms of the Lagrangian coordinates

$$(3) \qquad u_2 = u_1 - \Omega y - \dot{c}, \qquad v_2 = v_1 + \Omega x - \dot{d}$$

and

$$(4) \qquad a_2 = a_1 - 2\Omega v_1 - \Omega^2 x - \ddot{c}, \qquad b_2 = b_1 + 2\Omega u_1 - \Omega^2 y - \ddot{d},$$

where c and d are components of $\boldsymbol{V}(t)$ in the directions x and y of the inertial frame 1, and the dot $(\cdot)$ denotes differentiation with reference to time t.

The stream functions ψ_1 and ψ_2 in the two motions are now related by

$$\psi_2 = \psi_1 + \tfrac{1}{2}\Omega^2(x^2 + y^2) + \dot{c}y - \dot{d}x.$$

Since $\boldsymbol{V}(t)$ is independent of the space coordinates, $\operatorname{curl} \boldsymbol{V} = 0$ and hence $\boldsymbol{V}$ can be expressed as the gradient of a scalar φ,

$$\boldsymbol{V} = \nabla(cx + dy + ez + f).$$

The equations of motion (1) in the two motions in the directions x and y can be written as

$$\varrho a_r = \frac{\partial}{\partial x}[S_{xx}]_r + \frac{\partial}{\partial y}[S_{xy}]_r - \sigma\mu_e^2 H^2 u_r,$$

$$\varrho b_r = \frac{\partial}{\partial x}[S_{xy}]_r + \frac{\partial}{\partial y}[S_{yy}]_r - \sigma\mu_e^2 H^2 v_r,$$

together with the boundary condition

$$(u_r, v_r)_n = -\frac{\partial\psi_r}{\partial s}, \qquad r = 1, 2,$$

on Γ, where s and n denote the directions of the tangent and outward drawn normal, respectively, at a point P of Γ, the contour of the cylinder, and $(u_r, v_r)_n$ is the velocity along n.

From the principle of material indifference, the deviatoric stress tensor for an isotropic incompressible medium in motion remains unaltered by the superposition of rigid motions to the inertial frame. Hence, in the two motions 1 and 2, $T_1 = T_2$. From [5] we obtain

$$\varrho(a_2 - a_1) = -\frac{\partial}{\partial x}(p_2 - p_1) - \sigma\mu_e^2 H^2(u_2 - u_1)$$

and

$$\varrho(b_2 - b_1) = -\frac{\partial}{\partial y}(p_2 - p_1) - \sigma\mu_e^2 H^2(v_2 - v_1).$$

Employing (3) and (4) and rearranging, we get

$$\frac{\partial}{\partial x}(p_2 - p_1) = \varrho\left[2\Omega\frac{\partial\psi_1}{\partial x} + \Omega^2 x + \ddot{c}\right] + \sigma\mu_e^2 H^2(\Omega y + \dot{c})$$

and

$$\frac{\partial}{\partial y}(p_2 - p_1) = \varrho\left[2\Omega\frac{\partial\psi_1}{\partial y} + \Omega^2 y + \ddot{d}\right] + \sigma\mu_e^2 H^2(-\Omega x + \dot{d})$$

which yield on integration

$$p_2 - p_1 = \varrho(2\Omega\psi_1 + \tfrac{1}{2}\Omega^2\boldsymbol{r}\cdot\boldsymbol{r} + \ddot{\boldsymbol{V}}\cdot\boldsymbol{r}) + \sigma\mu_e^2 H^2(\boldsymbol{V}\cdot\boldsymbol{r} - 2\Omega A) + C,$$

where A is the area enclosed by the contour Γ, and C is the constant of integration, independent of the space coordinates (x, y).

We thus have: if a plane motion of an incompressible medium be dynamically possible in an inertial frame of reference, the same motion is also possible in any other frame of reference which is rotating with a constant angular velocity about an axis normal to the plane of motion and also when this frame undergoes a time dependent but spacially independent translation. This result is thus equivalent to the introduction of a conservative body force. We thus establish the Taylor result holding for any isotropic conducting medium in the presence of a constant magnetic field and a superposition of a three-dimensional translation.

3. Force system on the cylinder. The reaction (F_r, G_r) per unit length of the cylinder and the couple H_r exerted about the centre of mass (x_0, y_0) of A are given by

$$F_r = \int_\Gamma \{[S_{xx}]_r \cos(n, x) + [S_{xy}]_r \sin(n, x)\}\, dA,$$

$$G_r = \int_\Gamma \{[S_{xy}]_r \cos(n, x) + [S_{yy}]_r \sin(n, x)\}\, dA,$$

and

$$H_r = \int_\Gamma \big(\eta\{[S_{xx}]_r \cos(n, x) + [S_{xy}]_r \sin(n, x)\} - \\ - \xi\{[S_{xy}]_r \cos(n, x) + [S_{yy}]_r \sin(n, x)\}\big)\, dA,$$

where $\xi = x_0 - x$, and $\eta = y_0 - y$. Employing these results, we obtain

$$F_2 - F_1 = -A[\varrho(\Omega^2 x_0 + 2\Omega V_0 + \ddot{c}) + \sigma\mu_e^2 H^2 \dot{c}],$$
$$G_2 - G_1 = A[\varrho(\Omega^2 y_0 - 2\Omega U_0 + \ddot{d}) + \sigma\mu_e^2 H^2 \dot{d}],$$

and

$$H_2 - H_1 = -A[\varrho\{\Omega^2(cy_0 + dx_0) + \ddot{c}d - c\ddot{d}\} + \sigma\mu_e^2 H^2(\dot{c}d - c\dot{d})],$$

where (U_0, V_0) is the velocity of the centre of mass of A in the motion 1. Let M be the mass per unit length of the cylinder in plane motion. Then the forces on the cylinder in the second motion are statically equivalent (i) to the forces in the first motion and (ii) to another force with components

$$-M[\Omega^2 x_0 + 2\Omega V_0 + \ddot{c} + \sigma\mu_e^2 H^2 \dot{c}/\varrho]$$

and

$$+M[\Omega^2 y_0 + 2\Omega U_0 + \ddot{d} + \sigma\mu_e^2 H^2 \dot{d}/\varrho]$$

together with a couple of magnitude

$$-M[\Omega^2(cy_0 + dx_0) + \ddot{c}d - c\ddot{d} + \sigma\mu_e^2(\dot{c}d - c\dot{d})/\varrho]$$

about the axis of rotation.

References

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O RUCHU OBROTOWYM I POSTĘPOWYM W DWUWYMIAROWYM PRZEPŁYWIE IZOTROPOWEJ CIECZY PRZEWODZĄCEJ

STRESZCZENIE

Autor analizuje zagadnienie Taylora dwuwymiarowego ruchu nieskończonego, obracającego się walca w cieczy jednorodnej, nieściśliwej, przez porównanie ogólnego dwuwymiarowego ruchu walca w cieczy z ruchem powstałym z poprzedniego przez nałożenie na cały układ dodatkowego ruchu obrotowego o stałej prędkości kątowej. Celem tej noty jest zbadanie tego zagadnienia w przypadku dowolnej, izotropowej, nieściśliwej cieczy przewodzącej w obecności stałego pola magnetycznego. W istotny sposób wykorzystuje się założenia, że ciecz jest słabo przewodząca, amagnetyczna i pozbawiona ładunków, że ruch jest dwuwymiarowy, działające pole magnetyczne prostopadłe do płaszczyzny ruchu walca i że można zaniedbać zaburzenia tego pola wywołane w nim przez ruch cieczy.