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A. KOZEK (Wrocław)

INVERSE CAUCHY PROBLEM AND UNBIASED ESTIMATION

0. Introduction. In this paper we consider an *inverse Cauchy problem* which consists in solving a certain integral equation. This problem has two different faces: one is concerned with the construction of the minimum-risk unbiased estimators and the other with a problem in the theory of heat conduction. The literature on the first problem is rich (see, for example, [2], [3] and [6]-[12]). In Section 1 we give a description of the second problem. Section 1 contains also a formulation of the main problem of this paper. In Section 2 we give the notation and definitions. In Sections 3 and 4 we prove theorems which are helpful in the proof of the main theorem of this paper, i. e. in the proof of Theorem 2 given in Section 5. In Theorem 2 we give the solution of the inverse Cauchy problem. Next, in Section 6, we use the obtained results to the construction of minimum-risk unbiased estimators of functions of parameters in the cases of the normal distribution with both unknown and known variances (Theorems 3 and 4, respectively). In Section 7 we give examples of applications of the theorems proved in Section 6 to the construction of minimum-risk unbiased estimators.

1. Inverse Cauchy problem. The main problem considered in this paper is to find a solution $\varphi(v, u)$ of the integral equation

$$g(\theta, t) = \int_{R^n} \left(\frac{1}{2\pi t}\right)^{n/2} \exp\left[-\frac{1}{2t}\sum_{i=1}^{n}(x_i-\theta)^2\right]\varphi(\bar{x}, s^2)\,dx, \tag{1.1}$$

where $g(\theta, t)$ is a known function defined on $R\times(0, T)$, $T>0$, R^n is an n-dimensional Euclidean space, and

$$\bar{x} = \frac{1}{n}\sum x_i \quad \text{and} \quad s^2 = \sum (x_i-\bar{x})^2.$$

We propose to call this problem the *inverse Cauchy problem* for the following heat equation:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) f(\theta_1, \ldots, \theta_n, t) = 0. \tag{1.2}$$

The term "inverse Cauchy problem" can be justified as follows.

Let $g(\theta, t)$ stand for the temperature in the space R^n at a point θ on the line $\theta_1 = \theta_2 = \ldots = \theta_n$ at the instant t. The solution of (1.1) consists in finding a function $\varphi(\bar{x}, s^2)$ which may stand for the cylindrically distributed temperature at the initial instant $t = 0$. Therefore, the above-mentioned problem is "inverse" to the Cauchy problem which consists in finding the temperature $f(\theta_1, \ldots, \theta_n, t)$ at the point $(\theta_1, \ldots, \theta_n)$ at the instant t provided the initial temperature at the point $(\theta_1, \ldots, \theta_n)$ is given by a function $\psi(\theta_1, \ldots, \theta_n)$.

In Section 3 (cf. Theorem 1) we prove certain properties of the pair $g(\theta, t)$ and $\varphi(\bar{x}, s^2)$ which satisfies equation (1.1). Then, the question arises: whether a pair g, φ, having these properties, is a solution of (1.1). An answer to this question is given in Theorem 2 and Corollary 5.1 (Section 5). Theorem 2 and Corollary 5.1 give the solution of the inverse Cauchy problem (1.1).

2. Definitions and notation. Let T be a fixed positive number, let $D = R \times [0, \infty)$ and $k(t) = 1/2(T-t)$. Further, let $L_{n,k(t)}$, $L_{n,T}$, $L_{k(t)}$ and L_T stand for sets of functions defined as follows:

1. $L_{n,k(t)}$ stands for the set of functions defined on $R^n \times (0, T)$ for which

$$\|f\|^2_{L_{n,k(t)}} = \int_{R^n} |f(x_1, \ldots, x_n, t)|^2 \exp\left[-k(t) \sum x_i^2\right] dx < +\infty$$

for $t \in (0, T)$.

2. $L_{n,T}$ stands for the set of functions defined on R^n for which

$$\|\psi\|^2_{L_{n,T,\varepsilon}} = \int_{R^n} |\psi(x_1, \ldots, x_n)|^2 \exp\left[-\left(\frac{1}{2T} + \varepsilon\right) \sum x_i^2\right] dx < +\infty$$

for every $\varepsilon > 0$.

3. $L_{k(t)}$ stands for the set of functions defined on $D \times (0, T)$ for which

$$\|F\|^2_{L_{k(t)}} = \int_D |F(v, u, t)|^2 u^{(n-3)/2} \exp[-k(t)(nv^2 + u)] dv du < +\infty$$

for $t \in (0, T)$.

4. L_T stands for the set of functions defined on D for which

$$\|\varphi\|^2_{L_{T,\varepsilon}} = \int_D |\varphi(v, u)|^2 u^{(n-3)/2} \exp\left[-\left(\frac{1}{2T} + \varepsilon\right)(nv^2 + u)\right] dv du < +\infty$$

for every $\varepsilon > 0$.

A function $F(v, u, t)$ defined on $D\times(0, T)$ is called *holomorphic* provided F is extendable as a holomorphic function to the set

$$D_1 = \{(z_1, z_2, z_3)\colon z_1 = v_1+iv_2, z_2 = u_1+iu_2, z_3 = t_1+it_2$$
$$\text{and } (v_1, u_1, t_1)\in R\times(0, \infty)\times(0, T), (v_2, u_2, t_2)\in R^3\}.$$

A function $g(v, t)$ defined on $R\times(0, T)$ is called *holomorphic* provided g is extendable as a holomorphic function to the set

$$D_2 = \{(z_1, z_2)\colon z_1 = v_1+iv_2, z_2 = t_1+it_2$$
$$\text{and } (v_1, t_1)\in R\times(0, T), (v_2, t_2)\in R^2\}.$$

Moreover, let

$$N(v, t; \lambda) = \left(\frac{1}{2\pi t}\right)^{1/2}\exp\left[-\frac{1}{2t}(\lambda-v)^2\right], \tag{2.1}$$

$$\Gamma(u, t, n; \eta) = \frac{1}{\sqrt{2^n}\pi t}\exp\left[-\frac{1}{2t}(u+\eta)\right]\left(\frac{\eta}{t}\right)^{(n-2)/2}\sum_{k=0}^{\infty}\frac{(\eta u/t^2)^k}{(2k)!}\frac{\Gamma(k+1/2)}{\Gamma(k+n/2)}, \tag{2.2}$$

$$G_n(v, u, t; \lambda, \eta) = N(v, t/n; \lambda)\Gamma(u, t, n-1; \eta), \tag{2.3}$$

where $\Gamma(\cdot)$ stands for the gamma function.

Remark. If $X_1, \ldots, X_n$ $(n>1)$ are independently distributed normal random variables with means $\theta_1, \ldots, \theta_n$, respectively, $(-\infty<\theta_i<\infty)$ and identical variances $t>0$, then

$$N\left(\frac{1}{n}\sum\theta_i, \frac{t}{n}; \lambda\right), \quad \Gamma\left(\sum(\theta_i-\bar{\theta})^2, t, n-1; \eta\right)$$

and

$$G_n\left(\frac{1}{n}\sum\theta_i, \sum(\theta_i-\bar{\theta})^2, t; \lambda, \eta\right)$$

are the density functions of $\bar{X}$, S^2 and $(\bar{X}, S^2)$, respectively, where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i \quad\text{and}\quad S^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2.$$

Random variables $\bar{X}$ and S^2 are independently distributed for each $(\theta_1, \ldots, \theta_n, t)$.

3. Properties of solutions to the inverse Cauchy problem.

THEOREM 1. *Let* $\varphi \in L_T$ *and*

$$g(\theta, t) = \int_D \varphi(\lambda, \eta) G_n(\theta, 0, t; \lambda, \eta) d\lambda d\eta \quad \text{for } (\theta, t) \in R \times (0, T), \tag{3.1}$$

$$F(v, u, t) = \int_D \varphi(\lambda, \eta) G_n(v, u, t; \lambda, \eta) d\lambda d\eta \quad \text{for } (v, u, t) \in D \times (0, T). \tag{3.2}$$

Then

(a) φ *is a solution of equation* (1.1) *if and only if* φ *is a solution of equation* (3.1) *with identical* $g(\cdot, \cdot)$;

(b) *there exists at most one solution of equation* (1.1);

(c) $g(\cdot, \cdot)$ *is holomorphic*;

(d) *at all continuity points of* φ,

$$\lim_{t \to 0} F(v, u, t) = \varphi(v, u); \tag{3.3}$$

(e) *it follows that*

$$F(v, u, t) = \sum_{k=0}^{\infty} \frac{\Gamma\big((n-1)/2\big)}{k!\,\Gamma\big((n-1)/2+k\big)} \left(\frac{u}{2}\right)^k \mathcal{T}^k g(v, t), \tag{3.4}$$

where

$$\mathcal{T} = \frac{\partial}{\partial t} - \frac{1}{2n} \frac{\partial^2}{\partial v^2}, \tag{3.5}$$

and the series in (3.4) *is absolutely and almost uniformly convergent*;

(f) $F(v, u, t)$ *is holomorphic*;

(g) *the relation*

$$F(v, 0, t) = g(v, t) \tag{3.6}$$

holds.

Proof. The kernel of the integral on the right-hand side of (3.1) is the density function of n independent normal variables with the same mean θ and variance t. Thus, in view of the Remark in Section 2, part (a) is obvious.

Since $G_n(\theta, 0, t; \lambda, \eta)$ is an exponential family of distributions and (λ, η) is the complete sufficient statistic for $(\theta, t) \in R \times (0, T)$ (see [5], Theorem 1 of Chapter IV), it follows from

$$\int \varphi(\lambda, \eta) G_n(\theta, 0, t; \lambda, \eta) d\lambda d\eta = 0, \quad (\theta, t) \in R \times (0, T),$$

that $\varphi = 0$ almost everywhere. Therefore, equation (3.1) has at most one solution, where $g(\cdot, \cdot)$ denotes the same function as in equation (1.1). Hence, the inverse Cauchy problem (1.1) has at most one solution.

Part (c) is an immediate consequence of a theorem on exponential families of distributions (see [5], Theorem 9 of Chapter II).

Next we prove part (d). Let us remark that

$$(3.7)\quad F\left(\frac{1}{n}\sum\theta_i,\ \sum(\theta_i-\bar{\theta})^2,\ t\right) = \left(\frac{1}{2\pi t}\right)^{n/2}\int_{R^n}\varphi(\bar{x}, s^2)\exp\left[-\frac{1}{2t}\sum_{i=1}^{n}(x_i-\theta_i)^2\right]dx$$

is the Poisson integral. According to a theorem of Weierstrass (see [4], Section 56.8) we have

$$(3.8)\quad \lim_{t\to 0}F\left(\frac{1}{n}\sum\theta_i,\ \sum(\theta_i-\bar{\theta})^2,\ t\right) = \psi(\theta_1,\ldots,\theta_n) = \varphi\left(\bar{\theta},\ \sum(\theta_i-\bar{\theta})^2\right)$$

at each continuity point of $\psi(\theta_1,\ldots,\theta_n)$. But for every point $(v, u)\in D$ there exists $\theta\in R^n$ such that

$$v = \frac{1}{n}\sum\theta_i \quad\text{and}\quad u = \sum(\theta_i-\bar{\theta})^2.$$

Hence,

$$\lim_{t\to 0}F(v, u, t) = \varphi(v, u)$$

at each continuity point of φ. Thus, the proof of part (d) is complete d

Let $G_n(v, u, t;\ \lambda, \eta)$ and $\mathscr{T}$ be given by (2.3) and (3.5), respectively. We shall now prove that

$$(3.9)\quad G_n(v, u, t;\ \lambda, \eta) = \sum_{k=0}^{\infty}\frac{\Gamma\big((n-1)/2\big)}{k!\,\Gamma\big((n-1)/2+k\big)}\left(\frac{u}{2}\right)^k\mathscr{T}^kG_n(v, 0, t;\ \lambda, \eta).$$

Remark that

$$(3.10)\quad G_n(v, u, t;\ \lambda, \eta) = G_n(v, 0, t;\ \lambda, \eta)\frac{\Gamma\big((n-1)/2\big)}{\sqrt{\pi}}\exp\left[-\frac{u}{2t}\right]\sum_{k=0}^{\infty}\frac{(u\eta/t^2)^k}{(2k)!}\frac{\Gamma(k+1/2)}{\Gamma\big(k+(n-1)/2\big)}$$

and that the series appearing on the right-hand side is absolutely and almost uniformly convergent with respect to (u, t) on the set $R\times(0, \infty)$. Next, replace $\exp[-u/2t]$ by its Taylor expansion with respect to u, and change the order of summation. After an easy calculation we get

$$G_n(v,u,t;\ \lambda,\eta)$$

$$=\Gamma\left(\frac{n-1}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\big((n-1)/2+k\big)}\left(\frac{u}{2}\right)^k\left[(-1)^k\frac{\Gamma\big((n-1)/2+k\big)}{t^k}\times\right.$$

$$\left.\times\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{\Gamma\big(j+(n-1)/2\big)}\left(\frac{\eta}{2t}\right)^j\right]G_n(v,0,t;\ \lambda,\eta).$$

It is now easy to prove by induction on k that

$$\mathcal{T}^kG_n(v,0,t;\ \lambda,\eta)$$

$$=\left[(-1)^k\frac{\Gamma\big((n-1)/2+k\big)}{t^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{\Gamma\big(j+(n-1)/2\big)}\left(\frac{\eta}{2t}\right)^j\right]\times$$

$$\times G_n(v,0,t;\ \lambda,\eta).$$

Thus, the proof of formula (3.9) is completed.

Because $\varphi\in L_T$, integral (3.1) is absolutely and almost uniformly convergent on $R\times(0,T)$. Moreover, because $\lambda^k\eta^l\varphi(\lambda,\eta)\in L_T$ for $k,l=0,1,2,\ldots$, we can differentiate under the sign of the integral. Therefore,

$$\mathcal{T}^kg(v,t)=\int\limits_D\varphi(\lambda,\eta)\mathcal{T}^kG_n(v,0,t;\ \lambda,\eta)\,d\lambda\,d\eta.$$

Hence, by the absolutely and almost uniformly convergence of series (3.9) and absolutely and almost uniformly convergence of integral (3.2) on $D\times(0,T)$, we obtain (3.4).

To prove part (f) it is enough to note that, in view of (3.7) and of Theorem 9 of [5], Chapter II,

$$F\left(\frac{1}{n}\sum\theta_i,\sum(\theta_i-\bar{\theta})^2,t\right)$$

is extendable as a holomorphic function to the set $(\theta_1,\ldots,\theta_n)\in C^n$, $\mathrm{Re}\,t\in(0,T)$, $\mathrm{Im}\,t\in R$.

Since part (g) is obvious, the proof is completed.

4. Cauchy problem for the heat equation. The Cauchy problem, as it is well known, consists in finding a function $f\in L_{n,k(t)}$ with continuous partial derivatives of the second order with respect to θ_i $(i=1,\ldots,n)$ and of the first order with respect to t and, moreover, satisfying the conditions

$$(4.1)\quad \left(\frac{\partial}{\partial t}-\frac{1}{2}\Delta\right)f(\theta_1,\ldots,\theta_n,t)=0,\quad (\theta_1,\ldots,\theta_n,t)\in R^n\times(0,T),$$

$$(4.2)\qquad \lim_{t\to 0}\|f-\psi\|_{L_{n,k(t)}}=0,$$

where $\psi\in L_{n,T}$.

We need the following theorem which may be easily deduced from Theorem 1.3 proved in [1]:

THEOREM (Eidelman). *The Poisson integral*

$$(4.3)\quad f(\theta_1,\ldots,\theta_n,t)$$
$$=\left(\frac{1}{2\pi t}\right)^{n/2}\int_{R^n}\psi(x_1,\ldots,x_n)\exp\left[-\frac{1}{2t}\sum(x_i-\theta_i)^2\right]dx$$

is a solution of the Cauchy problem and satisfies the inequality

$$(4.4)\qquad \|f\|_{L_{n,k(t)}}\leqslant C\|\psi\|_{L_{n,T,0}},$$

where C is a constant.

In the class of functions $f\in L_{n,k(t)}$, which satisfy the condition

$$(4.5)\qquad \int_0^T\|f\|_{L_{n,k(t)}}dt<\infty,$$

the solution of the Cauchy problem is unique.

In the proof of Theorem 1 (part (d)) we noticed that if ψ is a function of

$$\bar{x}=\frac{1}{n}\sum x_i\quad\text{and}\quad s^2=\sum(x_i-\bar{x})^2$$

only, i. e. if

$$(4.6)\qquad \psi(x_1,\ldots,x_n)=\varphi(\bar{x},s^2),$$

then f is a function of $(1/n)\sum\theta_i$, $\sum(\theta_i-\bar{\theta})^2$ and t only, i. e.

$$(4.7)\qquad f(\theta_1,\ldots,\theta_n,t)=F\left(\frac{1}{n}\sum\theta_i,\sum(\theta_i-\bar{\theta})^2,t\right),$$

where $F(v,u,t)$ has continuous derivatives of the second order with respect to v and u, and of the first order with respect to t. By differentiation we infer that

$$(4.8)\qquad \left(\frac{\partial}{\partial t}-\frac{1}{2}\Delta\right)F\left(\frac{1}{n}\sum\theta_i,\sum(\theta_i-\bar{\theta})^2,t\right)=0$$

if and only if

$$\left(\frac{\partial}{\partial t}-\frac{1}{2n}\frac{\partial^2}{\partial v^2}-(n-1)\frac{\partial}{\partial u}-2u\frac{\partial^2}{\partial u^2}\right)F(v,u,t)=0. \tag{4.9}$$

Corollary 4.1. *Let $\varphi \in L_T$. Then*

$$F(v,u,t)=\int_D \varphi(\lambda,\eta)G_n(v,u,t;\lambda,\eta)\,d\lambda\,d\eta \tag{4.10}$$

is a solution of the following problem:

$$\left(\frac{\partial}{\partial t}-\frac{1}{2n}\frac{\partial^2}{\partial v^2}-(n-1)\frac{\partial}{\partial u}-2u\frac{\partial^2}{\partial u^2}\right)F(v,u,t)=0, \tag{4.11}$$

$$(v,u,t)\in R^2\times(0,T);$$

$$\lim_{t\to 0}\|F(v,u,t)-\varphi(v,u)\|_{L_{k(t)}}=0, \qquad (v,u,t)\in D\times(0,T). \tag{4.12}$$

In the class of functions $F(v,u,t)$, which satisfy the condition

$$\int_0^T \|F(v,u,t)\|_{L_{k(t)}}\,dt<\infty \tag{4.13}$$

and have continuous derivatives of the second order with respect to v and u, and of the first order with respect to t, the solution of problem (4.11)-(4.12) *is unique.*

In view of the connections of equation (3.2) with (3.7) and of equations (4.6)-(4.9) with the Cauchy problem (Eidelman's Theorem), we obtain the assertions of Corollary 4.1.

Let us remark that the function $F(v,u,t)$ appearing in Corollary 4.1 may be considered as the solution of the Cauchy problem cylindrically symmetric with respect to the line $\theta_1=\ldots=\theta_n$.

Remark. In the case of a spherically symmetric solution of the Cauchy problem it is easy to obtain an analogon of Corollary 4.1, where G_n should be replaced by $\Gamma(u,t,n;\eta)$ and an analogon of part (e) of Theorem 1, where

$$\Gamma(u,t,n;\eta)=\sum_{k=0}^{\infty}\frac{\Gamma(n/2)}{k!\,\Gamma(n/2+k)}\left(\frac{u}{2}\right)^k\frac{\partial^k}{\partial t^k}\Gamma(0,t;n;\eta),$$

while $\Gamma(u,t,n;\eta)$ is given by (2.2).

We can easily deduce from Corollary 4.1 and Theorem 1, part (a), the following

Corollary 4.2. *Let $F\in L_{k(t)}$ and $\varphi\in L_T$. If F is a solution of problem* (4.11)-(4.12) *and if* (4.13) *is satisfied, then $\varphi(\bar{x},s^2)$ is the unique solution of the inverse Cauchy problem* (1.1) *with $g(\theta,t)=F(\theta,0,t)$.*

COROLLARY 4.3. *Let* $F \in L_{k(t)}$ *satisfy* (4.13). *Suppose the limit*

$$\lim_{t\to 0} F(v, u, t) = \varphi(v, u) \tag{4.14}$$

exists at almost every point (v, u), $\varphi \in L_T$ *and* φ *is continuous at almost every point* (v, u). *If* F *is a solution of problem* (4.11)-(4.12), *then* $\varphi(\bar{x}, s^2)$ *is the unique solution of the inverse Cauchy problem* (1.1) *with* $g(\theta, t) = F(\theta, 0, t)$.

In the next sections we shall use Corollaries 4.2 and 4.3 to the construction of the solution of the inverse Cauchy problem (1.1). Moreover, the corollaries will be used in the proof of Theorem 2.

5. Solution of the inverse Cauchy problem. Let us suppose that there exists a solution of the inverse Cauchy problem (1.1). Then, it follows from part (c) of Theorem 1 that $g(\theta, t)$ is holomorphic.

Now, let us consider the inverse problem. Suppose g is a holomorphic function, series (3.4) is absolutely and almost uniformly convergent and the function φ, continuous at almost every point of D, is defined almost everywhere by (3.3).

In Theorem 2 and Corollary 5.1 we prove that under weak restrictions the function φ is a solution of the inverse Cauchy problem.

THEOREM 2. *Let* $g(v, t)$ *be a holomorphic function and let series* (3.4) *and their derivatives be absolutely and almost uniformly convergent on* $R^2 \times (0, T)$. *Let* F, *given by* (3.4), *be an element of* $L_{k(t)}$. *Finally, let* $\varphi \in L_T$ *and satisfy the condition*

$$\lim_{t\to 0} \|F - \varphi\|_{L_{k(t)}} = 0. \tag{5.1}$$

If

$$\int_0^T \|F\|_{L_{k(t)}} dt < \infty,$$

then $\varphi(\bar{x}, s^2)$ *is the unique solution of the inverse Cauchy problem and condition* (d) *of Theorem* 1 *holds.*

Proof. To prove this theorem it is enough to note that the function F given by (3.4) satisfies equation (4.11). Hence, F is a solution of problem (4.11)-(4.12). The assertion of Theorem 2 follows now from Corollary 4.2.

COROLLARY 5.1. *Let* φ *be defined almost everywhere by* (3.3) *and continuous at almost every point* (v, u). *If the assumptions of Theorem* 2 *regarding* g, F, *and* φ *are satisfied, then* $\varphi(\bar{x}, s^2)$ *is the unique solution of the inverse Cauchy problem.*

Remark. A counter-example showing that assumption (5.1) cannot be omitted will be given in Section 7.

Evidently, Theorem 2 and Corollary 5.1 provide a manner of deriving the unique solution of (1.1).

The right-hand side of (3.1), which is the expected value of the function $\varphi(v, u)$, may be considered as a mapping A from L_T into a set of holomorphic functions $A\varphi = g$.

On the set of functions $g(\cdot, \cdot)$ satisfying the assumptions of Corollary 5.1 we can define for the mapping A the inverse mapping A^{-1} having the property $A^{-1}g = \varphi$. Note, that A^{-1} is of the form

$$A^{-1}g = Bg|_{t=0},$$

where

$$B = \sum_{k=0}^{\infty} \frac{\Gamma(n-1)/2}{k!\Gamma\big((n-1)/2+k\big)} \left(\frac{u}{2}\right)^k \mathscr{T}^k,$$

while $\mathscr{T}$ is given by (3.5).

6. Application to the problem of unbiased estimation. Let $X_1, \ldots, X_n$ $(n > 1)$ be n independent and identically distributed normal variables with mean θ $(-\infty < \theta < \infty)$ and variance t $(0 < t < T)$. Let $g(\theta, t)$ be a known function. The general problem of the theory of unbiased estimation is to find functions $\psi_0(x_1, \ldots, x_n)$, called *estimators*, having the property

$$\text{(6.1)} \qquad \mathrm{E}_{\theta,t} L(\psi_0;\ \theta, t) = \inf \mathrm{E}_{\theta,t} L(\psi;\ \theta, t), \qquad (\theta, t) \in R \times (0, T),$$

where $L(\cdot;\ \theta, t)$, called the *loss function*, is convex for each θ, t. The infimum in (6.1) is taken in the set of estimators $\psi(\cdot)$ satisfying

$$\text{(6.2)} \qquad \int_{R^n} \psi(x_1, \ldots, x_n) \left(\frac{1}{2\pi t}\right)^{n/2} \exp\left[-\frac{1}{2t} \sum (x_i - \theta)^2\right] dx$$

$$= \mathrm{E}_{\theta,t}\psi = g(\theta, t) \quad \text{ for } (\theta, t) \in R \times (0, T).$$

An estimator satisfying (6.2) is called *unbiased*, whereas ψ_0 satisfying (6.1) and (6.2) is called a *uniformly minimum-risk estimator* in the class of unbiased estimators or, for short, *UMR-estimator*.

In view of Rao-Blackwell's theorem, it is sufficient to consider in (6.1) estimators based on sufficient statistics

$$\bar{X} = \frac{1}{n} \sum X_i \quad \text{ and } \quad S^2 = \sum (X_i - \bar{X})^2$$

only. Since $\bar{X}$ and S^2 are complete, there exists at most one unbiased estimator of $g(\theta, t)$ based on $\bar{X}$ and S^2. Hence, in view of (6.2) and (1.1), we have the following

THEOREM 3. *Suppose $X_1, \ldots, X_n$ $(n > 1)$ are n independent and identically distributed normal variables with mean $\theta \in R$ and variance $t \in (0, T)$. Further, let the loss function $L(\cdot;\ \theta, t)$ be convex for every (θ, t). Then $\varphi(\bar{x}, s^2)$*

is a UMR-estimator of $g(\theta, t)$ if and only if $\varphi(\cdot, \cdot)$ is a solution of the inverse Cauchy problem (1.1).

Remark 1. From Theorem 3 it follows that Theorem 2 and Corollary 5.1 may be used to the construction of the UMR-estimators of $g(\theta, t)$.

Remark 2. If a function F is known and the assumptions of Corollaries 4.2 or 4.3 are satisfied, then the function φ is a UMR-estimator of $g(\theta, t)$.

Now, let us consider the problem of construction of UMR-estimators for the case where the variance is known and equal t_0. Let $g(v)$ be a holomorphic function. Remark that series (3.4) is now of the form

$$(6.3)\quad F(v, u, t) = F(v, u) = \sum_{k=0}^{\infty} \frac{\Gamma((n-1)/2)}{k!\,\Gamma((n-1)/2+k)} \left(\frac{u}{2}\right)^k \left(-\frac{1}{2n}\right)^k \frac{\partial^{2k}}{\partial v^{2k}} g(v).$$

Suppose the series is absolutely and almost uniformly convergent. Let us write

$$(6.4)\quad F_k(v, u) = u^k \frac{\partial^{2k}}{\partial v^{2k}} g(v).$$

If, for $k = 1, 2, \ldots$, the functions $F_k(\bar{x}, s^2)$ have finite expected values, then there exist the conditional expectations $\mathrm{E}_{\theta,t_0}(F_k(\bar{x}, s^2) \mid \bar{X})$ for $k = 1, 2, \ldots$

Since $\bar{X}$ and S^2 are independent, we obtain

$$\mathrm{E}_{\theta,t_0}(F_k(\bar{x}, s^2) \mid \bar{X}) = \mathrm{E}_{\theta,t_0}(S^2)^k \left(\frac{\partial^{2k}}{\partial v^{2k}} g(v)\right)_{v=\bar{x}}$$

$$= (2t_0)^k \frac{\Gamma((n-1)/2+k)}{\Gamma((n-1)/2)} \left(\frac{\partial^{2k}}{\partial v^{2k}} g(v)\right)_{v=\bar{x}}.$$

Suppose that F given by (6.3) is an element of $L_{k(t)}$. Then, $F(\bar{x}, s^2, t)$ is integrable term by term and we obtain

$$\mathrm{E}_{\theta,t_0}(F \mid \bar{X}) = \sum_{k=0}^{\infty} \frac{\Gamma((n-1)/2)}{k!\,\Gamma((n-1)/2+k)} \left(-\frac{1}{4n}\right)^k \mathrm{E}_{\theta,t_0}(F_k \mid \bar{X})$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{t_0}{2n}\right)^k \left(\frac{\partial^{2k}}{\partial v^{2k}} g(v)\right)_{v=\bar{x}}.$$

Let us write

$$(6.5)\quad h(v) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{t_0}{2n}\right)^k \frac{\partial^{2k}}{\partial v^{2k}} g(v)$$

and

$$\varphi(v, u) = F(v, u) = F(v, u, t), \tag{6.6}$$

where F, defined by (6.3), does not depend on t.

It is easy to see that if

$$\int_0^T \|\varphi\|_{L_{k(t)}} dt < \infty, \tag{6.7}$$

then the assumptions of Theorem 2 on F and φ are satisfied. Thus, it follows from Theorem 3 that in the case where (θ, t) is unknown $\varphi(\bar{x}, s^2)$ is a UMR-estimator of $g(\theta)$. Since

$$h(\bar{x}) = \mathrm{E}_{\theta,t_0}(\varphi(\bar{x}, s^2) \,|\, \bar{X}), \tag{6.8}$$

$h(\bar{x})$ is an unbiased estimator of $g(\theta)$ provided that the variance t_0 is known. In view of Rao-Blackwell's Theorem and the completeness of $\bar{X}$, $h(\bar{x})$ is the unique UMR-estimator of $g(\theta)$. Thus, we proved the following

THEOREM 4. *Let $X_1, \ldots, X_n$ $(n > 1)$ be n independent and identically distributed normal variables with unknown mean $\theta \in R$ and known variance t_0. Let $L(\cdot; \theta)$, the loss function, be convex for every $\theta \in R$ and let $g(\cdot)$ be holomorphic. Finally, let $F \in L_{k(t)}$ and $F_k \in L_{k(t)}$ be given by* (6.3) *and* (6.4), *respectively* $(T > t_0)$. *If φ, given by* (6.6), *satisfies* (6.7), *then $h(\bar{x})$ is the unique UMR-estimator of $g(\theta)$.*

7. Examples.

(a) As we already mentioned, assumption (5.1) in Corollary 5.1 cannot be omitted. In fact, let

$$g(v, t) = \sqrt{\frac{n}{2\pi t}} \exp\left[-\frac{n}{2t} v^2\right].$$

Then, in view of (3.4), we have $F(v, u, t) = g(v, t)$.

Since $\varphi(v, u) = 0$ at each point (v, u) with $v \neq 0$, it follows that

$$\|F - \varphi\|^2_{L_{k(t)}} = \|g\|^2_{L_{k(t)}} = \mathrm{const} \frac{(T-t)^{n/2}}{\sqrt{t(2T-t)}}$$

and

$$\lim_{t \to 0} \frac{(T-t)^{n/2}}{\sqrt{t(2T-t)}} = \infty.$$

But $\varphi(\bar{x}, s^2) = 0$ is not a solution of the inverse Cauchy problem.

(b) Now, we shall construct the UMR-estimator for $\theta^s t^r$. Let $g(v, t) = v^s t^r$. In view of (3.4), we have

$$F(v, u, t)$$
$$= \Gamma\left(\frac{n-1}{2}\right) \sum_{k=0}^{r+[s/2]} \frac{1}{k!\Gamma\big((n-1)/2+k\big)} \left(\frac{u}{2}\right)^k t^r v^s \sum_{j=0}^{k} \left(-\frac{1}{2n}\right)^j \times$$
$$\times (k-j)!(2j)! \binom{k}{j} \binom{r}{k-j} \binom{s}{2j} t^{j-k} v^{-2j}.$$

Moreover, let

$$(7.1)\quad \varphi(v, u) = F(v, u, 0)$$
$$= \Gamma\left(\frac{n-1}{2}\right) \frac{r!}{2^r} u^r v^s \sum_{j=0}^{[s/2]} \frac{(-1)^j}{(r+j)!\Gamma\big((n-1)/2+r+j\big)} \left(\frac{1}{2n}\right)^j (2j)! \times$$
$$\times \binom{r+j}{j} \binom{s}{2j} \left(\frac{u}{2v^2}\right)^j.$$

Because the assumptions of Theorem 2 regarding g, F and φ are satisfied, it follows from Theorem 3 that the function $\varphi(\bar{x}, s^2)$ given by (7.1) is the unique UMR-estimator of $\theta^s t^r$.

It is possible to show in a similar manner that

$$\sin\bar{x} \sum_{k=0}^{\infty} \frac{\Gamma\big((n-1)/2\big)}{k!\Gamma\big((n-1)/2+k\big)} \left(\frac{s^2}{4n}\right)^k$$

and

$$\cos\bar{x} \sum_{k=0}^{\infty} \frac{\Gamma\big((n-1)/2\big)}{k!\Gamma\big((n-1)/2+k\big)} \left(\frac{s^2}{4n}\right)^k$$

are the UMR-estimators of $\sin\theta$ and $\cos\theta$, respectively.

(c) Let $F(v, u, t)$ satisfy (4.9). Further let

$$g(v, t) = F(v, 0, t) \quad \text{and} \quad \varphi(v, u) = \lim_{t\to 0} F(v, u, t).$$

As pointed out in Remark 2 of Section 6, if the assumptions of Corollary 4.3 are satisfied, then $\varphi(\bar{x}, s^2)$ is a UMR-estimator of $g(\theta, t)$. It is easy to see that by these means UMR-estimators can be derived from the functions

$$F_1(v, u, t) = \sin v \exp\left[-\frac{t}{2n}\right],$$

$$F_2(v, u, t) = \cos v \exp\left[-\frac{t}{2n}\right],$$

$$F_3(v, u, t) = \exp\left[av + \frac{a^2 t}{2n}\right],$$

$$F_4(v, u, t) = \left(1 - \frac{2a}{n-1}t\right)^{-(n-1)/2} \exp\left[\frac{au}{1-2at/(n-1)}\right],$$

where a can be complex.

(d) In case where the variance t_0 is known, we can apply Theorem 4. Take $g(\theta) = \theta^s$ as an example. Since the assumptions of Theorem 4 are satisfied, $h(\bar{x})$ given by (6.5) is the UMR-estimator of θ^s. Evidently, in the considered case, we have

$$h(v) = v^s \sum_{j=0}^{[s/2]} \frac{(2j)!}{j!} \left(-\frac{t_0}{2nv^2}\right)^j.$$

(e) Because the given formulas for UMR-estimators involve some series, the methods proposed in this paper may be inconvenient in practice for some functions $g(\theta, t)$, whereas other methods may be simpler. For example, using Kolmogorov-Linnik's method [3] (see also [10]) it is easy to show that

$$h_A(\bar{x}) = \int_A \left(\frac{n-p}{n}\right)^{(p-1)/2} \exp\left[\frac{p}{n-p} \frac{1}{2t_0} \sum_{i=1}^{p} (y_i - \bar{y})^2\right] \times$$

$$\times \left(\frac{1}{2\pi t_0}\right)^{p/2} \exp\left[-\frac{1}{2t_0} \sum_{i=1}^{p} (y_i - \bar{x})^2\right] dy$$

is the UMR-estimator of

$$g_A(\theta) = \int_A \left(\frac{1}{2\pi t_0}\right)^{p/2} \exp\left[-\frac{1}{2t_0} \sum_{i=1}^{p} (y_i - \theta)^2\right] dy,$$

where $A \subset R^p$ is a measurable set and $p < n$.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WROCŁAW

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A. KOZEK (Wrocław)

O ODWROTNYM PROBLEMIE CAUCHY'EGO I NIEOBCIĄŻONEJ ESTYMACJI

STRESZCZENIE

W pracy rozważane jest równanie całkowe (1.1), gdzie funkcja g jest znana. Funkcja $\varphi(\bar{x}, s^2)$, będąca rozwiązaniem tego równania, może być interpretowana albo jako początkowy rozkład temperatury w przestrzeni R^n, powodujący, że na prostej $\theta_1 = \ldots = \theta_n$ w chwili t będzie temperatura $g(\theta, t)$, albo jako nieobciążony estymator funkcji g, zależny od statystyk dostatecznych.

W twierdzeniu 2 podano sposób znajdowania rozwiązania równania (1.1). W zastosowaniu do teorii estymacji otrzymano metody znajdowania nieobciążonych estymatorów, zależnych od statystyk dostatecznych w przypadku rozkładu normalnego.
