ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIV, 1 (1974)

Z. A. ŁOMNICKI* (Codsall, Staffs.)

SOME REMARKS ON SYSTEM RELIABILITY**

When investigating the reliability of a system it is necessary to realize that we are often compelled to take into account not only the *failure to operate* but also the fact that the device may operate inadvertently when it should not do so and we shall call it *failure to idle*. The ordnance people use the expression "duds" (ammunition which does not explode when fired) and "prematures" (ammunition which explodes unexpectedly). Often the expressions "open" and "closed" ("short") failures are used borrowed from electrical engineers but this does not seem to be suitable since it may suggest that such failures occur only in electrical problems whilst anybody who applies the reliability concepts in practice knows perfectly well that the existence of these two kinds of failure is very important. A pilot responsible for the aeroplane should be sure that the devices at his disposal when operated by him will put the aeroplane's control surfaces into the proper position causing the plane to move in the proper direction but he must also know that, without his interference, these devices will not cause the aeroplane to change direction unnecessarily. Similarly, properly designed monitoring devices should give a proper warning when something goes wrong in the system but they should at the same time be constructed in such a way that they should not give an unnecessary ("nuisance") warning when everything is in order. (The danger is that, when the nuisance warning occurs too frequently, the genuine alarm may be disregarded as in the case of the shepherd boy who cried "wolf" too often.) No wonder that in ordinary everyday work both kinds of failure are constantly considered by the engineers. Unfortunately, very often the discrimination between them is not stressed strongly enough or even ignored by the textbooks and there are few papers on reliability discussing these two kinds of failure in detail.

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* Fellow of Royal Statistical Society.

** Paper presented at the symposium on queueing theory (Wrocław and Karpacz, January 11-18, 1973), organized by the Institute of Mathematics of the Polish Academy of Sciences and the Mathematical Institute of the University of Wrocław.

Barlow, Hunter and Proschan [1] discussed the case where the failure of each component was distributed according to $F(t)$ with the conditional probability of the failure to operate given by p and of the failure to idle by $q = 1-p$. It was assumed that the function $F(t)$ is a negative exponential $F(t) = 1-e^{-\lambda t}$ the same for all components and that the failures of components are mutually independent. The problem solved by them was to determine according to the value of p what should be the number of parallel components which would maximize the expected system life. (See also [2].)

Let us investigate under the same assumptions a different problem when we have at our disposal m components but, apart from combining them in parallel, we are also allowed to use them for building any other two-terminal series-parallel system. It is obvious that if $p = 1$, then we have only failures to operate and in such a case the highest expected life of the system will be obtained when all these components are put into parallel. Vice versa, if $p = 0$, we have only failures to idle and the best arrangement would be to put these m components in series. The problem arises what would be the best solution if p has a different value somewhere between 0 and 1.

The series-parallel structures built of m identical components were discussed by Mac Mahon [8]; some generalizations of his results can be found in Riordan and Shannon [9]. Knödel [5] and Carlitz and Riordan [3] discussed the number of different structures when all the components

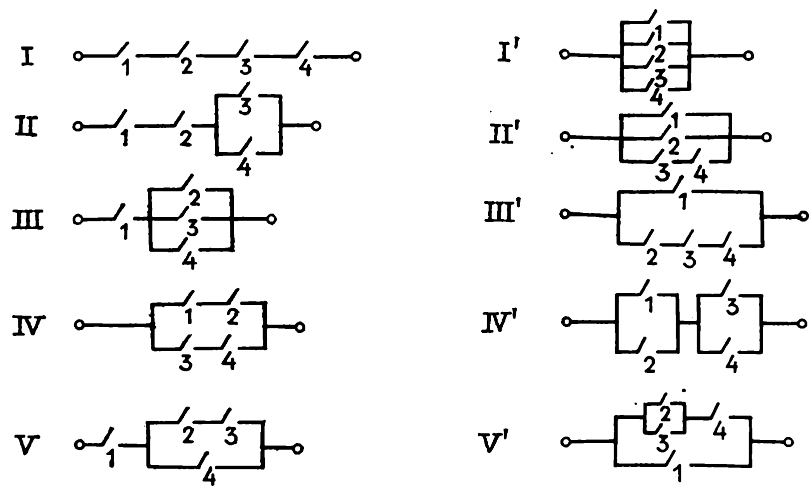

Fig. 1. Two-terminal series-parallel structures built of 4 components

are different and some further generalization can be found in Łomnicki [6]. Thus, for example, if $m = 4$ we can build from four identical components ten different structures shown in Fig. 1.

If a is the probability of a component failing to operate and b the probability of a component failing to idle, then it is easy to evaluate for every structure the probabilities $u(a)$ and $v(b)$ of these two kinds of failure. This may be done by writing in the Boolean form the *hindrance* (or the *admission*) *function* displaying all the possible combinations of component failures which cause the system to fail to operate (or to idle) and then by applying the so-called Inclusion-Exclusion Theorem (see, e. g., [4]).

For instance, for structure IV the failure F to operate is given in the Boolean form as the hindrance function

$$F = (A_1+A_2)(A_3+A_4) = A_1A_3+A_1A_4+A_2A_3+A_2A_4,$$

where A_i is the failure to operate of the i-th component. The probability of this failure being for all components equal to a, we have from the Inclusion-Exclusion Theorem

$$u(a) = 4a^2-(4a^3+2a^4)+4a^4-a^4 = 4a^2-4a^3+a^4.$$

Similarly, the admission function showing the failure to idle is given in the Boolean form by the expression

$$G = B_1B_2+B_3B_4,$$

where B_i is the failure to idle of the i-th component, and b being the probability of failure to idle of any component

$$v(b) = 2b^2-b^4.$$

It is easily seen that, for every structure,

$$1-u(a) = v(1-a). \tag{1}$$

The probabilities of these two kinds of failure for structures shown in Fig. 1 are given in Table 1.

TABLE 1

Structure	$u(a)$	$v(b)$
I	$4a-6a^2+4a^3-a^4$	b^4
II	$2a-2a^3+a^4$	$2b^3-b^4$
III	$a+a^3-a^4$	$3b^2-3b^3+b^4$
IV	$4a^2-4a^3+a^4$	$2b^2-b^4$
V	$a+2a^2-3a^3+a^4$	$b^2+b^3-b^4$
V′	$a^2+a^3-a^4$	$b+2b^2-3b^3+b^4$
IV′	$2a^2-a^4$	$4b^2-4b^3+b^4$
III′	$3a^2-3a^3+a^4$	$b+b^3-b^4$
II′	$2a^3-a^4$	$2b-2b^3+b^4$
I′	a^4	$4b-6b^2+4b^3-b^4$

Let us assume, following Barlow and Proschan [2], that $a = p - pe^{-\lambda t}$ and $b = q - qe^{-\lambda t}$. Then, in view of (1),

$$u(a) = 1 - v(1-a) = 1 - v(q + pe^{-\lambda t}) \quad \text{and} \quad v(b) = v(q - qe^{-\lambda t}),$$

and the corresponding expected times to failure are the following:

$$\int_0^\infty [v(q+pe^{-\lambda t}) - v(q)]\,dt \quad \text{and} \quad \int_0^\infty [v(q) - v(q - qe^{-\lambda t})]\,dt.$$

If we have $2N$ such systems with N of them operating and, consequently, subjected to failures to operate and N of them not operating and, consequently, subjected to failures to idle, then the expected time to failure of all these systems will be

$$N \int_0^\infty [v(q+pe^{-\lambda t}) - v(q - qe^{-\lambda t})]\,dt. \tag{2}$$

If each of these systems is composed of one component only, then (2) would give, as the expected time to failure,

$$N \int_0^\infty [(q+pe^{-\lambda t}) - (q - qe^{-\lambda t})]_1^{dt} = N \int_0^\infty e^{-\lambda t}\,dt = N/\lambda.$$

Hence, the ratio R_S of the expected time to failure of a system S to the expected time to failure of the system built of one component is given by

$$R_S = \lambda \int_0^\infty [v(q+pe^{-\lambda t}) - v(q - qe^{-\lambda t})]\,dt,$$

where $v(b)$ is the probability of failure to idle corresponding to the structure S.

If the structure is such that $v(b) = b^m$, then the corresponding ratio R is equal to

$$S_m = \lambda \int_0^\infty [(q+pe^{-\lambda t})^m - (q - qe^{-\lambda t})^m]\,dt$$

and

$$S_m = \sum_{k=1}^{m} \binom{m}{k} q^{m-k} [p^k + (-1)^{k+1} q^k] \frac{1}{k}.$$

Thus $S_1 = 1$ and

$$\begin{aligned} S_m - qS_{m-1} &= \sum_{k=1}^{m-1} \left[\binom{m}{k} - \binom{m-1}{k}\right] q^{m-k} [p^k + (-1)^{k+1} q^k] \frac{1}{k} + [p^m + (-1)^{m+1} q^m] \frac{1}{m} \\ &= \frac{1}{m} \sum_{k=1}^{m} \binom{m}{k} q^{m-k} [p^k + (-1)^{k+1} q^k] = \frac{1}{m}(p+q)^m + \frac{1}{m}(q-q)^m = \frac{1}{m}. \end{aligned}$$

Hence, by mathematical induction,

$$S_m = \sum_{k=0}^{m-1} \frac{q^k}{m-k} = \frac{1}{m} + \frac{q}{m-1} + \frac{q^2}{m-2} + \ldots + q^{m-1}.$$

The above-mentioned result simplifies the necessary calculations. Thus, for instance, in the case of structure IV of Fig. 1 we have $v(b) = 2b^2 - b^4$ so that the ratio R_{IV} is equal for this structure to

$$R_{IV} = 2\left(\frac{1}{2}+q\right) - \left(\frac{1}{4}+\frac{q}{3}+\frac{q^2}{2}+q^3\right) = \frac{3}{4}+\frac{5q}{3}-\frac{q^2}{2}-q^3.$$

For every structure S built of m components this improvement ratio can be calculated as a function of q or $p = 1-q$. Denoting this ratio by $R_S(p)$, we define the function

$$R_m(p) = \max_S R_S(p),$$

where the maximum is taken over all series-parallel systems built of m components. This function measures the advantage obtained by using m components in the case where p has a given value. The range of p-values for which $R_m(p)$ coincides with $R_S(p)$ shows for which values of p this particular system S should be chosen in preference to others as yielding the longest life of the system.

TABLE 2

Structure	$R_S(q)$	Structure	$R_S(q)$
I	$\frac{1}{4}+\frac{q}{3}+\frac{q^2}{2}+q^3$	I′	$\frac{25}{12}-\frac{13q}{3}+\frac{7q^2}{2}-q^3$
II	$\frac{5}{12}+\frac{2q}{3}+\frac{3q^2}{2}-q^3$	II′	$\frac{19}{12}-\frac{2q}{3}-\frac{3q^2}{2}+q^3$
III	$\frac{3}{4}+\frac{11q}{6}-\frac{5q^2}{2}+q^3$	III′	$\frac{13}{12}+\frac{q}{6}+\frac{q^2}{2}-q^3$
IV	$\frac{3}{4}+\frac{5q}{3}-\frac{q^2}{2}-q^3$	IV′	$\frac{11}{12}+\frac{7q}{3}-\frac{7q^2}{2}+q^3$
V	$\frac{7}{12}+\frac{7q}{6}+\frac{q^2}{2}-q^3$	V′	$\frac{5}{4}+\frac{5q}{6}-\frac{5q^2}{2}+q^3$

The ratios R_S for the series-parallel structures built of four components (shown in Fig. 1) are given in Table 2.

The ratios $R_S(p)$ and the function $R_m(p)$ for $m = 4$ are shown in Fig. 2. Of course, in view of duality between structures I and I′, II and II′ etc., it is sufficient to study only a half of them. (Indeed, it is easy to

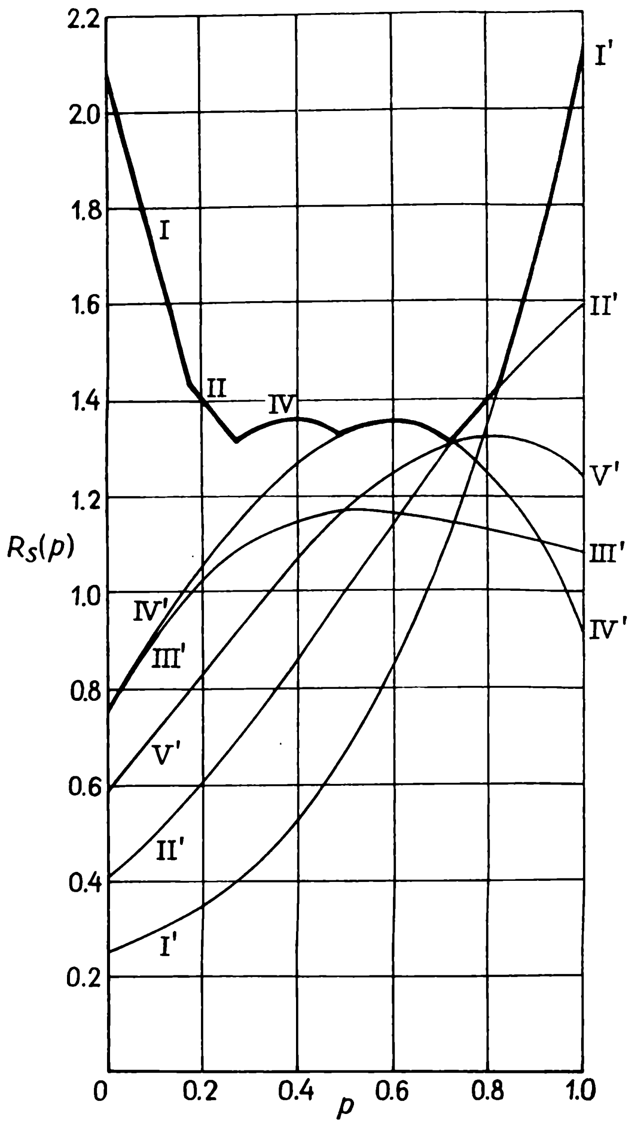

Fig. 2. Improvement ratios as functions of p for $m = 4$

verify that the ratio for structure I′ is given by the same function as for structure I if we write $1-p$ for q.) From Fig. 2 it is seen that it is sufficient to consider only 6 structures, that is I, II, IV, IV′, II′, I′, out of possible 10 structures as leading to the improved life time. Similarly, for $m = 5$ from 24 possible series-parallel structures only 10 deserve to be considered. For $m = 6$ we have 66 possibilities (see, e. g., [6]) from which only 12 produce an improvement. To find what is the situation for larger values of m would require the aid of a computer.

It would be interesting to investigate what changes will occur if instead of an exponential failure distribution some other distributions like Weibull or Gamma are used, what happens if instead of m indentical components we have m_i components of type i, where $\sum m_i = m$, as discussed in [6] and [7], and what results will be obtained if both kinds of failure are subjected to different distributions. It should be also added that the engineers are slightly unhappy about a very great number of possible series-parallel structures and they would welcome some method allowing them to identify this special, more useful class of series-parallel structures and to concentrate on these selected structures when looking for a solution.

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THE STONE HOUSE, OAKEN LANES,
CODSALL, STAFFS., ENGLAND

Received on 8. 2. 1973

Z. A. ŁOMNICKI (Codsall, Staffs.)

UWAGI O NIEZAWODNOŚCI SYSTEMÓW

STRESZCZENIE

W pracy analizowana jest klasa systemów o szeregowo-równoległej strukturze elementów. Zakłada się niezależność elementów oraz wykładniczy rozkład czasu ich pracy. Awarie elementów mogą być przy tym dwojakie: z prawdopodobieństwem warunkowym p — mogą to być awarie polegające na błędnym działaniu elementu

(failure to operate) — i z prawdopodobieństwem warunkowym $1-p$ — mogą to być awarie polegające na niespodziewanym działaniu elementu, gdy powinien on spoczywać *(failure to idle)*. Systemy charakteryzuje się przy użyciu funkcji niezawodności oraz oczekiwanego czasu pracy. Następnie znajduje się — w zależności od parametru p — struktury optymalne, maksymalizujące oczekiwany czas pracy systemu. Systemy czteroelementowe stanowią ilustrację przedstawionych pojęć i metod.