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# ON THE BINDING NUMBER OF SOME HALLIAN GRAPHS

**1. Definitions and notation.** We consider only finite undirected graphs without loops or multiple edges. Our terminology and notation will be standard except for indicated. For a graph $G = (V, E)$ and a set $X \subseteq V$ we denote by $\Gamma_G(X)$ (or, briefly, $\Gamma(X)$) the set of vertices joined to vertices in $X$. A set of independent edges that cover all vertices of a graph is called a 1-*factor* of that graph. By a (1, 2)-*factor* of a graph $G$ we mean a set of independent edges or vertex disjoint cycles which cover all vertices of $G$. Obviously, we can restrict the cycles in the above definition to odd ones.

A graph $G$ is *hallian* if $|\Gamma(X)| \geqslant |X|$ for any set $X \subseteq V(G)$ or, equivalently, if $G$ has a (1, 2)-factor. Obviously, $G$ is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph $G$ is *k-hallian* if for any set $A$ of vertices of order at most $k$ the subgraph of $G$ induced by the set $V(G)\setminus A$ is hallian. The largest $k$ such that $G$ is $k$-hallian is called the *hallian index* of $G$ and is denoted by $h(G)$. The *vertex connectivity* $\varkappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. Clearly,

$$h(G) \leqslant \delta(G) - 1 \quad \text{and} \quad \varkappa(G) \leqslant \delta(G),$$

where $\delta(G)$ denotes the minimum degree among the vertices of $G$.

For concepts not defined here, see [1].

**2. Some properties of hallian graphs.** The main purpose of this section is to show relations between orders of $\Gamma(X)$ and $X$, $X \subseteq V(G)$, for $k$-hallian and $l$-connected graphs. Let $G$ be a given graph and let

$$\mathscr{F}_G = \{X \subseteq V(G): X \neq \emptyset \text{ and } \Gamma(X) \neq V(G)\}.$$

LEMMA 1. *Let $G$ be a graph on $n$ vertices. If $G$ is $l$-connected, then*

$$|\Gamma(X)\setminus X| \geqslant l$$

*for any set $X \in \mathscr{F}_G$.*

Proof. For $l=0$ the lemma is obvious. Let $l \geqslant 1$ and $X \in \mathscr{F}_G$. There exists a vertex $v \notin \Gamma(X)$. If $v \in X \setminus \Gamma(X)$, then $\Gamma(v)$ is disjoint from $X$, and so

$$|\Gamma(X) \setminus X| \geqslant |\Gamma(v)| \geqslant \delta(G) \geqslant l.$$

If $v \in V \setminus (X \cup \Gamma(X))$, then $\Gamma(X) \setminus X$ is a cutset of $G$, so $|\Gamma(X) \setminus X| \geqslant l$.

THEOREM 1. *Let $G$ be an $l$-connected and $k$-hallian graph on $n$ vertices. Then*

$$|\Gamma(X)| \geqslant |X| + r$$

*for any set $X \in \mathscr{F}_G$, where $r = \min\{l, k\}$.*

Proof. Let $X \in \mathscr{F}_G$. By Lemma 1 we have

$$|\Gamma(X) \setminus X| \geqslant r,$$

so choose $A \subseteq \Gamma(X) \setminus X$ with $|A| = r$. The graph $G' = \langle V(G) \setminus A \rangle$ is $(k-r)$-hallian, hence hallian, whence $|\Gamma_{G'}(X)| \geqslant |X|$. Thus

$$|\Gamma_G(X)| = |\Gamma_{G'}(X)| + |A| \geqslant |X| + r,$$

as required.

COROLLARY 1. *Let $G$ be a hallian graph on $n$ vertices and let*

$$r = \min\{h(G), \varkappa(G)\}.$$

*Then there exists a set $X_0 \in \mathscr{F}_G$ such that*

$$|\Gamma(X_0)| = |X_0| + r.$$

Proof. We consider two cases.

Case 1. $r = h(G)$. Then there exists a set $A \subseteq V(G)$, $|A| = r+1$, such that the graph $G' = \langle V(G) \setminus A \rangle$ is not hallian. Thus there exists $X \subseteq V(G) \setminus A$ such that $|\Gamma_{G'}(X)| < |X|$, and

$$|\Gamma_G(X)| \leqslant |\Gamma_{G'}(X)| + |A| < |X| + r + 1.$$

So $|\Gamma(X)| \leqslant |X| + r$. Since $\Gamma(X) \neq V(G)$, the converse inequality follows from Theorem 1, and the result is proved in this case.

Case 2. $r = \varkappa(G)$. Let $Y \subseteq V(G)$ be a cutset of $r$ vertices and let $X$ be the set of vertices on one component of $G \setminus Y$. Then, evidently, $|\Gamma(X)| \leqslant |X| + r$. Since the converse inequality follows again from Theorem 1, the result is also proved in this case.

THEOREM 2. *Let $G$ be a graph on $n$ vertices. If any set $X \in \mathscr{F}_G$ satisfies $|\Gamma(X)| \geqslant |X| + k$, then $G$ is $k$-hallian and $k$-connected.*

Proof. For $k = 0$ the theorem is clear. Let $k \geqslant 1$. Suppose $G$ is not $k$-connected or $G$ is not $k$-hallian. Then $\varkappa(G) < k$ or $h(G) < k$. Let

$$r = \min\{\varkappa(G), h(G)\}.$$

By Corollary 1 there exists a set $X_0 \in \mathscr{F}_G$ such that $|\Gamma(X_0)| = |X_0| + r$, where $r < k$, a contradiction.

COROLLARY 2. *If $k'$ is the largest integer for which the assertion of Theorem 2 holds, then $h(G) = k'$ or $\varkappa(G) = k'$.*

**3. On the binding number of hallian graphs.** The *binding number* of $G$, denoted by $\operatorname{bind}(G)$, is defined by

$$\operatorname{bind}(G) = \min_{X \in \mathscr{F}_G} \frac{|\Gamma(X)|}{|X|}.$$

The binding number was intensively investigated by Woodall; see [4] and its references. If $\operatorname{bind}(G)$ is large, then the vertices of $G$ are well bound together, i.e., $G$ has a lot of edges fairly well distributed. If the binding number is large, then the minimum degree, the connectivity, the chromatic number are also large. Clearly, $\operatorname{bind}(G) = 0$ if and only if $G$ has an isolated vertex.

Woodall has proved in [4] the following

PROPOSITION 1. *If $G$ is a graph on $n$ vertices, then*

$$\operatorname{bind}(G) \leqslant \frac{n-1}{n-\delta(G)}.$$

PROPOSITION 2. $\operatorname{bind}(K_n) = n-1$ *for* $n \geqslant 1$.

PROPOSITION 3. *If $n \geqslant 3$, then*

$$\operatorname{bind}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ (n-1)/(n-2) & \text{if } n \text{ is odd.} \end{cases}$$

PROPOSITION 4. *If $n \geqslant 1$, then*

$$\operatorname{bind}(P_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ (n-1)/(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

We note that $G$ is hallian if and only if $\operatorname{bind}(G) \geqslant 1$. Using Theorem 1 and the next lemma we shall obtain a lower bound for $\operatorname{bind}(G)$, where $G$ is a hallian graph.

LEMMA 2. *Let $G$ be a graph on $n$ vertices. If $X \in \mathscr{F}_G$, then*

$$|X| \leqslant n - \delta(G).$$

Proof. Let $v \in V(G) \setminus \Gamma(X)$. Then $X \cap \Gamma(v) = \emptyset$, and $|\Gamma(v)| \geqslant \delta(G)$. The result is immediate.

THEOREM 3. *If a graph $G$ on $n$ vertices is $k$-hallian and $l$-connected, and $r = \min\{k, l\}$, then*

$$\operatorname{bind}(G) \geqslant \frac{n-\delta(G)+r}{n-\delta(G)}.$$

Proof. If $X \in \mathcal{F}_G$, then $|\Gamma(X)| \geqslant |X|+r$ by Theorem 1. Thus, by the above and Lemma 2,

$$\frac{|\Gamma(X)|}{|X|} \geqslant \frac{|X|+r}{|X|} \geqslant \frac{n-\delta(G)+r}{n-\delta(G)}.$$

The result follows.

LEMMA 3. *If a graph $G$ on $n$ vertices has $h(G) = \delta(G)-1$ and $\varkappa(G) \geqslant h(G)$, then*

$$\operatorname{bind}(G) = \frac{n-1}{n-\delta(G)}.$$

Proof. As a consequence of Theorem 3 we have

$$\operatorname{bind}(G) \geqslant \frac{n-1}{n-\delta(G)}$$

and by Proposition 1 we obtain the required equality.

**4. The binding number of the Cartesian product of some graphs. An application of the preceding results.** Let $G_1$ and $G_2$ be graphs with vertex sets

$$V(G_1) = \{x_1, \ldots, x_n\} \quad \text{and} \quad V(G_2) = \{y_1, \ldots, y_m\}.$$

The Cartesian product of graphs $G_1$ and $G_2$ is a graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and

$$\{(x_i, y_j), (x_k, y_l)\} \in E(G_1 \times G_2)$$

if and only if $i = k$ and $\{y_j, y_l\} \in E(G_2)$ or $j = l$ and $\{x_i, x_k\} \in E(G_1)$.

Kane et al. [2] have given the following

CONJECTURE. *Let $n \geqslant 3$ and $m \geqslant 3$. Then*

$$\operatorname{bind}(C_n \times C_m) = \frac{mn-1}{mn-4}$$

*if $mn$ is odd.*

Using Theorem 1 and Lemma 3 we will prove this Conjecture. Moreover, in a similar way we calculate $\operatorname{bind}(P_n \times C_m)$, $\operatorname{bind}(C_n \times K_m)$ and $\operatorname{bind}(P_n \times K_m)$.

The connectivity of the Cartesian product of two connected graphs is described by Sabidussi [3].

PROPOSITION 5 ([3]). *If $G_1$ is $l_1$-connected and $G_2$ is $l_2$-connected, then $G_1 \times G_2$ is $(l_1+l_2)$-connected.*

COROLLARY 3. *The graph $(C_n \times C_m)$ is 4-connected for $n \geqslant 3$, $m \geqslant 3$.*

COROLLARY 4. *The graph $(P_n \times C_m)$ is 3-connected for $n \geqslant 2$, $m \geqslant 3$.*

COROLLARY 5. *The graph $(C_n \times K_m)$ is $(m+1)$-connected for $n \geqslant 3$, $m \geqslant 3$.*

COROLLARY 6. *The graph* $(P_n \times K_m)$ *is* $m$*-connected for* $n \geqslant 2$, $m \geqslant 3$.

Moreover, if $G$ is $l$-connected and $\delta(G) = l$, then $\varkappa(G) = l$. Hence

$$\varkappa(C_n \times C_m) = 4, \quad \varkappa(P_n \times C_m) = 3, \quad \varkappa(C_n \times K_m) = m+1, \quad \varkappa(P_n \times K_m) = m.$$

LEMMA 4. *If* $H$ *is a spanning subgraph of* $G$ *and* $H$ *is* $k$*-hallian, then* $G$ *is* $k$*-hallian. Moreover,* $h(H) \leqslant h(G)$.

COROLLARY 7. *Let* $G_1$ *and* $G_2$ *be hallian graphs. Then*

$$h(G_1 \times G_2) \geqslant \max\{h(G_1), h(G_2)\}.$$

Moreover, if either $G_1$ or $G_2$ (but not both) is hallian, then $h(G_1 \times G_2)$ is greater than or equal to the hallian index of this graph.

LEMMA 5. *If* $n \geqslant 2$, $m \geqslant 3$, *then*

$$h(P_n \times C_m) = \begin{cases} 2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let $G = P_n \times C_m$. Since

$$h(C_m) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

we have

$$h(G) \geqslant \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

If $m$ is even, then removing any vertex $x$ will destroy the hallian property since $G-x$ has an odd number of vertices but no odd cycles. Thus $h(G) = 0$. Assume that $m$ is odd. Let $A \subseteq V(G)$ and $|A| = 2$. If $A$ consists of vertices of two copies of $C_m$, then the vertices of $G-A$ are covered by two even paths $C_m - x$ and odd cycles, the remaining copies of $C_m$. If $A$ consists of vertices of the same copy $C_m$, then it is not difficult to see that the vertices of $C_m - A$ induce exactly one odd path and at most one even path. One of the end vertices of the odd path and the corresponding vertex of an adjacent copy $C_m$ can be covered by an edge (see Fig. 1). In this way vertices of $C_m - A$ and vertices of an adjacent copy $C_m$ can be covered by even paths (Fig. 1), so $G - A$ is again hallian. Thus $h(G) \geqslant 2$. On the other hand, $\delta(G) = 3$, so $h(G) = 2$.

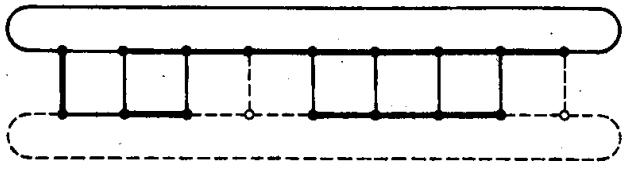

Fig. 1

LEMMA 6. *Let* $n \geqslant 3$, $m \geqslant 3$. *Then*

$$h(C_n \times C_m) = \begin{cases} 3 & \text{if } m \text{ or } n \text{ is odd,} \\ 0 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. Let $G = C_n \times C_m$. By Lemmas 4 and 5 we obtain

$$h(G) \geqslant \begin{cases} 2 & \text{if } m \text{ or } n \text{ is odd,} \\ 0 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

If $m$ and $n$ are both even, then the result follows by the argument of Lemma 5. So suppose that one of them, say $m$, is odd. Let $A \subseteq V(G)$ and $|A| = 3$. If $A$ consists of vertices of three copies of $C_m$, then it is clear that $G-A$ has a cover by even paths $C_m - A$ and odd cycles $C_m$. If $A$ consists of two vertices of the same copy $C_m$ and one from another copy, then the vertices of $G-A$ can be covered by even paths and odd cycles; see the proof of Lemma 5 (noting now that each copy of $C_m$ has two others adjacent to it). If $A$ consists of vertices of the same copy $C_m$, then the vertices of $C_m - A$ induce only even paths or exactly two odd paths and at most one even path. In the first possibility it is clear that the vertices of $G-A$ are covered by even paths and odd cycles. In the second possibility the vertices of $C_m - A$ and all vertices of the two copies adjacent to it can be covered by even paths (see Fig. 2) so that the graph $G-A$ is still hallian. Thus $h(G) \geqslant 3$. But $\delta(G) = 4$, and so $h(G) = 3$.

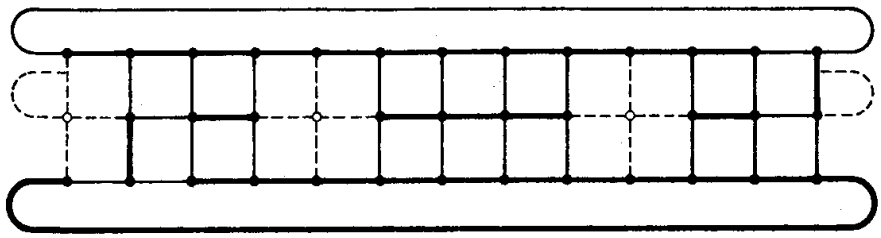

Fig. 2

THEOREM 4. *If* $n \geqslant 3$, $m \geqslant 3$, *then*

$$\operatorname{bind}(C_n \times C_m) = \begin{cases} \dfrac{mn-1}{mn-4} & \text{if } m \text{ or } n \text{ is odd,} \\ 1 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. Let $G = C_n \times C_m$. If $m$ or $n$ is odd, then by Corollary 3 and Lemma 6 the graph $G$ satisfies the hypothesis of Lemma 3 and the result follows. If $m$ and $n$ are even, then by Lemma 6 and Corollary 1 we have $h(G) = 0$ and $\varkappa(G) = 4$. According to Corollary 1 there exists a set $X_0 \in \mathcal{F}_G$ such that $|\Gamma(X_0)| = |X_0|$, so $\operatorname{bind}(G) = 1$.

Using Lemmas 3 and 5 and Corollaries 1 and 4 we prove in the same way as Theorem 4 the following

THEOREM 5. *If* $n \geqslant 2$, $m \geqslant 3$, *then*

$$\operatorname{bind}(P_n \times C_m) = \begin{cases} \dfrac{mn-1}{mn-3} & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Now we will calculate $\operatorname{bind}(P_n \times K_m)$ and $\operatorname{bind}(C_n \times K_m)$. First we notice that the following lemma is true:

LEMMA 7. *If* $n \geqslant 2$, $m \geqslant 3$, *then* $h(P_n \times K_m) = m-1$.

Proof. Let $G = P_n \times K_m$. By Lemma 4 we have $h(P_n \times K_m) \geqslant m-1$. Let $A \subseteq V(G)$, $|A| = m-1$. Note that removing at most $m-2$ vertices from each copy of $K_m$ leaves a hallian graph, since the remaining vertices of each $K_m$ can be covered by a single edge or a cycle. If all vertices of $A$ are of the same copy $K_m$, then the remaining vertex can be paired by an edge with a vertex in an adjacent copy of $K_m$, and it is easy to see that the graph $G-A$ is hallian. Thus $h(G) \geqslant m-1$. Since $\delta(G) = m$, it follows that $h(G) = m-1$.

LEMMA 8. *If* $n \geqslant 3$, $m \geqslant 3$, *then* $h(C_n \times K_m) = m$.

Proof. Let $G = C_n \times K_m$ and suppose that at most $m$ vertices of $G$ are removed. The resulting graph is hallian by the argument of Lemma 7 unless all the vertices removed are from the same copy of $K_m$, in which case the result is obvious. Thus $h(G) \geqslant m$. But $\delta(G) = m+1$, and so $h(G) = m$.

From Lemmas 7 and 3 we obtain

THEOREM 6. *If* $n \geqslant 2$, $m \geqslant 3$, *then*

$$\operatorname{bind}(P_n \times K_m) = \frac{mn-1}{mn-n}.$$

From Lemmas 8 and 3 we obtain

THEOREM 7. *If* $n \geqslant 3$, $m \geqslant 3$, *then*

$$\operatorname{bind}(C_n \times K_m) = \frac{mn-1}{mn-n-1}.$$

We express our sincere thanks to the referee for his useful suggestions towards the improvement of this paper. He brought to our attention that the Conjecture of Kane et al. [2] has been also settled in the affirmative by D. R. Guichard (see Ars Combinatoria 19 (1985), pp. 175–178).

## References

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