ZASTOSOWANIA MATEMATYKI
VII (1963)

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AN APPROXIMATION TO ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KIND BY MEANS OF ELEMENTARY FUNCTIONS AND SOME APPLICATIONS TO PHYSICAL PROBLEMS

1. Statement of the problem. The integration of the non-linear differential equation

$$\frac{d^2x}{dt^2}+a^2\sin x = 0, \tag{1.1}$$

or

$$\frac{d^2x}{dt^2}+\beta\,\mathrm{sh}\,\gamma x = 0, \tag{1.2}$$

or

$$\frac{d^2x}{dt^2}+\beta x+\gamma x^3 = 0, \tag{1.3}$$

where $x = x(t)$ and a, β, γ are constants, leads to non-elementary elliptic functions

$$F(\varphi, k) = \int_0^{\varphi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \tag{1.4}$$

and

$$E(\varphi, k) = \int_0^{\varphi}\sqrt{1-k^2\sin^2\psi}\,d\psi. \tag{1.5}$$

Equation (1.1) describes a non-harmonic motion, for instance, that of a mathematical or physical pendulum with a large amplitude angle, where the linearization $\sin x \approx x$ is not legitimate. This equation, as well as the remaining two, are well known in the theory of elasticity (stability of flexible rods under considerable loads) in theoretical mechanics (some problems of dynamical stability) and in electrodynamics (vibration of certain non-linear electric systems). In addition, elliptic integrals may be met with in the theory of some surface phenomena connected with the so-called contact problems and also in the theory of structure of an atomic nucleus. Thus, an exact solution of many problems of the physics of solids and of technical physics is not possible unless we use the functions $F(\varphi, k)$ and $E(\varphi, k)$, i.e. incomplete elliptic integrals of the first and second kind, respectively. Both integrals are functions of two variables: the modulus k and the amplitude φ, determined in the region D:

$$\begin{cases} 0 \leqslant k \leqslant 1, \\ 0 \leqslant \varphi \leqslant \pi/2. \end{cases} \tag{1.6}$$

In the particular case of $\varphi = \pi/2$ we obtain complete elliptic integrals of the first and second kind, i.e. $K(k) = F(\pi/2, k)$ and $E(k) = E(\pi/2, k)$, which are also non-elementary functions and cannot be expressed (in an exact manner) by a finite number of elementary functions. This means that quantities described by means of elliptic integrals cannot, as a rule, be expressed directly in the form of an explicit relation or in the form of an effective formula. This constitutes a considerable difficulty and, therefore, many authors have tried to find some approximate relations to replace the exact ones in practical applications, thus permitting effective solution without resorting to tables. The problem becomes still more complicated if inverse functions $E_{-1}(\beta)$ or $K_{-1}(\beta)$ are concerned. Let us consider, as an example, the exact equations

$$T = 4\sqrt{L/g}K(k) \tag{1.7}$$

and

$$\delta = \frac{2l}{\beta} K_{-1}^{1/2}(\beta). \tag{1.8}$$

The first one represents the vibration period of a mathematical pendulum for finite amplitude angles and the second—the deflection of an axially compressed bar as a function of the load (for the notations, cf. the end part of this paper, where some simple examples are given).

The usual procedure of replacing a non-elementary function by its expansion in power series, the higher terms being rejected, is, in the author's opinion, not always satisfactory. Such a method, commonly used by many authors, gives, at most, a relatively good approximation of the exact solution in the neighbourhood of one point ($k = 0$ or $k = 1$, for instance), while inside the region the method of power series expansion is not reasonable.

In the present paper we shall show certain possibilities of approximating the elliptic integrals $F(\varphi, k)$ and $E(\varphi, k)$ by simple elementary functions $\bar{F}(\varphi, k)$ and $\bar{E}(\varphi, k)$ constituting a generalization of the Puwein formula. The essential feature of the approximation now proposed is that the approximating functions are required to differ very slightly from the exact original values not only at some particular points of the region in which the function is determined but also in the entire region D.

2. Approximation procedure. In 1954 a very convenient approximate formula was proposed by M. G. Puwein [6] for the period T of the mathematical pendulum

$$T = 2\pi\sqrt{L/g\cos(\varphi_0/2)}, \tag{2.1}$$

where L is the length of the pendulum, g is the gravity acceleration and φ_0 is the amplitude angle. This equation has no theoretical derivation. Its application is justified only by the fact that the power series expansion of the exact equation (1.7),

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}\sin^2\frac{\varphi_0}{2}+\frac{9}{64}\sin^4\frac{\varphi_0}{2}+\ldots\right), \tag{2.2}$$

has the first two terms identical with those of the expansion of (2.1) and the difference in the third term is 1/64 only, the expansion of (2.1) having the form

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}\sin^2\frac{\varphi_0}{2}+\frac{10}{64}\sin^4\frac{\varphi_0}{2}+\ldots\right). \tag{2.3}$$

Attention should be paid to the fact, not mentioned by Puwein, that the difference between the fourth terms of the expansions (2.2) and (2.3) is also insignificant.

Let us now point out a relation between the Puwein equation (2.1) and the full elliptic integral of the first kind $K(k)$, where, in the case of the pendulum, the modulus of the elliptic integral is $k = \sin(\varphi_0/2)$. To this effect we confront (1.7) with (2.1), bearing in mind the obvious relation

$$\cos\frac{\varphi_0}{2} = \sqrt{1-k^2}. \tag{2.4}$$

We have

$$4\sqrt{L/g}K(k) = 2\pi\sqrt{L/g}\sqrt[4]{1/(1-k^2)}. \tag{2.5}$$

Hence we obtain directly

$$\bar{K}(k) = \pi/2\sqrt[4]{1-k^2}, \tag{2.6}$$

where $\bar{K}(k)$ is the required elementary function, which, as will be shown, constitutes a good approximation of the full elliptic integral of the first kind almost in the entire interval $0 \leqslant k \leqslant 1$.

Equation (2.6) may be generalized by constructing a function of two variables $\bar{F}(\varphi, k)$ so that

$$\begin{aligned} \bar{F}(0, k) &\equiv F(0, k), \\ \bar{F}(\varphi, 0) &\equiv F(\varphi, 0); \end{aligned} \tag{2.7}$$

this means that we require that the exact (original) function and the approximating function (cf. Fig. 1) should have the same value along two edges of the region D, namely $k = 0$ and $\varphi = 0$, and that the value

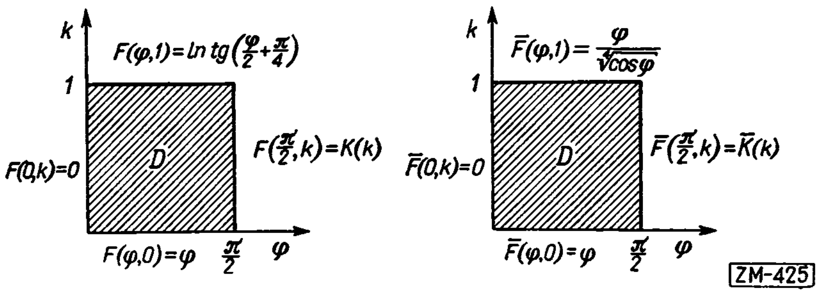

Fig. 1

of both functions should differ as little as possible along the remaining edges $k = 1$ and $\varphi = \pi/2$. Of course, for $\varphi = \pi/2$ we should obtain equation (2.6). As the result of many trials the formula

$$\bar{F}(\varphi, k) = \varphi/\sqrt[4]{1-k^2(1-\cos\varphi)}, \tag{2.8}$$

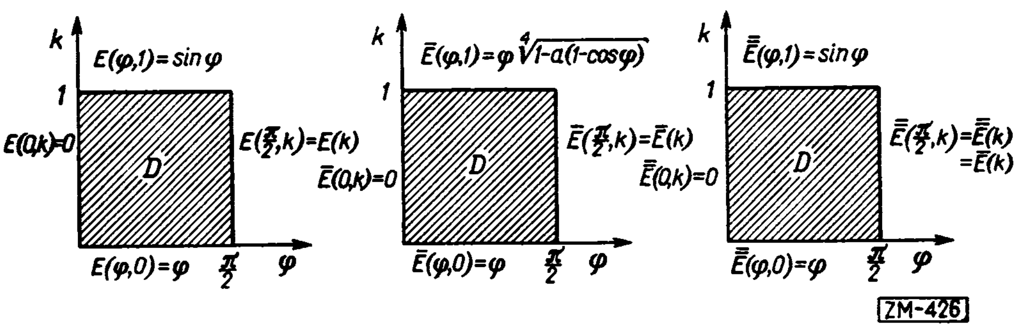

Fig 2

whose derivation is omitted, has been found to be the best one [1]. Similarly, for the incomplete elliptic integral of the second kind (cf. Fig. 2) we have found two expressions

$$(2.9)\qquad \begin{aligned} \bar{E}(\varphi, k) &= \varphi\sqrt[4]{1-ak^2(1-\cos\varphi)}\,, \\ \bar{\bar{E}}(\varphi, k) &= \varphi\sqrt[4]{1-ak^2}+k^2\left(\sin\varphi-\frac{2}{\pi}\varphi\right), \end{aligned}$$

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[1] There are various methods of approximating a function of one variable, such as Chebyshev's or Hermite's approximation, the minimum square deviation procedure, etc. No such method is available, in general, for functions of two or more variables.

where a is a constant: $a = 1-2^4/\pi^4 = 0{,}835742$. The accuracy of the equations just presented will be verified numerically and by the method of power series expansion of the two functions. The difference between corresponding terms of these expansions will be decisive. In general, for a function of one variable, we can write

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}, \tag{2.10}$$

and

$$\bar{f}(x) = \sum_{n=0}^{\infty} \bar{f}^{(n)}(x_0)\frac{(x-x_0)^n}{n!}, \tag{2.11}$$

where $f(x)$ is the exact function, $\bar{f}(x)$ the approximate function and x_0 the point where the functions are expanded in Taylor's series. Comparing the n-th terms of these expansions,

$$(x-x_0)^n R_n = \frac{(x-x_0)^n}{n!}[\bar{f}^{(n)}(x_0)-f^{(n)}(x_0)], \quad n = 0, 1, 2, \ldots, \tag{2.12}$$

we can estimate the agreement between a few derivatives of the exact function and the corresponding derivative of the approximate function. Thus, we shall be able to appraise the accuracy of the approximation proposed.

For $k = 1$, $\varphi_0 = 0$ we obtain

$$\begin{aligned} F(\varphi, 1) &= \ln \operatorname{tg}\left(\frac{\varphi}{2}+\frac{\pi}{4}\right) = \varphi + 0{,}1667\varphi^3 + 0{,}0417\varphi^5+\ldots \\ \bar{F}(\varphi, 1) &= \varphi/\sqrt[4]{\cos\varphi} = \varphi + 0{,}1250\varphi^3 + 0{,}0248\varphi^5+\ldots \end{aligned} \tag{2.13}$$

Hence

$$\begin{aligned} R_0 &= 0{,}0000, & R_3 &= -0{,}0417, \\ R_1 &= 0{,}0000, & R_4 &= 0{,}0000, \\ R_2 &= 0{,}0000, & R_5 &= -0{,}0169, \\ & & &\ldots\ldots\ldots \end{aligned}$$

For $\varphi = \pi/2$ (complete elliptic integral of the first kind) and $k_0 = 0$, we obtain

$$\begin{aligned} F(\pi/2, k) = K(k) &= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \\ &= \tfrac{1}{2}\pi(1+0{,}2500k^2+0{,}1407k^4+0{,}0977k^6+\ldots) \\ \bar{F}(\pi/2, k) = \bar{K}(k) &= \pi/2\sqrt[4]{1-k^2} \\ &= \tfrac{1}{2}\pi(1+0{,}2500k^2+0{,}1563k^4+0{,}1171k^6+\ldots) \end{aligned} \tag{2.14}$$

where

$$R_0 = 0{,}0000, \quad R_4 = +0{,}0156,$$
$$R_1 = 0{,}0000, \quad R_5 = 0{,}0000,$$
$$R_2 = 0{,}0000, \quad R_6 = +0{,}0194,$$
$$R_3 = 0{,}0000, \quad \ldots\ldots\ldots$$

The appraisal procedure for the approximation to the incomplete elliptic integral of the second kind is similar. For $k = 1$ and $\varphi_0 = 0$ we have

(2.15)
$$E(\varphi, 1) = \sin\varphi = \varphi - 0{,}1667\varphi^3 + 0{,}0083\varphi^5 - \ldots$$
$$\bar{E}(\varphi, 1) = \varphi\sqrt[4]{1 - a(1-\cos\varphi)} = \varphi - 0{,}1045\varphi^3 - 0{,}0033\varphi^5 - \ldots$$

Hence

$$R_0 = 0{,}0000, \quad R_4 = 0{,}0000,$$
$$R_1 = 0{,}0000, \quad R_5 = -0{,}0116,$$
$$R_2 = 0{,}0000,$$
$$R_3 = +0{,}0622, \quad \ldots\ldots\ldots$$

For the full elliptic integral of the second kind, i.e. along the right-hand edge of the region D, $\varphi = \pi/2$, and in the neighbourhood of the point $k_0 = 0$, we find

(2.16)
$$E(\pi/2, k) = E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\psi}\, d\psi = \tfrac{1}{2}\pi(1 - 0{,}2500k^2 - 0{,}0469k^4 - 0{,}0195k^6 - \ldots)$$
$$\bar{E}(\pi/2, k) = \bar{E}(k) = \tfrac{1}{2}\pi\sqrt[4]{1 - ak^2} = \tfrac{1}{2}\pi(1 - 0{,}2089k^2 - 0{,}0659k^4 - 0{,}0318k^6 - \ldots)$$

where

$$R_0 = 0{,}0000, \quad R_4 = -0{,}0186,$$
$$R_1 = 0{,}0000, \quad R_5 = 0{,}0000,$$
$$R_2 = +0{,}0401, \quad R_6 = -0{,}0123,$$
$$R_3 = 0{,}0000, \quad \ldots\ldots\ldots$$

The last function under consideration $\bar{\bar{E}}(\varphi, k)$, yields the exact value for the edge $k = 1$, i.e. $\bar{\bar{E}}(\varphi, 1) \equiv E(\varphi, 1) = \sin\varphi$. For $\varphi = \pi/2$ we find $\bar{\bar{E}}(k) = \frac{1}{2}\pi\sqrt[4]{1 - ak^2}$, which is identical with the second expression of (2.16).

The accuracy of the equations proposed, i.e. (2.8), (2.9) and (2.10), inside the region D has been investigated numerically by means of computers and tables of elliptic integrals of high accuracy [3]. The results are shown in Fig. 3. Using these data one might plot the diagram of the error function

$$\Delta(\varphi, k) = \frac{\bar{F}(\varphi, k) - F(\varphi, k)}{F(\varphi, k)} \cdot 100\ \% \tag{2.17}$$

representing a certain surface and defined similarly to the elliptic integrals discussed in the region D. The function $\Delta(\varphi, k)$ is bounded with the exception of one point $\varphi = \pi/2$, $k = 1$ where $\Delta(\varphi, k) = +\infty$. This is an unexpected result, if it is borne in mind that for $\varphi = \pi/2$, $k = 1$ we

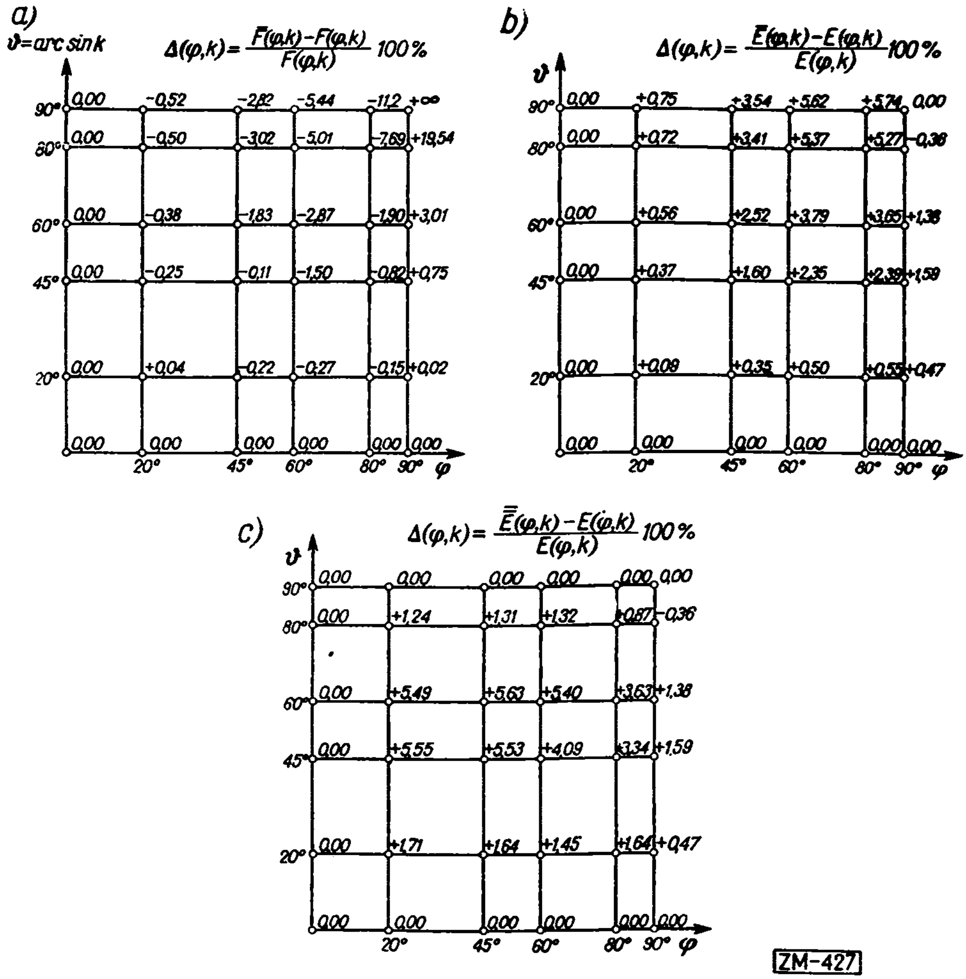

Fig. 3

have both $F(\varphi, k) = +\infty$ and $\overline{F}(\varphi, k) = +\infty$; therefore it might be concluded that the approximate equation yields the correct value. It can easily be seen, however, that the limit of the expression

$$\frac{\overline{F}(\pi/2, 1) - F(\pi/2, 1)}{F(\pi/2, 1)} \tag{2.18}$$

does not exist. For,

$$(2.19)\qquad \lim_{\substack{k=1\\ \varphi\to\pi/2}}\left(\frac{\bar{F}}{F}-1\right)=\lim_{\varphi\to\pi/2}\left[\frac{\varphi}{\sqrt[4]{\cos\varphi}\,\ln\operatorname{tg}\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)}-1\right]=+\infty.$$

It should also be mentioned that by making use of the equations given it is easy to express any non-elementary function

$$(2.20)\qquad W=W[F(\varphi,k),E(\varphi,k)].$$

in the form of a finite number of elementary functions

$$(2.21)\qquad \bar{W}=\bar{W}[\bar{F}(\varphi,k),\bar{E}(\varphi,k)].$$

The degree of accuracy of the approximation may change, which, how ever, can easily be foretold.

Fig. 4 shows the ranges of applicability of all the three approximate equations.

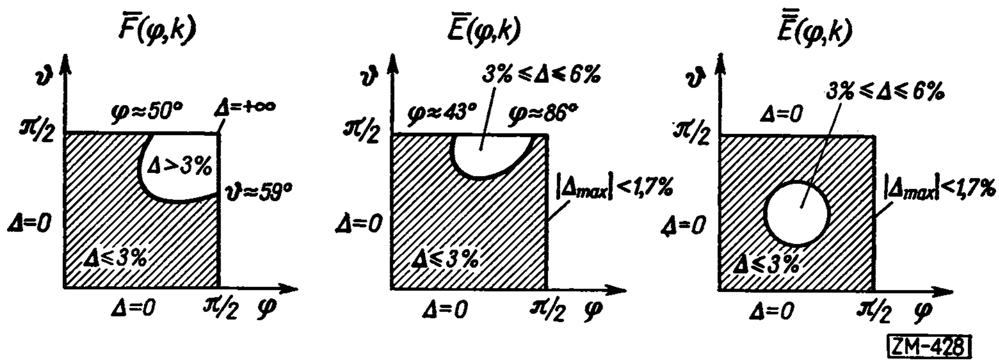

Fig. 4

3. Examples of application:

3.1. The question is now as follows: what is the amplitude angle of the mathematical pendulum if the period for a given length L is prescribed to be $T=1$ sec? To solve this problem with the exact formula we must write, making use of (1.7),

$$(3.1)\qquad K\left(\sin^2\frac{\varphi_0}{2}\right)=\frac{T}{4}\sqrt{\frac{g}{L}}.$$

Hence, after inverting the function $K\left(\sin^2\frac{\varphi_0}{2}\right)$ and rearranging, we have

$$(3.2)\qquad \varphi_0=2\arcsin K_{-1}^{1/2}\left(\frac{T}{4}\sqrt{\frac{g}{L}}\right)$$

Equation (3.2) represents the relation $\varphi_0 = \varphi_0(L)$, i.e. it expresses the amplitude angle in terms of the length of the pendulum. Hence, for a given value of L, we should calculate (3.2). This expression and the further calculation procedure are very inconvenient.

If the Puwein equation (or equation (2.6), which is a generalization of (2.1) is used, we have

$$\varphi_0 = 2\arccos\frac{4\pi^2 L}{gT^2}. \tag{3.3}$$

Hence for $L = 10$ cm, $T = 1$ sec, for instance, we easily find $\varphi_0 = 46°50'$.

3.2. Let us now solve a problem of the non-linear theory of buckling. The deflection of the bar (Fig. 5), after the loss of stability of the rectilinear form of equilibrium, may be obtained by integrating the non-linear differential equation (1.1). Of course, the deflection δ is a function of the load P. Let us write the final equations, [5]:

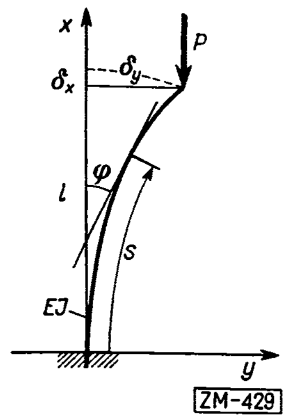

Fig. 5

$$\delta_y = \frac{2l}{\beta} K_{-1}^{1/2}(\beta), \tag{3.4}$$

$$\delta_x = 2l[1 - E(k)/K(k)], \tag{3.5}$$

where the modulus of the elliptic integrals $E(k)$, $K(k)$ is

$$k = k(m) \tag{3.6}$$

and $m = P/P_E$ is the dimensionless force. The remaining symbols are: $\beta = l\sqrt{P/EJ}$, EJ — rigidity of the bar, l — length of the bar, $P_E = \pi^2 EJ/4l^2$ — critical force (according to Euler's formula), δ_x, δ_y — deflection of the bar in the directions x and y respectively. The knowledge of δ_y is of particular practical importance. We calculate the dimensionless deflection $\vartheta_y = \delta_y/l$, bearing in mind that $\beta = \pi\sqrt{m}/2$ and

$$\beta = K(k) \approx \bar{K}(k) = \pi/2\sqrt[4]{1-k^2}. \tag{3.7}$$

The assumption $K(k) \approx \bar{K}(k)$ enables us to invert function $K(k)$, what would not be otherwise possible, thus we find modulus k explicite

$$k = K_{-1}^{1/2}\left(\frac{\pi}{2}\sqrt{m}\right) = \frac{\sqrt{m^2-1}}{m}. \tag{3.8}$$

Hence and from (3.4) we obtain

$$\vartheta_y = \frac{\delta_y}{l} = \frac{4}{\pi}\sqrt{\frac{m^2-1}{m^3}}. \tag{3.9}$$

This equation has been derived for the first time in [7] in a somewhat different manner. This reference contains also a detailed numerical analysis of the accuracy of equation (3.9) by confronting it with the exact solution (3.4) and several other approximate equations obtained in various ways by a number of scientists: Timoshenko, Mises, Rshanitsyn, Poeschl, Biezeno, Grammel, Ponomariev, Naleszkiewicz, Życzkowski. Let us mention paper [8], by M. Życzkowski, in which the function $K(k)$ is expanded in a power series and the function $K_{-1}(\beta)$ is found by inverting the series. This method, although a correct one, requires a lot of labour. Equation (3.9) turns out to be the most accurate of all the equations found in literature and the simplest one from the formal point of view. For $m = 3$, i.e. for the force $P = 3P_E$, equation (3.9) gives the error $\Delta = +2{,}0\%$. For m approaching unity this error is very small.

Similarly, by replacing the integrals $E(k)$ and $K(k)$ in (3.5) by the functions $\bar{E}(k)$ and $\bar{K}(k)$, we obtain the deflection in the x-direction (reduction of the distance between the ends of the bar):

$$(3.10) \qquad \vartheta_x = \frac{\delta_x}{l} = \frac{2}{m}\left[m - \sqrt[4]{m^2 - a(m^2-1)}\right], \qquad a = 0{,}835742.$$

For m near unity we can set $a = 1$; then $\vartheta_x = 2(m-1)/m$. Equation (3.10) is less accurate than expression (3.9). For $m = 2{,}160$, i.e. for the

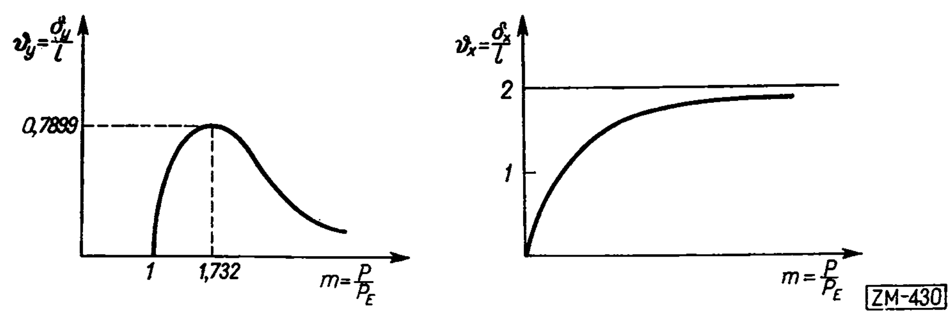

Fig. 6

load P more than twice as high as the critical load P_E, the value of ϑ^x calculated by means of (3.10) is charged with the error $\Delta = -3{,}6\%$. For $m \to \infty$, i.e. for infinitely increasing force, equations (3.9) and (3.10) both yield values in agreement with the exact solution, i.e. $\vartheta_y \to 0$, $\vartheta_x \to 2$. The relevant diagrams are shown in Fig. 6.

3.3. The intensity of the magnetic field produced at the point 0 by a linear current i in a conductor in the form of an ellipse (Fig. 7) is expressed, according to the Biot-Savart law, by the formula

$$(3.11) \qquad H_0 = \frac{4pi}{b}\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\psi}\,d\psi,$$

where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse and p is a coefficient depending on the choice of the system of units. Setting $E(e) = \bar{E}(e)$ we obtain the simple approximation (²)

$$H_0 = \frac{2\pi p i}{b}\sqrt[4]{1 - \alpha e^2}, \qquad \alpha = 0{,}835742. \tag{3.12}$$

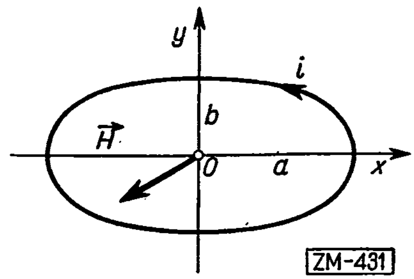

Fig. 7

A similar procedure can be used to compute the magnetic intensity due to a linear current in a circular conductor, for a point 0 not lying at the centre of the circle (in the plane of the circle or not). The corresponding equations will not be quoted here.

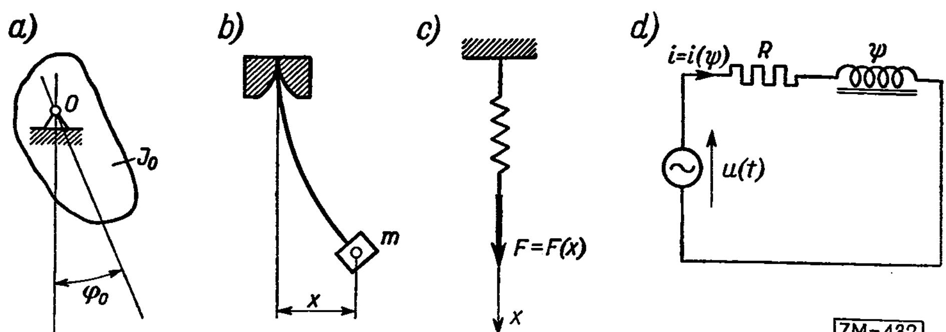

Fig. 8

3.4. Figure 8 represents four non-linear systems described by the differential equations mentioned at the beginning. Three of them are mechanical systems: the physical pendulum, the elastic pendulum with a programmed length and a vibrating system with a non-linear spring

$$F = \alpha x + \beta x^3 \tag{3.13}$$

(²) The maximum error for (3.12) is not greater than 1,7 %.

or

$$F = a\,\mathrm{sh}\,\beta x, \tag{3.14}$$

where a, β are constants, $x = x(t)$ is the displacement, F is the force, and t is the time. The fourth system is a C, ψ—electric system, fed with a sinusoidal excitation voltage; the non-linearity of the coil may be prescribed, for example

$$i = a\psi + \beta\psi^3, \tag{3.15}$$

where $i = i(t)$ is the intensity of the current, C is the capacity, and ψ is the magnetic flow.

All the above cases reduce, after integration of the corresponding differential equation, to [2]:

$$\gamma t = F(\varphi, k), \tag{3.16}$$

where γ is a constant. In this manner the time dependence of the displacement of the pendulum, the deformation of the spring and the current impulse of the system are described by means of an incomplete elliptic integral of the first kind. In particular, for $\varphi = \pi/2$ we obtain a complete elliptic integral of the first kind $K(k)$ (for the computation of the vibration period T, for instance). Making use of the equations of section 2 and setting $F(\varphi, k) = \bar{F}(\varphi, k)$ or $K(k) = \bar{K}(k)$ we can obtain, in each particular case, the required effective approximate formulas.

The author wishes to thank Prof. dr M. Życzkowski for his helpful suggestions.

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Praca wpłynęła 24. 1. 1963

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APROKSYMACJA CAŁEK ELIPTYCZNYCH PIERWSZEGO I DRUGIEGO RODZAJU PRZEZ FUNKCJE ELEMENTARNE ORAZ NIEKTÓRE ICH ZASTOSOWANIA W FIZYCE

STRESZCZENIE

Całkowanie pewnych nieliniowych równań różniczkowych prowadzi do użycia nieelementarnych funkcji $F(\varphi, k)$ oraz $E(\varphi, k)$, to jest niepełnych całek eliptycznych I-go i II-go rodzaju, odpowiednio. Występowanie funkcji $F(\varphi, k)$ oraz $E(\varphi, k)$ uniemożliwia w wielu przypadkach podanie efektywnych wzorów końcowych, co jest kłopotliwe zwłaszcza tam, gdzie zachodzi potrzeba odwracania funkcji nieelementarnych, na przykład — znajdywania modułu pełnej całki eliptycznej $K(k)$ lub $E(k)$. W pracy przedstawiono możliwości aproksymacji funkcji $F(\varphi, k)$ oraz $E(\varphi, k)$ przez proste funkcje elementarne $\bar{F}(\varphi, k)$, $\bar{E}(\varphi, k)$, które stanowią uogólnienie przybliżonego wzoru Puweina (1954). Dokładność uzyskanych wzorów sprawdzono metodą rozwijania w szereg potęgowy „oryginału" (funkcji nieelementarnej) oraz jej odpowiednika — funkcji elementarnej. Ponadto, na podstawie dostatecznie dokładnych obliczeń numerycznych zbudowano funkcję błędu $\Delta = \Delta(\varphi, k)$ określoną w obszarze $0 < \varphi < \pi/2$, $0 < k < 1$.

W zakończeniu pracy podano przykłady zastosowań proponowanych funkcji przy rozwiązywaniu pewnych nieliniowych zagadnień elektrodynamiki oraz teorii sprężystości.

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АПРОКСИМАЦИЯ ЭЛЛИПТИЧЕСКИХ ИНТЕГРАЛОВ ПЕРВОГО И ВТОРОГО РОДА ЭЛЕМЕНТАРНЫМИ ФУНКЦИЯМИ И ИХ НЕКОТОРЫЕ ПРИМЕНЕНИЯ В ФИЗИКЕ

РЕЗЮМЕ

Интегрирование некоторых нелинейных дифференциальных уравнений ведёт к использованию неэлементарных функций $F(\varphi,E)$ и $E(\varphi,k)$ — неполных эллиптических интегралов 1-го и 2-го рода соответственно. Участие функций $F(\varphi, k)$ и $E(\varphi,k)$ во многих случаях не даёт возможности написания эффективных конечных формул, что особенно невыгодно в тех случаях, когда является необходимым иметь дело с обратными неэлементарными функциями, например — в случае нахождения модула полного эллиптического интеграла $K(k)$ или $E(k)$. В работе представлены возможности апроксимации функции $F(\varphi,k)$ и $E(\varphi,k)$ посредством простых элементарных функций $(\bar{F}\varphi,k)$, $\bar{E}(\varphi,k)$ являющихся обобщением приближенной формулы Пувейна (1954). Точность полученных формул была проверена методом разложения „оригинала" (неэлементарной функции) и соответствующей элементарной функции в степенной ряд. Кроме этого, на основе достаточно точных нумеричных расчётов была построена функция ошибок $\Delta = \Delta(\varphi,k)$ определенная в области $0 < \varphi < \pi/2$, $0 < k < 1$.

В заключительной части работы приведены примеры применений предлагаемых функций для решения некоторых нелинейных вопросов электродинамики и теории упругости.