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STEADY-STATE DISTRIBUTIONS OF PIECEWISE MARKOV PROCESSES

1. Introduction. In the paper of Kuczura [4] the relations between the stationary probability distribution of the state of the piecewise Markov process and the stationary distributions of some imbedded Markov chains have been found. In the proof the imbedded renewal process and the key renewal theorem have been used (Smith [5]). Here we give different proofs of Kuczura's results and also some generalizations using in proof the method of extension of a piecewise Markov process to a Markov process and assuming, first of all, the stationarity of the extended Markov process. These two methods have been shown in [1]-[3] by applying to the analysis of some processes in queueing theory.

2. Definitions and notations. We introduce the definition of the piecewise Markov process in Kuczura's sense. Let S be a discrete state space. The stochastic process $Y(t)$ is called a *piecewise Markov process* if

(i) there exists a sequence of random variables $\ldots < w_{-1} < w_0 < w_1 < w_2 < \ldots$ such that, for every $n \in N$, where $N = \{\ldots, -1, 0, 1, 2, \ldots\}$, $\{Y(t), w_{n-1} \leqslant t < w_n\}$ is the homogeneous Markov process in which we have

$$P_{ij}^{(k)}(t) = \Pr\{Y(s+t) = j \mid Y(s) = i,\ Y(w_{n-1}) = k\}$$

for any s and t with $w_{n-1} \leqslant s < s+t < w_n$,

(ii) $p_{ij} = \Pr\{Y(w_n) = j \mid Y(w_n - 0) = i\}$ are the elements of the transition matrix (p_{ij}), and

(iii) $F_k(u) = \Pr\{w_n - w_{n-1} < u \mid Y(w_{n-1}) = k\}$, $i, j, k \in S$, are the distribution functions with $F_k(0+) = 0$.

Remark. A. Kuczura considers the process $Y(t)$ for $t \geqslant 0$ assuming $w_0 = 0$. In considerations concerning stationary processes this restriction can be omitted.

The change of the state of the process in regenerative moments w_n, $n \in N$, is called a *regenerative transition* (the transition to the same state is possible), and the transition without regenerative moments is called *Markovian*.

The intervals $w_{n-1} \leqslant t < w_n$, $n \in N$, are called *Markovian segments*. It follows from the above-given assumptions that, if at the beginning of the segment the process is in the state k, then the distribution F_k of the segment length and the Markovian transition matrix $\big(P_{ij}^{(k)}(t)\big)$ may depend upon k, but they are independent of n.

Denote by $(\theta_{ij}^{(k)})$ the Markovian transition intensity matrix. Obviously,

$$\theta_{ij}^{(k)} = \lim_{t\to 0} \frac{P_{ij}^{(k)}(t)}{t}, \quad i \neq j,$$

$$\theta_{i}^{(k)} = \lim_{t\to 0} \frac{1-P_{ii}^{(k)}(t)}{t} = \sum_{j\neq i} \theta_{ij}^{(k)}, \quad i, j, k \in S.$$

Let us define the semi-Markov process

$$Z(t) = Y(w_{n-1}), \quad w_{n-1} \leqslant t < w_n, \ n \in N.$$

The extended process $\big(Y(t), Z(t)\big)$ is also a piecewise Markov process with a state space $S\times S$ and with identical regenerative moments w_n, $n \in N$. Obviously, for any s and t with $w_{n-1} \leqslant s < s+t < w_n$, we have

$$\begin{aligned} (2.1)\quad P_{ik;jl}(t) &= \Pr\{Y(s+t) = j,\ Z(s+t) = l \mid Y(s) = i,\ Z(s) = k\} \\ &= \delta_{kl} P_{ij}^{(k)}(t), \end{aligned}$$

$$\begin{aligned} (2.2)\quad p_{ik;jl} = \Pr\{Y(w_n) = j,\ Z(w_n) = l \mid Y(w_n-0) = i,& \\ Z(w_n-0) = k\} &= \delta_{lj} p_{ij}, \end{aligned}$$

$$(2.3)\quad F_{kk}(u) = \Pr\{w_n - w_{n-1} < u \mid Y(w_{n-1}) = k,\ Z(w_{n-1}) = k\} = F_k(u),$$

where $i, j, k, l \in S$, and δ is Kronecker's delta.

From equalities (2.1)-(2.3) it follows that the analysis of the piecewise Markov process is equivalent to the analysis of the extended process $\big(Y(t), Z(t)\big)$.

3. Imbedded Markov chains. Let $\big(Y(t), Z(t)\big)$ be the extended piecewise Markov process characterized by the triple $\big(F_k, (p_{ij}), (\theta_{ij}^{(k)})\big)$. Let us consider the following sequences of random variables:

$$R_n^* = \big(Y(w_n-0), Z(w_n-0)\big), \quad R_n = Y(w_n-0), \quad S_n = Y(w_n), \quad n \in N.$$

These sequences are homogeneous Markov chains, since w_n, $n \in N$, are regenerative moments. Let us introduce the following notation for the transition probabilities:

$$\begin{aligned} r_{ij}^{(k)} &= \Pr\{R_n^* = (j, k) \mid R_{n-1}^* = (i, m)\}, \\ r_{ij} &= \Pr\{R_n = j \mid R_{n-1} = i\}, \\ s_{ij} &= \Pr\{S_n = j \mid S_{n-1} = i\}, \quad i, j, k, m \in S. \end{aligned}$$

It is easy to see that these probabilities can be expressed by the above-given formulas as follows:

$$(3.1)\qquad r_{ij}^{(k)} = \int_0^\infty p_{ik} P_{kj}^{(k)}(u)\, dF_k(u),$$

$$r_{ij} = \sum_k r_{ij}^{(k)},$$

$$(3.2)\qquad s_{ij} = \int_0^\infty \sum_k P_{ik}^{(i)}(u) p_{kj}\, dF_i(u), \qquad i, j, k \in S.$$

Consequently, the steady-state probability distributions of the imbedded Markov chains, i.e.

$$\varrho_j^{(k)} = \Pr\{R_n^* = (j, k)\}, \qquad \varrho_j = \Pr\{R_n = j\}, \qquad \sigma_j = \Pr\{S_n = j\}, \qquad j, k \in S,$$

satisfy the homogeneous systems of equations

$$(3.3)\qquad \varrho_j^{(k)} = \sum_i \sum_m r_{ij}^{(k)} \varrho_i^{(m)},$$

$$\varrho_j = \sum_i r_{ij} \varrho_i,$$

$$(3.4)\qquad \sigma_j = \sum_i s_{ij} \sigma_i, \qquad j, k \in S,$$

the entire probability conditions

$$\sum_j \sum_k \varrho_j^{(k)} = 1, \qquad \sum_j \varrho_j = 1, \qquad \sum_j \sigma_j = 1,$$

and the equality

$$(3.5)\qquad \varrho_j = \sum_k \varrho_j^{(k)}, \qquad j \in S.$$

Since the random variables R_n and S_n are related by a regenerative transition, the distributions $\{\varrho_j\}$ and $\{\sigma_j\}$ are related as follows:

$$(3.6)\qquad \sigma_j = \sum_i p_{ij} \varrho_i, \qquad j \in S.$$

4. Steady-state distributions. Now we proceed to the analysis of $(Y(t), Z(t))$ as a process of continuous parameter. Let us introduce the following notation for the steady-state probabilities:

$$q_j^{(k)} = \Pr\{Y(t) = j, Z(t) = k\}, \qquad q_j = \Pr\{Y(t) = j\}, \qquad j, k \in S.$$

Let $X(t)$ be the time length from the moment t to the nearest regenerative moment. Let us write

$$P_j^{(k)}(x) = \Pr\{Y(t) = j, Z(t) = k, X(t) < x\}, \qquad j, k \in S,\ x > 0.$$

LEMMA 1. *The stochastic process* $(Y(t), Z(t), X(t))$ *is Markovian. The probabilities* $P_j^{(k)}(x)$ *satisfy the system of differential equations*

$$(4.1)\quad P_j^{(k)\prime}(x) - P_j^{(k)\prime}(0) - \theta_j^{(k)} P_j^{(k)}(x) + \\ + \sum_{i \neq j} \theta_{ij}^{(k)} P_i^{(k)}(x) + \delta_{jk} F_k(x) \sum_i p_{ij} \sum_l P_i^{(l)\prime}(0) = 0, \quad j, k \in S.$$

Proof. From the analysis of the state of the process at the moments $t+h$ and t $(h > 0)$ we obtain

$$\Pr\{Y(t+h) = j, Z(t+h) = k, X(t+h) < x\}$$
$$= (1 - \theta_j^{(k)} h) \Pr\{Y(t) = j, Z(t) = k, h \leqslant X(t) < x+h\} +$$
$$+ \sum_{i \neq j} \theta_{ij}^{(k)} h \Pr\{Y(t) = i, Z(t) = k, h \leqslant X(t) < x+h\} +$$
$$+ \delta_{jk} F_k(x + ah) \sum_l \sum_i p_{ij} \Pr\{Y(t) = i, Z(t) = l, X(t) < h\} + o(h),$$
$$j, k \in S, \quad a \in (0, 1).$$

Dividing sidewise by h and taking the limit for $h \to 0$, we obtain the system of differential equations (4.1).

LEMMA 2. *The steady-state distribution function of the process* $X(t)$ *is of the form*

$$(4.2)\quad P(x) = \Pr\{X(t) < x\} = \nu \sum_j \sigma_j \int_0^x \big(1 - F_j(u)\big) du,$$

and we have

$$(4.3)\quad P_j^{(k)\prime}(0) = \nu \varrho_j^{(k)}, \quad j, k \in S,$$

where

$$(4.4)\quad \frac{1}{\nu} = \sum_i \frac{\sigma_i}{\nu_i} \quad \textit{with } \frac{1}{\nu_i} = \int_0^\infty \big(1 - F_i(t)\big) dt.$$

Proof. An addition of (4.1) side by side over j and k gives the differential equation

$$P'(x) = P'(0) + \sum_j F_j(x) \sum_i p_{ij} \sum_l P_i^{(l)\prime}(0).$$

Since $P(0) = 0$ and $P(\infty) = 1$, this equation has the solution

$$(4.5)\quad P(x) = \sum_i \sum_l P_i^{(l)\prime}(0) \sum_j p_{ij} \int_0^x \big(1 - F_j(u)\big) du,$$

where

$$\sum_i \sum_l P_i^{(l)\prime}(0) \sum_j p_{ij} \frac{1}{\nu_j} = 1. \tag{4.6}$$

Let us write

$$\lim_{h\to 0} \frac{1}{h} \Pr\{X(t) < h\} = \nu.$$

It is easy to see that

$$P_j^{(k)\prime}(0) = \lim_{h\to 0} \frac{1}{h} \Pr\{X(t) < h\} \Pr\{Y(t) = j,\, Z(t) = k \mid X(t) < h\} = \nu \varrho_j^{(k)}.$$

From this and from (4.6), using (3.5) and (3.6), we obtain (4.3) and (4.4). Substituting (4.3) into (4.5), we obtain (4.2).

THEOREM 4.1. *For the extended piecewise Markov process* $(Y(t), Z(t))$, *the steady-state probability distributions satisfy the relations*

$$\nu \varrho_j + \sum_k \theta_j^{(k)} q_j^{(k)} = \nu \sigma_j + \sum_{i\neq j} \sum_k \theta_{ij}^{(k)} q_i^{(k)}, \qquad j, k \in S. \tag{4.7}$$

Proof. Adding equalities (4.1) over k and taking the limit for $x \to \infty$, since $P_j^{(k)\prime}(x) \to 0$, $F_k(x) \to 1$ and $P_i^{(k)}(x) \to q_i^{(k)}$, we obtain

$$-\sum_k P_j^{(k)\prime}(0) - \sum_k \theta_j^{(k)} q_j^{(k)} + \sum_{i\neq j} \theta_{ij}^{(k)} q_i^{(k)} + \sum_i p_{ij} \sum_l P_i^{(l)\prime}(0) = 0, \qquad j \in S.$$

Substituting (4.3), in view of (3.5), (3.6) and (4.4), we obtain (4.7).

COROLLARY 4.1. *If the Markovian transition intensity matrices are identical*, $\theta_{ij}^{(k)} = \theta_{ij}$, *then*

$$\nu \varrho_j + \theta_j q_j = \nu \sigma_j + \sum_{i\neq j} \theta_{ij} q_i, \qquad j \in S. \tag{4.8}$$

THEOREM 4.2. *The steady-state probabilities of the process* $(Y(t), Z(t), X(t))$ *are given by*

$$P_j^{(k)}(x) = \nu \sigma_k \int_0^\infty \big(F_k(x+t) - F_k(t)\big) P_{kj}^{(k)}(t)\, dt. \tag{4.9}$$

Proof. Let us consider the following Chapman-Kolmogorov system of differential equations for the transition probabilities $P_{kj}^{(k)}(t)$:

$$P_{kj}^{(k)\prime}(t) = -\theta_j^{(k)} P_{kj}^{(k)}(t) + \sum_{i\neq j} \theta_{ij}^{(k)} P_{ki}^{(k)}(t), \qquad k, j \in S. \tag{4.10}$$

Multiplying the left-hand side of (4.10) by $\nu\sigma_k\big(F_k(x+t)-F_k(t)\big)$ and integrating, we obtain

$$L = \nu\sigma_k \int\limits_0^\infty \big(F_k(x+t)-F_k(t)\big)\,dP_{kj}^{(k)}(t)$$

$$= \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{kj}^{(k)}(t)\big|_0^\infty - \int\limits_0^\infty \nu\sigma_k P_{kj}^{(k)}(t)\,d\big(F_k(x+t)-F_k(t)\big)$$

$$= -\delta_{kj}\nu\sigma_k F_k(x) - \frac{d}{dx}\Big(\int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{kj}^{(k)}(t)\,dt\Big) +$$

$$+\int\limits_0^\infty \nu\sigma_k P_{kj}^{(k)}(t)\,dF_k(t),$$

where the last term on the right-hand side, by (3.6), (3.5), (3.1) and (3.3), equals $\nu\varrho_j^{(k)}$.

Applying the same transformation to the right-hand side of (4.10), we obtain

$$R = \int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)\Big(-\theta_j^{(k)}P_{kj}^{(k)}(t)+\sum_{i\neq j}\theta_{ij}^{(k)}P_{ki}^{(k)}(t)\Big)\,dt$$

$$= -\theta_j^{(k)}\int\limits_0^\infty \nu\sigma_k\big(F_k(x+k)-F_k(t)\big)P_{kj}^{(k)}(t)\,dt +$$

$$+\sum_{i\neq j}\theta_{ij}^{(k)}\Big(\int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{ki}^{(k)}(t)\,dt\Big).$$

Comparing $L = R$ we have

$$\frac{d}{dx}\Big(\int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{kj}^{(k)}(t)\,dt\Big) -$$

$$-\nu\varrho_j^{(k)} - \theta_j^{(k)}\Big(\int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{kj}^{(k)}(t)\,dt\Big) +$$

$$+\sum_{i\neq j}\theta_{ij}^{(k)}\Big(\int\limits_0^\infty \nu\sigma_k\big(F_k(x+t)-F_k(t)\big)P_{ki}^{(k)}(t)\,dt\Big) + \delta_{jk}\nu\sigma_k F_k(x) = 0.$$

From this, using (4.3) and (3.6), it follows that the solution of the system of equations (4.1) has form (4.9).

COROLLARY 4.2. *The steady-state probabilities of the process* $(Y(t), Z(t))$ *and* $Y(t)$ *are equal to*

$$q_j^{(k)} = \int_0^\infty \nu\sigma_k\big(1-F_k(t)\big)P_{kj}^{(k)}(t)\,dt, \tag{4.11}$$

$$q_j = \sum_k q_j^{(k)} = \int_0^\infty \nu \sum_k \sigma_k\big(1-F_k(t)\big)P_{kj}^{(k)}(t)\,dt, \qquad j \in S. \tag{4.12}$$

Relation (4.8) has been proved in [4], expression (4.7) is a generalization of these relations for the case of Markovian transition intensity matrices $(\theta_{ij}^{(k)})$ dependent upon k.

5. Imbedded renewal process. The moments of the regenerative entries into the state k ($k \in S$) for the piecewise Markov process $Y(t)$ constitute the renewal process. Let μ_k denote the mean time between successive signals in this renewal process. The calculation of $\{\mu_k\}$ is equivalent to the calculation of the steady-state probability distribution $\{\sigma_k\}$.

THEOREM 5.1. *For the piecewise Markov process* $Y(t)$, *the sequences* $\{\mu_j\}$ *and* $\{\sigma_j\}$ *satisfy the relations*

$$\frac{1}{\mu_j} = \nu\sigma_j, \qquad j \in S. \tag{5.1}$$

COROLLARY 5.1. *The sequence* $1/\mu_j$ *satisfies the homogeneous system of equations*

$$\frac{1}{\mu_j} = \sum_i s_{ij}\frac{1}{\mu_i}, \qquad j \in S, \tag{5.2}$$

with the condition

$$\sum_j \frac{1}{\nu_j \mu_j} = 1. \tag{5.3}$$

Formula (5.2) follows from (3.4) after substitution of (5.1), and formula (5.3) follows from (5.1) and (4.4).

Proof of theorem 5.1. Let F_{ij} denote the probability distribution of the time length from the moment of the regenerative entry into the state i to the moment of the regenerative entry into the state j and let

$$\mu_{ij} = \int_0^\infty \big(1-F_{ij}(t)\big)\,dt, \qquad i, j \in S.$$

It is easy to verify that

$$1-F_{ij}(x) = 1-F_{ii}(x) + \sum_k \sum_{l\neq j} \int_0^x P_{ik}^{(i)}(t)p_{kl}\big(1-F_{lj}(x-t)\big)\,dF_i(t), \qquad i, j \in S,$$

which, after side by side integration, leads to the equalities

$$\mu_{ij} = \frac{1}{\nu_i} + \sum_{l \neq j} \mu_{lj} \int_0^\infty \sum_k P_{ik}^{(k)}(t) p_{kl} dF_i(t), \qquad i, j \in S.$$

In view of (3.2) we have

$$\mu_{ij} = \frac{1}{\nu_i} + \sum_{l \neq j} s_{il} \mu_{lj}, \qquad i, j \in S.$$

Multiplying these equalities sidewise by σ_i and summing them over i, using (3.4) we obtain, after reduction,

$$\sigma_j \mu_{jj} = \sum_i \frac{\sigma_i}{\nu_i} = \frac{1}{\nu}, \quad \text{where } \mu_{jj} = \mu_j.$$

This completes the proof of theorem 5.1.

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STACJONARNE PROCESY PRZEDZIAŁAMI MARKOWSKIE

STRESZCZENIE

W pracy przedstawiamy nowe dowody oraz pewne uogólnienia wyników A. Kuczury, dotyczących relacji między granicznymi rozkładami prawdopodobieństwa stanu procesu przedziałami markowskiego, zdefiniowanego w [4], i granicznymi rozkładami prawdopodobieństwa łańcuchów Markowa włożonych w chwilach regeneracji. Zakładając stacjonarność rozszerzonego procesu wykorzystujemy w dowodach metodę rozbudowy procesu przedziałami markowskiego do procesu Markowa.