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M. KRAKOWSKI (Łódź)

ON CERTAIN TRANSCENDENTAL FUNCTIONS

0. Summary. Some properties of the functions $\Phi(u, v)$, $\Omega(u, v)$ and $\Psi(u, v)$ defined by equations (1), (2) and (3) are considered. Series expansions, approximate expressions and asymptotic formulae are derived for these functions. Furthermore, integrals and particular solutions of some ordinary differential equations connected with the functions considered are derived. The functions discussed appear in some technical problems. The technical application of the functions Ω and Ψ is illustrated by two examples. [6]

1. General equations. The transcendental functions Ω and Ψ are defined by equations [6]

$$\Omega(u, v) = \frac{1}{2}[e^{-u}\Phi(u, v)+e^{u}\Phi(-u, v)], \tag{1}$$

$$\Psi(u, v) = \frac{1}{2}[e^{-u}\Phi(u, v)-e^{u}\Phi(-u, v)], \tag{2}$$

where

$$\Phi(u, v) = \int_{-u}^{\infty} \frac{e^{-\tau}d\tau}{\sqrt{\tau^2+v^2}} \tag{3}$$

on the assumption that u and $v \neq 0$ are complex.

It is easy to show that $\Omega(u, v)$ is even whereas $\Psi(u, v)$ is odd with respect to u; hence

$$\Omega(-u, v) = \Omega(u, v), \qquad \Psi(-u, v) = -\Psi(u, v).$$

The functions Ω and Ψ occur in problems associated with current propagation along conductors which are in direct contact with the earth. Such problems are considered in [2]-[7], and in other publications.

In technical problems, u and v are real or complex with the requirement that $\arg u = \arg v$. Tables of $\Omega(u, v)$ and $\Psi(u, v)$ are encountered in a number of publications dealing with earth currents. Extensive tables for real u and v are published in [5]. Short tables for complex u and v are given for $\arg u = \arg v = \pi/6$ and $\pi/4$ in [6], whereas for $\arg u = \arg v = 2\pi/9$ in [4].

Some properties of the functions Ω and Ψ are discussed in the sequel.

2. Derivatives. From equation (1) we obtain

$$\frac{\partial \Omega}{\partial u} = \frac{1}{2}\left[-e^{-u}\Phi(u, v) + e^{-u}\frac{\partial \Phi(u, v)}{\partial u} + e^{u}\Phi(-u, v) + e^{u}\frac{\partial \Phi(-u, v)}{\partial u}\right]$$

and substitution of

$$\frac{\partial \Phi(u, v)}{\partial u} = \frac{e^{u}}{\sqrt{u^2+v^2}}, \qquad \frac{\partial \Phi(-u, v)}{\partial u} = -\frac{e^{-u}}{\sqrt{u^2+v^2}},$$

which results from (3), gives

$$\frac{\partial \Omega}{\partial u} = -\Psi(u, v). \tag{4}$$

Similarly, we derive

$$\frac{\partial \Psi}{\partial u} = -\Omega(u, v) + \frac{1}{\sqrt{u^2+v^2}}. \tag{5}$$

Equations (4) and (5) have been derived in [5].

The differential equation for Ω as function of u is

$$\frac{d^2\Omega}{du^2} - \Omega(u, v) = -\frac{1}{\sqrt{u^2+v^2}}$$

which results from (4) and (5). It is easy to verify that boundary conditions for the function Ω are

$$\Omega(0, v) = \int_0^\infty \frac{e^{-\tau}\,d\tau}{\sqrt{\tau^2+v^2}} = \frac{\pi}{2}\left[\boldsymbol{H}_0(v) - Y_0(v)\right] \tag{6}$$

and

$$\left.\frac{\partial \Omega}{\partial u}\right|_{u=0} = -\Psi(0, v) = 0,$$

where $\boldsymbol{H}_0(v)$ is Struve's function of order zero and $Y_0(v)$ is the Bessel function of the second kind, order zero. The integral which appears in (6) results from formula no. 3.387.7 in [1] for $\nu = 1/2$ and $\mu = 1$.

3. Series expansion of function Φ. The function Φ given by (3) may be represented as

$$\Phi(u, v) = \int_{-u}^{0} \frac{e^{-\tau} d\tau}{\sqrt{\tau^2 + v^2}} + \int_{0}^{\infty} \frac{e^{-\tau} d\tau}{\sqrt{\tau^2 + v^2}},$$

whence,

$$\Phi(u, v) = \frac{\pi}{2} [\boldsymbol{H}_0(v) - Y_0(v)] + \int_{0}^{u} \frac{e^x dx}{\sqrt{x^2 + v^2}}, \tag{7}$$

if equation (6) is used. Substitution of the series expansion for e^x into the last integral gives

$$\int_{0}^{u} \frac{e^x dx}{\sqrt{x^2 + v^2}} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{0}^{u} \frac{x^k dx}{\sqrt{x^2 + v^2}}.$$

It is easy to verify that

$$\int_{0}^{u} \frac{x^k dx}{\sqrt{x^2 + v^2}} = \frac{1}{k} \left[u^{k-1} \sqrt{u^2 + v^2} - (k-1) v^2 \int_{0}^{u} \frac{x^{k-2} dx}{\sqrt{x^2 + v^2}} \right],$$

where the first two integrals for $k = 0$ and $k = 1$ are

$$\int_{0}^{u} \frac{x^0 dx}{\sqrt{x^2 + v^2}} = \ln \frac{\sqrt{u^2 + v^2} + u}{v},$$

$$\int_{0}^{u} \frac{x\, dx}{\sqrt{x^2 + v^2}} = \sqrt{u^2 + v^2} - v.$$

Using the equations derived, we obtain

$$\int_{0}^{u} \frac{e^x dx}{\sqrt{x^2 + v^2}} = J_0(v) \ln \frac{\sqrt{u^2 + v^2} + u}{v} - \frac{\pi}{2} \boldsymbol{H}_0(v) + \sum_{k=1}^{\infty} \frac{A_k}{k!}, \tag{8}$$

since the terms involving the logarithm constitute the Bessel function $J_0(v)$ and the terms including v^k only constitute Struve's function $\boldsymbol{H}_0(v)$. The coefficients A_k which appear in (8) are determined by

$$A_0 = 0,\ A_1 = \sqrt{u^2 + v^2},\ \ldots,\ A_k = \frac{1}{k} [u^{k-1} \sqrt{u^2 + v^2} - (k-1) v^2 A_{k-2}]. \tag{9}$$

Inserting equation (8) into (7), we have

$$\Phi(u, v) = J_0(v)\ln\frac{\sqrt{u^2+v^2}+u}{v} - \frac{\pi}{2}Y_0(v) + \sum_{k=1}^{\infty}\frac{A_k}{k!}, \tag{10}$$

whence, retaining the first terms in the series expansion, we obtain

$$\Phi(u, v) = \left(1-\frac{v^2}{4}\right)\ln\frac{\sqrt{u^2+v^2}+u}{v} - \left(1-\frac{v^2}{4}\right)\ln\frac{\varkappa v}{2} - \tag{11}$$

$$-\frac{v^2}{4} + \sqrt{u^2+v^2} + \frac{1}{4}u\sqrt{u^2+v^2} + \ldots,$$

where $\varkappa = 1{,}7811\ldots$ ($\ln\varkappa = 0{,}5772\ldots$ is Euler's constant). Equations (9) and (10) are given in [6].

When $|u|$ is sufficiently large in comparison with $|v|$, which is assumed to be sufficiently small, we have approximately

$$A_k = \frac{u^k}{k}, \qquad k = 1, 2, 3, \ldots,$$

according to (9). Then, by use of the series expansions of the Bessel functions $J_0(v)$ and $Y_0(v)$, we obtain

$$\Phi(u, v) \approx \ln\frac{1{,}12(\sqrt{u^2+v^2}+u)}{v^2} + \sum_{k=1}^{\infty}\frac{u^k}{k.\,k!}. \tag{12}$$

We retain the term $\sqrt{u^2+v^2}$ on the right-hand side of equation (12), which permits to calculate $\Phi(-u, v)$. The series which appears in (12) can be expressed in terms of the exponential integral function $Ei(u)$.

4. Approximate expressions for functions Ω and Ψ. From relationship (11) we derive

$$e^{-u}\Phi(u, v) = \left(1-u+\frac{u^2}{2}-\frac{v^2}{4}\right)\left(\ln\frac{\sqrt{u^2+v^2}+u}{v}+\ln\frac{1{,}12}{v}\right) -$$

$$-\frac{v^2}{4} + \sqrt{u^2+v^2} - \frac{3}{4}u\sqrt{u^2+v^2} + \ldots$$

if the series expansion of the exponential function is used. Hence,

$$e^{u}\Phi(-u, v) = -\left(1+u+\frac{u^2}{2}-\frac{v^2}{4}\right)\left(\ln\frac{\sqrt{u^2+v^2}+u}{v}-\ln\frac{1{,}12}{v}\right) -$$

$$-\frac{v^2}{4} + \sqrt{u^2+v^2} + \frac{3}{4}u\sqrt{u^2+v^2} + \ldots$$

and substitution into equations (1) and (2) gives

$$\Omega(u, v) = \left(1 + \frac{u^2}{2} - \frac{v^2}{4}\right) \ln \frac{1,12}{v} - u \ln \frac{\sqrt{u^2+v^2}+u}{v} - \frac{v^2}{4} + \sqrt{u^2+v^2} + \ldots,$$

$$\Psi(u, v) = \left(1 + \frac{u^2}{2} - \frac{v^2}{4}\right) \ln \frac{\sqrt{u^2+v^2}+u}{v} - u \ln \frac{1,12}{v} - \frac{3}{4} u \sqrt{u^2+v^2} + \ldots$$

These expressions may be used if $|u|$ and $|v|$ are sufficiently small.

5. Asymptotic formulae. Integration by parts of the integral (3) gives

$$\Phi(u, v) \sim \frac{e^u}{(u^2+v^2)^{1/2}} + \frac{ue^u}{(u^2+v^2)^{3/2}} + \frac{(2u^2-v^2)e^u}{(u^2+v^2)^{5/2}} + \frac{3u(2u^2-3v^2)e^u}{(u^2+v^2)^{7/2}} + \ldots$$

which is the asymptotic formula for the function Φ. Hence,

$$e^{-u}\Phi(u, v) \sim \frac{1}{(u^2+v^2)^{1/2}} + \frac{u}{(u^2+v^2)^{3/2}} + \frac{2u^2-v^2}{(u^2+v^2)^{5/2}} + \frac{3u(2u^2-3v^2)}{(u^2+v^2)^{7/2}} + \ldots$$

and

$$e^{u}\Phi(-u, v) \sim \frac{1}{(u^2+v^2)^{1/2}} - \frac{u}{(u^2+v^2)^{3/2}} + \frac{2u^2-v^2}{(u^2+v^2)^{5/2}} - \frac{3u(2u^2-3v^2)}{(u^2+v^2)^{7/2}} + \ldots$$

Inserting these equations into (1) and (2), we derive asymptotic formulae for the functions Ω and Ψ, namely

$$\begin{aligned} &\Omega(u, v) \sim \frac{1}{(u^2+v^2)^{1/2}} + \frac{2u^2-v^2}{(u^2+v^2)^{5/2}} + \ldots, \\ &\Psi(u, v) \sim u\left[\frac{1}{(u^2+v^2)^{3/2}} + \frac{3(2u^2-3v^2)}{(u^2+v^2)^{7/2}} + \ldots\right]. \end{aligned} \tag{13}$$

From the first relationship of (13) we obtain

$$\Omega(0, v) \sim \frac{1}{v} - \frac{1}{v^3} + \ldots$$

6. Integrals. The indefinite integrals of $\Omega(u, v)$ and $\Psi(u, v)$ are equal to

$$\int \Omega(u, v)\,du = -\Psi(u, v) + \ln(u + \sqrt{u^2+v^2}),$$

$$\int \Psi(u, v)\,du = -\Omega(u, v),$$

which follows from (4) and (5).

It is easy to show on the basis of equations (1) and (2) that the functions Ω and Ψ may be expressed as

$$\Omega(u,v) = \frac{1}{2}\left[\int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z-u)^2}} + \int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z+u)^2}}\right],$$

$$\Psi(u,v) = \frac{1}{2}\left[\int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z-u)^2}} - \int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z+u)^2}}\right],$$

whence,

$$\int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z-u)^2}} = \Omega(u,v)+\Psi(u,v),$$

(14)

$$\int_0^\infty \frac{e^{-z}dz}{\sqrt{v^2+(z+u)^2}} = \Omega(u,v)-\Psi(u,v).$$

The integral (6) results from (14) as a special case for $u = 0$.

It may be shown that

$$\int_{-\infty}^{+\infty} \frac{e^{-\Gamma|x\pm u|}}{\sqrt{u^2+a^2}}\,du = 2\Omega(\Gamma x, \Gamma a),$$

(15)

$$\int_{-\infty}^{+\infty} \frac{x\pm u}{|x\pm u|}\,\frac{e^{-\Gamma|x\pm u|}}{\sqrt{u^2+a^2}}\,du = 2\Psi(\Gamma x, \Gamma a)$$

for real x, $a > 0$ and $\mathrm{Re}(\Gamma) > 0$; e.g.

$$\int_{-\infty}^{+\infty} \frac{e^{-\Gamma|x+u|}}{\sqrt{u^2+a^2}}\,du = e^{\Gamma x}\int_{-\infty}^{-x} \frac{e^{-\Gamma u}du}{\sqrt{u^2+a^2}} + e^{-\Gamma x}\int_{-x}^{\infty} \frac{e^{-\Gamma u}du}{\sqrt{u^2+a^2}}$$

and substitution of $u = -\tau/\Gamma$ and $u = \tau/\Gamma$, respectively, into the integrals on the right-hand side of the last equation gives the first relationship of (15) if (1) is used.

Consider the integral

$$\int_z^\infty \Omega(u,v)e^{-au}du,$$

where $\mathrm{Re}(a) > 0$, $\mathrm{Re}(v) > 0$ and z is complex. Integrating the last integral by parts and considering (4), we have

$$\int_z^\infty \Omega(u,v)e^{-au}du = \frac{e^{-az}}{a}\,\Omega(z,v) - \frac{1}{a}\int_z^\infty \Psi(u,v)e^{-au}du.$$

Integration by parts of the integral which includes the function Ψ gives

$$\int_z^\infty \Omega(u, v)e^{-au}du = \frac{e^{-az}}{a}\Omega(z, v) - \frac{e^{-az}}{a^2}\Psi(z, v) + $$
$$+\frac{1}{a^2}\int_z^\infty \Omega(u, v)e^{-au}du - \frac{1}{a^2}\Phi(-az, av)$$

if (5) is used. Hence,

$$(16) \qquad \int_z^\infty \Omega(u, v)e^{-au}du$$
$$= \frac{a}{a^2-1}\left[e^{-az}\Omega(z, v) - \frac{e^{-az}}{a}\Psi(z, v) - \frac{1}{a}\Phi(-az, av)\right],$$

when $a \neq 1$.

We obtain similarly

$$(17) \qquad \int_{-\infty}^z \Omega(u, v)e^{au}du$$
$$= \frac{a}{a^2-1}\left[e^{az}\Omega(z, v) + \frac{e^{az}}{a}\Psi(z, v) - \frac{1}{a}\Phi(az, av)\right]$$

and

$$\int_z^\infty \Psi(u, v)e^{-au}du = \frac{a}{a^2-1}\left[e^{-az}\Psi(z, v) - \frac{e^{-az}}{a}\Omega(z, v) + \Phi(-az, av)\right],$$

$$\int_{-\infty}^z \Psi(u, v)e^{au}du = \frac{a}{a^2-1}\left[e^{az}\Psi(z, v) + \frac{e^{az}}{a}\Omega(z, v) - \Phi(az, av)\right]$$

for $\mathrm{Re}(a) > 0$, $\mathrm{Re}(v) > 0$ and complex z.

The integral

$$\int_{-\infty}^{+\infty} \Omega(\Gamma_1 u, \Gamma_1 a)e^{-\Gamma_2|x-u|}du$$
$$= e^{-\Gamma_2 x}\int_{-\infty}^x \Omega(\Gamma_1 u, \Gamma_1 a)e^{\Gamma_2 u}du + e^{\Gamma_2 x}\int_x^\infty \Omega(\Gamma_1 u, \Gamma_1 a)e^{-\Gamma_2 u}du$$

can be expressed in terms of (16) and (17) by substitution of $\eta = \Gamma_1 u$, and as the result we obtain

(18)

By use of the integrals which have been considered previously, we obtain

$$(19)\qquad \int_{-\infty}^{+\infty} \Psi(\Gamma_1 u, \Gamma_1 a) e^{-\Gamma_2|x-u|} du = \frac{2\Gamma_2}{\Gamma_2^2-\Gamma_1^2}[\Psi(\Gamma_1 x, \Gamma_1 a)-\Psi(\Gamma_2 x, \Gamma_2 a)],$$

$$(20)\qquad \int_{-\infty}^{+\infty} \frac{x-u}{|x-u|}\Omega(\Gamma_1 u, \Gamma_1 a) e^{-\Gamma_2|x-u|} du = \frac{2\Gamma_1}{\Gamma_2^2-\Gamma_1^2}[\Psi(\Gamma_1 x, \Gamma_1 a)-\Psi(\Gamma_2 x, \Gamma_2 a)],$$

$$(21)\qquad \int_{-\infty}^{+\infty} \frac{x-u}{|x-u|}\Psi(\Gamma_1 u, \Gamma_1 a) e^{-\Gamma_2|x-u|} du = \frac{2}{\Gamma_2^2-\Gamma_1^2}[\Gamma_1\Omega(\Gamma_1 x, \Gamma_1 a)-\Gamma_2\Omega(\Gamma_2 x, \Gamma_2 a)].$$

Equations (18)-(21) are valid for $\Gamma_1 \neq \Gamma_2$, $\mathrm{Re}(\Gamma_1) > 0$, $\mathrm{Re}(\Gamma_2) > 0$, $a > 0$ and real x,

Now, consider the integral

$$(22)\qquad \int_0^{\infty} \frac{K_0(au)\cos xu}{u^2+\Gamma^2} du = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{K_0(a|u|)}{u^2+\Gamma^2} e^{ixu} du,$$

where $K_0(au)$ is the modified Bessel function of second kind and order zero, whereas x is real, $a > 0$ and $\mathrm{Re}(\Gamma) > 0$. Using an integral representation of $K_0(au)$, the integral (22) becomes

$$\frac{1}{4}\int_{-\infty}^{+\infty} \frac{e^{ixu}}{u^2+\Gamma^2}\left[\int_{-\infty}^{+\infty} \frac{e^{iuv}\,dv}{\sqrt{v^2+a^2}}\right] du$$
$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{v^2+a^2}}\left[\int_0^{\infty} \frac{\cos(x+v)u}{u^2+\Gamma^2} du\right] dv = \frac{\pi}{4\Gamma}\int_{-\infty}^{+\infty} \frac{e^{-\Gamma|x+v|}}{\sqrt{v^2+a^2}} dv,$$

according to formula no. 3.723.2 in [1]. Hence,

$$(23)\qquad \int_0^{\infty} \frac{K_0(au)\cos xu}{u^2+\Gamma^2} du = \frac{\pi}{2\Gamma}\Omega(\Gamma x, \Gamma a)$$

if the first equation of (15) is applied.

Similarly, we obtain

$$\int_0^\infty \frac{uK_0(au)\sin xu}{u^2+\Gamma^2}\,du = \frac{\pi}{2}\Psi(\Gamma x, \Gamma a). \tag{24}$$

Equations (23) and (24) are valid for real $x, a > 0$ and $\mathrm{Re}(\Gamma) > 0$. Integral (23) has been derived in [7] for real Γ.

7. Differential equations. It is possible to obtain particular solutions of some differential equations.

A particular solution of the equation

$$\frac{d^2y}{dx^2} - \Gamma^2 y(x) = \frac{c}{\sqrt{x^2+a^2}}$$

is given by

$$y(x) = -\frac{c}{2\Gamma}\int_{-\infty}^{+\infty} \frac{e^{-\Gamma|x-v|}\,dv}{\sqrt{v^2+a^2}} = -\frac{c}{\Gamma}\Omega(\Gamma x, \Gamma a)$$

for real x, complex c, $a > 0$ and $\mathrm{Re}(\Gamma) > 0$ if the first equation of (15) is used.

A particular solution of the equation

$$\frac{d^2y}{dx^2} - \Gamma^2 y(x) = \frac{cx}{(x^2+a^2)^{3/2}} \tag{25}$$

takes the form

$$y(x) = -c\Psi(\Gamma x, \Gamma a) \tag{26}$$

under the same assumptions as previously.

Particular solutions of the equations

$$\frac{d^2y}{dx^2} - \Gamma_2^2 y(x) = c\Omega(\Gamma_1 x, \Gamma_1 a),$$

$$\frac{d^2y}{dx^2} - \Gamma_2^2 y(x) = c\Psi(\Gamma_1 x, \Gamma_1 a)$$

are

$$\begin{aligned} y(x) &= \frac{c}{\Gamma_2(\Gamma_1^2-\Gamma_2^2)}[\Gamma_2\Omega(\Gamma_1 x, \Gamma_1 a) - \Gamma_1\Omega(\Gamma_2 x, \Gamma_2 a)], \\ y(x) &= \frac{c}{\Gamma_1^2-\Gamma_2^2}[\Psi(\Gamma_1 x, \Gamma_1 a) - \Psi(\Gamma_2 x, \Gamma_2 a)], \end{aligned} \tag{27}$$

respectively, for real x, complex c, $a > 0$, $\mathrm{Re}(\Gamma_1) > 0$, $\mathrm{Re}(\Gamma_2) > 0$ and $\Gamma_1 \neq \Gamma_2$. It should be noted that equations (27) are derived by use of (18) and (19).

8. Applications.

8.1. *Currents and potentials along underground conductors.* Let us consider an underground conductor carrying the longitudal current $I(x)$ which flows in the positive direction of the x-axis lying along this conductor. We denote by $V(x)$ the potential which is produced along the conductor by $I(x)$. It is assumed that currents and potentials vary with time as $\exp(i\omega t)$, where ω is the angular frequency.

It is shown in [2] that approximate relationships between $I(x)$ and $V(x)$ form the differential equations

$$\frac{dV}{dx}+ZI(x) = E^0(x), \quad \frac{dI}{dx} = -YV(x), \tag{28}$$

where Z and Y are the unit-length impedance and leakage conductance of the underground conductor, respectively, whereas $E_0(x)$ is the electric intensity impressed along the conductor by external factors. When the conductor is buried in the vicinity of a point earth electrode which is placed on the earth surface at a point opposite to $x = 0$, the impressed electric intensity becomes

$$E^0(x) = \frac{I_0}{2\pi\gamma}\frac{x}{(x^2+a^2)^{3/2}}, \tag{29}$$

where I_0 denotes the current which leaves the earth electrode, γ — earth conductivity, and a — distance between earth electrode and conductor.

Elimination of $V(x)$ from (28) with (29) gives

$$\frac{d^2I}{dx^2}-\Gamma^2 I(x) = -\frac{YI_0}{2\pi\gamma}\frac{x}{(x^2+a^2)^{3/2}}, \tag{30}$$

where $\Gamma = \sqrt{ZY}$ with the requirement that $\mathrm{Re}(\Gamma) > 0$ is the propagation coefficient of the conductor.

Equation (30) is also valid for the conductor which lies on the earth surface, as has been shown in [6].

Now $I(x)$ becomes

$$I(x) = \frac{YI_0}{2\pi\gamma}\Psi(\Gamma x, \Gamma a) \tag{31}$$

as the solution of differential equation (25) which is given by (26) with $t = -YI_0/2\pi\gamma$.

The potential impressed by $I(x)$ in the conductor is derived from second equation of (28). Hence,

$$V(x) = -\frac{1}{Y}\frac{dI}{dx} = \frac{\Gamma I_0}{2\pi\gamma}\left[\Omega(\Gamma x, \Gamma a)-\frac{1}{\Gamma\sqrt{x^2+a^2}}\right]$$

if equation (5) is applied.

8.2. *Voltage between sheath and cable conductor.* The longitudal current $I(x)$ flows in the sheath of an underground cable. This current produces the voltage between the sheath and a cable wire as (see [3])

$$(32) \qquad U(x) = \frac{Z_s}{2}\left[e^{\Gamma_c x}\int_x^{\infty} I(u)e^{-\Gamma_c u}du - e^{-\Gamma_c x}\int_{-\infty}^{x} I(u)e^{\Gamma_c u}du\right],$$

where $\Gamma_c = \sqrt{(G+i\omega C)Z_c}$, $\mathrm{Re}(\Gamma_c) > 0$, Z_s is unit-length internal impedance of cable sheath, Z_c — unit-length impedance of cable wire, G — unit-length leakage conductance of cable insulation, C — unit-length capacitance between sheath and cable wire.

Equation (32) may be represented as

$$(33) \qquad U(x) = -\frac{Z_s}{2}\int_{-\infty}^{+\infty}\frac{x-u}{|x-u|} I(u)e^{-\Gamma_c|x-u|}du.$$

If the cable is burried near the point earth electrode which lies on the earth surface, the current in the sheath is given by (31). Hence,

$$U(x) = -\frac{YZ_sI_0}{4\pi\gamma}\int_{-\infty}^{+\infty}\frac{x-u}{|x-u|}\Psi(\Gamma u, \Gamma a)e^{-\Gamma_c|x-u|}du$$

and substitution of (21) with $\Gamma_1 = \Gamma$ and $\Gamma_2 = \Gamma_c$ gives

$$U(x) = \frac{YZ_s\Gamma I_0}{2\pi\gamma(\Gamma^2-\Gamma_c^2)}\left[\Omega(\Gamma x, \Gamma a) - \frac{\Gamma_c}{\Gamma}\Omega(\Gamma_c x, \Gamma_c a)\right].$$

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TECHNICAL UNIVERSITY
ŁÓDŹ

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M. KRAKOWSKI (Łódź)

O PEWNYCH FUNKCJACH TRANSCENDENTALNYCH

STRESZCZENIE

W pracy omówiono szereg własności funkcji $\Phi(u,v)$, $\Omega(u,v)$ i $\Psi(u,v)$ określonych przez równania (1), (2) i (3). Wyprowadzono szeregi potęgowe, wyrażenia przybliżone oraz wzory asymptotyczne dla tych funkcji. Wyznaczono kilka całek oraz podano rozwiązania szczególne kilku równań różniczkowych zwyczajnych, związanych z funkcjami Ω i Ψ. Rozpatrywane funkcje znajdują zastosowanie w niektórych zagadnieniach technicznych. Podano dwa przykłady ilustrujące zastosowanie praktyczne funkcji Ω i Ψ.

М. КРАКОВСКИ (Лодзь)

О НЕКОТОРЫХ ТРАНСЦЕНДЕНТНЫХ ФУНКЦИЯХ

РЕЗЮМЕ

В статье рассмотрены свойства функций $\Phi(u,v)$, $\Omega(u,v)$ и $\Psi(u,v)$ определенных уравнениями (1), (2) и (3). Для этих функций получено представления в видее степенных рядов, приближенные формулы и асимптотические разложения. Определено некоторые интегралы и получено частные решения некоторых обыкновенных дифференциальных уравнений связанных с функциями Ω и Ψ. Рассматриваемые функции применяются в некоторых технических проблемах. Технические применения функций Ω и Ψ иллюстрированы двумя примерами.
