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EXPLICIT FORMS OF BLOCK MATRICES IN UNBALANCED CROSS CLASSIFICATION

1. Introduction. The mathematical linear model of cross classification can be written as

$$\boldsymbol{y} = \sum_{u=1}^{z} \boldsymbol{X}_u \boldsymbol{\beta}_u + \boldsymbol{e},$$

where $\boldsymbol{y}$ is an $(N\times 1)$-vector of observations, $\boldsymbol{e}$ is an $(N\times 1)$-vector of errors, $\boldsymbol{\beta}_u$ is a vector of main or interaction effects, and $\boldsymbol{X}_u$ is a matrix that contains only zeros and ones.

The forms of block matrices $\boldsymbol{X}_u$ $(u = 1, 2, \ldots, z)$ are rather complicated and depend on the statistical design, the ordering of elements in the vector $\boldsymbol{y}$, the kind of effects in the vector $\boldsymbol{\beta}_u$, and on the ordering of effects in the vector $\boldsymbol{\beta}_u$. The block matrices $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_z$ as ordered Kronecker products of identity matrices and column vectors of ones in any case of the balanced mathematical model of cross and hierarchical classifications and in case of an arbitrary combination of these classifications are presented by Oktaba in [1]-[3]. The generalization of the results for the case of any unbalanced hierarchical classification is stated in [4] and [5].

The generalization for the case of any unbalanced cross classification is given in this paper.

2. The general linear model. Consider a cross classification design with k classifications A_s $(s = 1, 2, \ldots, k)$ such that the s-th classification A_s has a_s classes. Then the number of all combinations of classes is $t = a_1 a_2 \ldots a_k$. For the sake of uniqueness of representation of block matrices, it s necessary to order the classifications, observations and main and interaction effects.

Classifications are ordered arbitrarily and the order is kept in the indices of the classes of classifications.

The order of the combinations of the classes is given by the following relation between the numbers of classes and the order subscript:

$$v = \sum_{s=1}^{k-1} \Big[\Big(\prod_{n=s+1}^{k} a_n\Big)(j_s-1)\Big]+j_k, \tag{1}$$

where j_s denotes the order number of the class of the classification A_s.

Let y_{vi} be the i-th observation made on the v-th class combination, where $i = 1, 2, \ldots, n_v$ ($n_v \geqslant 1$ and some of the $n_v \geqslant 2$), and let N be the total number of observations. Then y_{vi} may be written as

$$y_{vi} = \eta_v + \varepsilon_{vi}, \tag{2}$$

where

$$\eta_v = \mu + \sum_{s=1}^{k} a_s(j_s) + \sum_{\substack{r \\ 1\leqslant r<s\leqslant k}}\sum_{s} a_{rs}(j_r, j_s) + \ldots + a_{12\ldots k}(j_1, j_2, \ldots, j_k) \tag{3}$$

and μ, $a_s(j_s)$, $a_{rs}(j_r, j_s)$, and $a_{12\ldots k}(j_1, j_2, \ldots, j_k)$ denote the general mean, the main effect, the interaction effect of the two classifications, and the interaction effect of the k classifications, respectively.

Let $\boldsymbol{y}$ be the column vector with elements y_{vi} ordered as

$$\boldsymbol{y}' = [y_{11}, y_{12}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{2n_2}, \ldots, y_{t1}, \ldots, y_{tn_t}], \tag{4}$$

where the prime indicates the transposition of the vector. Using matrix notation, we can write formulas (2) and (3) in the form

$$\boldsymbol{y} = \boldsymbol{X}_0\mu + \sum_{s=1}^{k} \boldsymbol{X}_{A_s}\boldsymbol{a}_{A_s} + \sum_{\substack{r \\ 1\leqslant r<s\leqslant k}}\sum_{s} \boldsymbol{X}_{A_rA_s}\boldsymbol{a}_{A_rA_s} + \ldots + \boldsymbol{X}_{A_1A_2\ldots A_k}\boldsymbol{a}_{A_1A_2\ldots A_k} + \boldsymbol{e}, \tag{5}$$

where $\boldsymbol{a}_{A_s}$, $\boldsymbol{a}_{A_rA_s}$, and $\boldsymbol{a}_{A_1A_2\ldots A_k}$ denote vectors with the elements that are the main effects, the interaction effects of the two classifications, and the interaction effects of the k classifications, respectively, and $\boldsymbol{X}_0, \boldsymbol{X}_{A_s}, \boldsymbol{X}_{A_rA_s}, \boldsymbol{X}_{A_1A_2\ldots A_k}$ are matrices that contain only zeros and ones placed so that (2) and (3) are true.

It is easy to verify (see (3)) that $\boldsymbol{X}_0$ is the $(N\times 1)$- matrix all elements of which are ones. For the sake of uniqueness of the formulas for the other matrices it is necessary to lay down a rule for the ordering of elements of the vectors $\boldsymbol{a}_{A_s}, \boldsymbol{a}_{A_rA_s}, \ldots, \boldsymbol{a}_{A_1A_2\ldots A_k}$.

Let $i_1, i_2, \ldots, i_r$ be the subsequence chosen from the sequence $1, 2, \ldots, k$ (i.e., $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k$) and let $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ be the corresponding subsequence chosen from $A_1, A_2, \ldots, A_k$. The symbol $\boldsymbol{a}_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ is reserved for the vector of the interaction effects of r classifications $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$. For $r = 1$ the symbol $\boldsymbol{a}_{A_{i_1}}$ denotes the vector of the main effects of the classification A_{i_1} ($i_1 = 1, 2, \ldots, k$). The interaction effects of the classes $j_{i_1}, j_{i_2}, \ldots, j_{i_r}$ of the classifications $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ are ordered within the vector $\boldsymbol{a}_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ by the order subscript

$$(6) \qquad w_{i_1 i_2 \ldots i_r} = a_{i_2} a_{i_3} \ldots a_{i_r}(j_{i_1}-1) + a_{i_3} a_{i_4} \ldots a_{i_r}(j_{i_2}-1) + \\ + \ldots + a_{i_r}(j_{i_{r-1}}-1) + j_{i_r} \quad (r = 2, 3, \ldots, k).$$

The main effects of the classes j_{i_1} of the classification A_{i_1} are ordered within the vector $\boldsymbol{a}_{A_1}$ by the order subscript

$$(7) \qquad w_{i_1} = j_{i_1}.$$

It is easy to verify that for $r = k$ formula (6) is the same as (1). Then the interaction effects of the k classifications $A_1, A_2, \ldots, A_k$ are ordered in the same manner as the combinations of the classes of the classifications $A_1, A_2, \ldots, A_k$. Hence the matrix $\boldsymbol{X}_{A_1A_2\ldots A_k}$ is $N \times t$ and can be found in the following way.

Rows of the matrix $\boldsymbol{X}_{A_1A_2\ldots A_k}$ can be identified by a pair of numbers (v, i), where v is the number of the combinations of classes and i is the number of the observations in the v-th combination. Columns of the matrix can be identified by the order subscript $w_{12\ldots k}$. Hence the element in the (v, i)-th row and in the $w_{12\ldots k}$-th column of the matrix $\boldsymbol{X}_{A_1A_2\ldots A_k}$ is equal to one if $v = w_{12\ldots k}$, and is equal to zero if $v \neq w_{12\ldots k}$. Then

$$(8) \qquad \boldsymbol{X}_{A_1A_2\ldots A_k} = \begin{bmatrix} \boldsymbol{J}_{n_1} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{n_2} & \ldots & \boldsymbol{0} \\ \ldots & \ldots & \ldots & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{J}_{n_t} \end{bmatrix},$$

where $\boldsymbol{J}_{n_s}$ denotes the column vector of n_s ones, and $\boldsymbol{0}$ denotes the column vector of zeros. The following theorem gives the method of constructing the other submatrices.

THEOREM. *In the mathematical model* (5) *for any k-way cross classification* $A_1 \times A_2 \times \ldots \times A_k$ ($k = 2, 3, \ldots$) *the submatrix* $\boldsymbol{X}_{A_1A_2\ldots A_r}$ *for the vector* $\boldsymbol{a}_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ ($r = 1, 2, \ldots, k-1$) *of the interaction effects is given by*

$$\boldsymbol{X}_{A_{i_1}A_{i_2}\ldots A_{i_r}} = \boldsymbol{X}_{A_1A_2\ldots A_k} \boldsymbol{X}^*_{A_{i_1}A_{i_2}\ldots A_{i_r}},$$

where

$$\boldsymbol{X}^*_{A_{i_1}A_{i_2}\ldots A_{i_r}} = \boldsymbol{J}_{a_1} \otimes \boldsymbol{J}_{a_2} \otimes \ldots \otimes \boldsymbol{J}_{a_{i_1-1}} \otimes \boldsymbol{I}_{a_{i_1}} \otimes \boldsymbol{J}_{a_{i_1+1}} \otimes \ldots \\ \ldots \otimes \boldsymbol{J}_{a_{i_2-1}} \otimes \boldsymbol{I}_{a_{i_2}} \otimes \boldsymbol{J}_{a_{i_2+1}} \otimes \ldots \otimes \boldsymbol{J}_{a_{i_r-1}} \otimes \boldsymbol{I}_{a_{i_r}} \otimes \boldsymbol{J}_{a_{i_r+1}} \otimes \ldots \otimes \boldsymbol{J}_{a_k},$$

and the matrix $X_{A_1A_2\ldots A_k}$ *is given by* (8), I_{a_s} *denotes the identity matrix* $a_{i_s}\times a_{i_s}$, $J_{a_{i_s}}$ *is the column vector of* a_{i_s} *ones, and the symbol* $\otimes$ *denotes the Kronecker product of matrices.*

Proof. Consider the cross classification $A_1\times A_2\times\ldots\times A_k$ with one observation in each of the t subclasses. Let observations y_{vi} be ordered as in (4) and let the main and interaction effects be ordered as in (6) and (7). Using the method of mathematical induction with respect to the number of classifications k, it can be proved that the submatrix $X_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ for the vector $a_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ of the interaction effects is equal to $X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ (see [1]). It follows immediately from (2)-(4) that the submatrix $X_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ in the unbalanced cross classification $A_1\times A_2\times\ldots\times A_k$ can be obtained from the matrix $X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ by repeating n_v times the row of the matrix $X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ with the order subscript v $(v=1,2,\ldots,t)$. It is easy to verify (see (8)) that this type of operation on the matrix $X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ can be realized by pre-multiplication of the matrix $X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}$ by the matrix $X_{A_1A_2\ldots A_k}$.

It is easy to verify that if all n_v are equal, say, to n, i.e. if the data are balanced, then (see [2])

$$X_{A_1A_2\ldots A_k}=I_t\otimes J_n \quad\text{and}\quad X_{A_{i_1i_2\ldots i_r}}=X^*_{A_{i_1}A_{i_2}\ldots A_{i_r}}\otimes J_n.$$

3. Example. Consider a 3-way cross classification design with the classifications A_1, A_2, A_3 such that the classification A_s $(s=1,2,3)$ has a_s classes. Combinations of the classes are ordered by the order subscript

$$v=a_2a_3(j_1-1)+a_3(j_2-1)+j_3,$$

where j_s denotes the order number of the class of the classification A_s. The vectors of the main and interaction effects are the following (see (6) and (7)):

$$\begin{aligned}
a'_{A_s} &= [\alpha_s(1),\alpha_s(2),\ldots,\alpha_s(a_s)] \quad (s=1,2,3),\\
a'_{A_rA_s} &= [\alpha_{rs}(1,1),\alpha_{rs}(1,2),\ldots,\alpha_{rs}(1,a_s),\alpha_{rs}(2,1),\\
&\quad \alpha_{rs}(2,2),\ldots,\alpha_{rs}(2,a_s),\ldots,\alpha_{rs}(a_r,1),\\
&\quad \alpha_{rs}(a_r,2),\ldots,\alpha_{rs}(a_r,a_s)] \quad (1\leqslant r<s\leqslant 3),\\
a'_{A_1A_2A_3} &= [\alpha_{123}(1,1,1),\ldots,\alpha_{123}(1,1,a_3),\alpha_{123}(1,2,1),\ldots,\\
&\quad \alpha_{123}(1,2,a_3),\ldots,\alpha_{123}(1,a_2,1),\ldots,\alpha_{123}(1,a_2,a_3),\\
&\quad \ldots,\alpha_{123}(a_1,1,1),\ldots,\alpha_{123}(a_1,1,a_3),\ldots,\\
&\quad \alpha_{133}(a_1,a_2,1),\alpha_{123}(a_1,a_2,2),\ldots,\alpha_{123}(a_1,a_2,a_3)].
\end{aligned}$$

Then

$$
\begin{aligned}
X_{A_1} &= X_{A_1A_2A_3}(I_{a_1}\otimes J_{a_2}\otimes J_{a_3}),\\
X_{A_2} &= X_{A_1A_2A_3}(J_{a_1}\otimes I_{a_2}\otimes J_{a_3}),\\
X_{A_1A_2} &= X_{A_1A_2A_3}(I_{a_1}\otimes I_{a_2}\otimes J_{a_3}),\\
X_{A_3} &= X_{A_1A_2A_3}(J_{a_1}\otimes J_{a_2}\otimes I_{a_3}),\\
X_{A_1A_3} &= X_{A_1A_2A_3}(I_{a_1}\otimes J_{a_2}\otimes I_{a_3}),\\
X_{A_2A_3} &= X_{A_1A_2A_3}(J_{a_1}\otimes I_{a_2}\otimes I_{a_3}).
\end{aligned}
$$

References

[1] W. Oktaba, *Iloczyn kroneckerowski macierzy w analizie wariancji dla zrównoważonych modeli matematycznych*, Trzecie Colloquium Metodologiczne z Agro-Biometrii, Warszawa 1973, p. 6-44.

[2] — *Explicit forms of block matrices in the analysis of variance*, Biometrische Z. 17 (1975), p. 27-32.

[3] — *Iloczyn kroneckerowski macierzy blokowych*, Piąte Colloquium Metodologiczne z Agro-Biometrii, Warszawa 1975, p. 8-27.

[4] — *Kronecker products of block matrices*, Bull. Acad. Polon. Sci., Sér. sci. math. astr. phys., 23 (1975), p. 587-592.

[5] — *Some properties of Kronecker matrix products*, Biometrische Z. 17 (1975), p. 475-485.

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MACIERZE BLOKOWE W MODELU LINIOWYM NIEZRÓWNOWAŻONEJ KLASYFIKACJI KRZYŻOWEJ

STRESZCZENIE

W pracy podano twierdzenie umożliwiające znalezienie wyraźnej postaci macierzy blokowych $X_1, X_2, \ldots, X_z$, odpowiadających efektom głównym i interakcyjnym w modelu liniowym niezrównoważonej klasyfikacji krzyżowej. Podane wzory stanowią uogólnienie wyników Oktaby [2], uzyskanych dla zrównoważonej klasyfikacji krzyżowej.
