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DETERMINATION OF MATRICES OF THE DESIRED 2-D CHARACTERISTIC POLYNOMIAL

Abstract. Sufficient conditions for the existence of matrices of the 2-D characteristic polynomial so that it is equal to the desired one are established. An algorithm for finding these matrices is proposed and illustrated by a numerical example.

1. Problem formulation. Let R_{mn} be the set of $(m \times n)$-matrices with entries from the real number field R. The two-dimensional (2-D) characteristic polynomial $p(z_1, z_2)$ of matrices $A, B, C \in R_{nn}$ is defined as (see [3])

$$p(z_1, z_2) := \det[I_n z_1 z_2 - A - Bz_1 - Cz_2],$$

where I_n is the identity matrix and z_1, z_2 are indeterminate (variables).

Let the desired 2-D characteristic polynomial be of the form

$$d(z_1, z_2) = \sum_{i=0}^{n} \sum_{j=0}^{n} d_{ij} z_1^i z_2^j \quad (d_{nn} = 1). \tag{1}$$

The problem can be stated as follows: Given $d(z_1, z_2)$, find matrices $A, B, C \in R_{nn}$ such that

$$p(z_1, z_2) = d(z_1, z_2). \tag{2}$$

This problem is a starting point, e.g., in the coefficient assignment problem (see [1] and [2]).

2. Problem solution. Note that the number of the given coefficients d_{ij} of (1) is equal to $(n+1)^2 - 1 = n^2 + 2n$. Therefore, we can assume the matrices A, B, C are of the form

$$A = -\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}, \quad B = -\begin{bmatrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}, \tag{3}$$

$$C = -\begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}.$$

THEOREM. *There exist matrices A, B, C of the form* (3) *such that* (2) *holds if the equations*

$$x^n - d_{n,n-1}x^{n-1} + d_{n,n-2}x^{n-2} + \dots + (-1)^n d_{n,0} = 0, \tag{4}$$

$$x^n - d_{n-1,n}x^{n-1} + d_{n-2,n}x^{n-2} + \dots + (-1)^n d_{0,n} = 0 \tag{5}$$

have both all their solutions in R and the solutions of each equation are equal at most by pairs (*there do not exist three or more solutions equal to each other*).

Proof. Carrying out the expansion of the determinant

$$\det[I_n z_1 z_2 - A - Bz_1 - Cz_2], \tag{6}$$

where A, B, C are of the form (3), and considering the terms in $z_1^n z_2^k$ successively for $k = n-1, n-2, \dots, 0$, we obtain the following set of equations:

$$\begin{aligned} &\sum_{i=1}^{n} b_{ii} = d_{n,n-1}, \\ &\sum_{1 \le i < j \le n} b_{ii} b_{jj} = d_{n,n-2}, \\ &\sum_{1 \le i < j < h \le n} b_{ii} b_{jj} b_{hh} = d_{n,n-3}, \\ &\dots\dots\dots\dots \\ &\prod_{i=1}^{n} b_{ii} = d_{n,0}. \end{aligned} \tag{7}$$

It is well known that the solutions b_{ii}, $i = 1, 2, \dots, n$, of (7) satisfy equation (4). Hence a necessary and sufficient condition for the existence of real b_{ii}, $i = 1, 2, \dots, n$, is the solvability in R of equation (4). Symmetrically, we obtain as the necessary and sufficient condition for the existence of real c_{ii}, $i = 1, 2, \dots, n$, the solvability in R of the dual equation (5). Following the expansion of (6) and considering the terms in $z_1^{n-l} z_2^k$ ($k = n-l, n-l-1, \dots, 0$) successively for $l = 1, 2, \dots, n-2$ (and symmetrically in $z_1^k z_2^{n-l}$), it is easy to prove that for $k = n-l$ we get a simple linear equation yielding directly a_{l+1}, and for $k = n-l-1, \dots, 0$ we obtain a set of $n-l$ linear equations in $b_{m,m-l}$, $m = l+1, \dots, n$, the right-hand side of which consists of d_{ij} and of already evaluated entries (i.e., $b_{m,m-i}$, $c_{m,m-i}$ for $i < l$ and a_j for j

$<i+1$). The determinant of the equations is of the form

$$D^{(l)} = (d_{ij}^{(l)})_{i,j=1,2,\ldots,n-l},$$

where

$$d_{1j} = 1 \quad \text{for } j = 1, \ldots, n-l,$$

$$d_{ij} = \sum \left(\prod_{h=1}^{i-1} b_{hh}\right) \quad \text{for } i = 2, \ldots, n-l,\ j = 1, \ldots, n-l,$$

the summation being taken over all $k_1, \ldots, k_{i-1}$ such that

$$1 \leqslant k_1 < k_2 < \ldots < k_{i-1} \leqslant n \quad \text{and} \quad k_1, \ldots, k_{i-1} \notin \{j, j+1, \ldots, j+l\}.$$

For $l = n-1$, we get one equation yielding directly $b_{n,1}$, i.e., $D^{(n-1)} = 1$. Finally, for $l = n$ we obtain $a_1 = (-1)^n d_{00}$. Subtracting the $(m-1)$-st column from the m-th column of $D^{(l)}$ successively for $m = n-l, \ldots, 2$, we obtain

$$D^{(l)} = \prod_{k=1}^{n-l-1} (b_{kk} + b_{k+l+1,k+l+1}) D^{(l+1)}.$$

As $D^{(n-1)} = 1$, all the $D^{(l)}$ are nonzero iff

$$b_{kk} \neq b_{k+l+1,k+l+1} \quad \text{for } l = 1, \ldots, n-2,\ k = 1, \ldots, n-l-1,$$

i.e., iff

$$b_{ii} \neq b_{mm} \quad \text{for } m \geqslant i+2.$$

Symmetrically, as a sufficient condition for the existence of all c_{ij} we get

$$c_{ii} \neq c_{mm} \quad \text{for } m \geqslant i+2.$$

The entries of the matrices A, B, C can be evaluated using the following

ALGORITHM.

Step 1. Solving the equation (4) find $b_{11}, b_{22}, \ldots, b_{nn}$ and check that there are no more than two equal solutions. Proceed in the same way with equation (5) obtaining c_{ii}, $i = 1, \ldots, n$.

Step 2. For $k = 1$, considering the terms in $z_1^{n-k} z_2^{n-k}$ in the expansion of (6) evaluate a_{k+1}.

Step 3. Considering successively the terms in $z_1^{n-k} z_2^l$, $l = n-k-1, \ldots, 0$, build the system of $n-l$ linear equations yielding $b_{m,m-k}$ and solve it.

Step 4. Considering the terms in $z_1^l z_2^{n-k}$, $l = n-k-1, \ldots, 0$, find $c_{m,m-k}$ in the same way.

Step 5. Repeat Steps 2, 3 and 4 successively for $k = 2, 3, \ldots, n-1$, finding all b_{ij}, c_{ij} and a_i for $i = k, k-1, \ldots, 2$.

Step 6. Considering constant terms, evaluate a_1 ($a_1 = (-1)^{n-1} d_{00}$).

This procedure is illustrated by a numerical example.

EXAMPLE. Solve the problem for

$$d(z_1, z_2) = z_1^3 z_2^3 + 2z_1^3 z_2^2 + 2z_1^2 z_2^3 - z_1^3 z_2 - z_1 z_2^3 + 3z_1^2 z_2^2 - 2z_1^2 z_2 - 44z_1 z_2^2$$
$$- 2z_1^3 - 2z_2^3 - z_1^2 + 2z_2^2 + z_1 z_2 + z_1 - z_2 + 2.$$

Step 1. In this case, (4) takes the form

$$x^3 - 2x^2 - x + 2 = (x-1)(x-2)(x+1)$$

and $b_{11} = 1$, $b_{22} = 2$, $b_{33} = -1$.

Similarly, (5) takes the form

$$x^3 - 2x^2 - x + 2 = (x+1)(x-1)(x-2)$$

and $c_{11} = -1$, $c_{22} = 1$, $c_{33} = 2$.

Step 2 ($k = 1$). Considering the terms in $z_1^2 z_2^2$, we obtain

$$d_{22} = a_3 + b_{11}(c_{22} + c_{33}) + b_{22}(c_{11} + c_{33}) + b_{33}(c_{11} + c_{22})$$

and $a_3 = -2$.

Step 3 ($k = 1$). Considering the terms in $z_1^2 z_2$, we get

$$d_{21} = (b_{22} + b_{11}) a_3 + b_{11}(b_{22} c_{33} + b_{33} c_{22}) + b_{22} b_{33} c_{11} - b_{21} - b_{32}$$

and considering the terms in z_1^2, we have

$$d_{20} = b_{11} b_{22} a_3 - b_{33} b_{21} - b_{11} b_{32}.$$

Hence $b_{21} + b_{32} = 1$, $b_{21} - b_{32} = 3$ and, consequently, $b_{21} = 2$, $b_{32} = -1$.

Step 4 ($k = 1$). By symmetry,

$$d_{12} = (c_{22} + c_{11}) a_3 + c_{11}(c_{22} b_{33} + c_{33} b_{22}) + c_{22} c_{33} b_{11} - c_{21} - c_{32},$$

$$d_{02} = c_{11} c_{22} a_3 - c_{33} c_{21} - c_{11} c_{32},$$

so that $c_{21} = 1$, $c_{32} = 2$.

Step 5.

a. Step 2 ($k = 2$):

$$d_{11} = a_3(b_{22} c_{11} + b_{11} c_{22}) - b_{21} c_{33} - c_{21} b_{33} - b_{32} c_{11} - c_{32} b_{11} - a_2,$$

which yields $a_2 = -5$.

b. Step 3 ($k = 2$):

$$d_{10} = -b_{21} a_3 - b_{11} a_2 + b_{31} \quad \text{and} \quad b_{31} = -8.$$

c. Step 4 ($k = 2$):

$$d_{01} = -c_{21} a_3 - c_{11} a_2 + c_{31} \quad \text{and} \quad c_{31} = 2.$$

Step 6. Finally, $a_1 = d_{00}$ yields $a_1 = 2$.

Thus, the desired matrices take the form

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -2 & 0 \\ 8 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & -2 & -2 \end{bmatrix}.$$

References

[1] T. Kaczorek, *Two-dimensional linear systems*, Springer, New York 1985.
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