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NEW SOLVABLE CASES OF m-MACHINE AND n-ELEMENT SEQUENCING PROBLEMS

0. This paper contains some generalizations of properties C and C_1 (see [2]) of the processing time matrix. The m-machine and n-element sequencing problems with the processing time matrix which has one of these properties can be reduced to the 2-machine and n-element problems for which an effective algorithm of solution is well known [3].

1. Consider a given set $M_1, M_2, \ldots, M_m$ of m machines and a set $N = \{1, 2, \ldots, n\}$ of n elements. Each element should be successively processed on machines $M_1, M_2, \ldots, M_m$.

The problem is to determine such a processing sequence of elements $\{i_1, i_2, \ldots, i_n\}$, being a permutation of the sequence $\{1, 2, \ldots, n\}$, as to minimize the time required to complete all jobs.

In this paper we are interested in determining the solution of the above-mentioned problem with the following additional constraints:

(a) Every element can be processed on not more than one machine, and any machine can process not more than one element at any time.

(b) The processing of each element must not be interrupted.

(c) The processing sequence of elements is the same on all machines.

(d) The set-up times are equal to zero.

Let us introduce the following notations:

$\bar{X}$ – the sequence $\{1, 2, \ldots, n\}$;

X – the permutation of the sequence $\bar{X}$;

$T(X) = (T_{ij}(X))$ – the matrix of processing times for the processing sequence X, i.e., $T_{ij_i}(X)$ is the processing time for an element j_i on a machine i, where $X = \{j_1, j_2, \ldots, j_n\}$ is a permutation of the sequence $\bar{X}$;

$t(X)$ – the objective function, i.e., the time required to complete all jobs in the processing sequence X;

X_0 – the optimal processing sequence, i.e.,

$$t(X_0) = \min_X t(X).$$

It is well known (see [1]) that

$$t(X) = \max_{1\leqslant u_1\leqslant\ldots\leqslant u_{m-1}\leqslant n} \sum_{j=1}^{u_1} T_{1j}(X) + \sum_{j=u_1}^{u_2} T_{2j}(X) + \ldots + \sum_{j=u_{m-1}}^{n} T_{mj}(X).$$

Let

$$q_u(X) = \sum_{j=1}^{n} T_{1j}(X) + \sum_{i=2}^{m-1} T_{iu}(X) + \sum_{j=u}^{n} T_{mj}(X),$$

$$Q(X) = \max_{1\leqslant u\leqslant n} q_u(X). \tag{1}$$

2. In this section we introduce, for the matrix $\boldsymbol{T}(\bar{X})$, new properties $\boldsymbol{M}$ and $\boldsymbol{M}_1$ which are extensions of the properties $\boldsymbol{C}$ and $\boldsymbol{C}_1$, respectively, from [2]. Next we shall prove that the optimum processing sequence $\boldsymbol{X}_0$ for the matrix $\boldsymbol{T}(\bar{X})$ which has the property $\boldsymbol{M}$ or $\boldsymbol{M}_1$ can be determined by the algorithms which describe the optimal sequence for the matrix $\boldsymbol{T}(\bar{X})$ having the property $\boldsymbol{C}$ or $\boldsymbol{C}_1$, respectively (see [2]). Let

$$H_{p,q,j} = \begin{cases} \sum_{i=p}^{q} T_{ij}(\bar{X}) & \text{for } p \leqslant q, \\ 0 & \text{otherwise}, \end{cases}$$

where $p, q = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and, for the sake of simplicity, let

$$R_{q,j} = H_{1,q,j}, \quad r_{q,j} = H_{2,q,j}, \tag{2}$$

$$D_{p,j} = H_{p,m,j}, \quad d_{p,j} = H_{p,m-1,j}. \tag{3}$$

Let $M(\lambda)$ denote the following property of the matrix $\boldsymbol{T}(\bar{X})$:

$$M(\lambda): \begin{cases} d_{p,l} \leqslant D_{p+1,k} \ (p = 2, 3, \ldots, m-1;\ k, l = 1, 2, \ldots, n;\ k \neq l) \\ \qquad\qquad \text{for } \lambda = 1, \\ r_{q+1} \leqslant R_{q,k} \text{ and } d_{p,l} \leqslant D_{p+1,k} \ (q = 1, 2, \ldots, \lambda-1; \\ \qquad p = \lambda+1, \ldots, m-1;\ k, l = 1, 2, \ldots, n;\ k \neq l) \\ \qquad\qquad \text{for } 2 \leqslant \lambda \leqslant m-2, \\ r_{q+1,l} \leqslant R_{q,k} \ (q = 1, 2, \ldots, m-2;\ k, l = 1, 2, \ldots, n;\ k \neq l) \\ \qquad\qquad \text{for } \lambda = m-1. \end{cases}$$

Definition 1. The matrix $\boldsymbol{T}(\bar{X})$ has the *property* $\boldsymbol{M}$ if there exists a number λ ($\lambda = 1, 2, \ldots, m-1$) such that the matrix $\boldsymbol{T}(\bar{X})$ has the property $M(\lambda)$.

It is evident that if the matrix $\boldsymbol{T}(\bar{X})$ has the property $\boldsymbol{M}$, then the matrix $\boldsymbol{T}(X)$ has the property $\boldsymbol{M}$ for each permutation X. The property $\boldsymbol{M}$ is an extension of the property $\boldsymbol{C}$ from [2]:

LEMMA 1. *If the matrix $\boldsymbol{T}(\bar{X})$ has the property $\boldsymbol{C}$, then it also has the property $\boldsymbol{M}$ for the same value of λ.*

The proofs are inserted in Section 3.

Example 1. The following matrix has the property $\boldsymbol{M}$ ($\lambda = 3$) however, it has not the property $\boldsymbol{C}$:

$$\begin{bmatrix} 49 & 46 & 51 & 39 & 41 & 44 & 53 & 38 & 50 & 50 \\ 36 & 38 & 37 & 32 & 31 & 24 & 30 & 39 & 34 & 37 \\ 31 & 28 & 31 & 36 & 35 & 46 & 38 & 25 & 34 & 30 \\ 23 & 14 & 17 & 19 & 24 & 30 & 28 & 21 & 20 & 15 \\ 32 & 35 & 34 & 30 & 31 & 28 & 41 & 36 & 38 & 42 \end{bmatrix}.$$

The following theorem states that the thesis of the first theorem from paper [2] holds under the weaker assumptions:

THEOREM 1. *If the matrix $\boldsymbol{T}(\bar{X})$ has the property $\boldsymbol{M}$, then $Q(X) = t(X)$ for each permutation X.*

Thus, if the matrix $\boldsymbol{T}(\bar{X})$ of the m-machine processing problem has the property $\boldsymbol{M}$, then the optimal solution of this problem can be obtained by Algorithm 1 from paper [2]. (This algorithm reduces the m-machine processing problem to the 2-machine one, for which an effective algorithm of solution is well known [3].)

Let us consider another group of m-machine processing problems for which there exists an effective method of solution. Let us write

$$L_2(X) = \sum_{i=1}^{\lambda-1} T_{i1}(X) + \sum_{j=1}^{n} T_{\lambda j}(X) + \sum_{i=\lambda+1}^{m} T_{in}(X),$$

where λ is one of the numbers $1, 2, \ldots, m$. Let $M_1(\lambda)$ denote the following property of the matrix $\boldsymbol{T}(\bar{X})$:

$$\boldsymbol{M}_1(\lambda): \begin{cases} H_{2,p+1,k} \leqslant H_{1,p,l} \ (p = 1, 2, \ldots, m-1;\ k, l = 1, 2, \ldots, n;\ k \neq l) \\ \qquad\qquad \text{for } \lambda = 1, \\ H_{\lambda+1,p+1,k} \leqslant H_{\lambda,p,l} \text{ and } H_{q,\lambda-1,k} \leqslant H_{q+1,\lambda,l} \ (q = 1, 2, \ldots, \lambda-1; \\ \quad p = \lambda, \ldots, m-1;\ k, l = 1, 2, \ldots, n;\ k \neq l) \ \text{ for } 2 \leqslant \lambda \leqslant m-1, \\ H_{q,m-1,k} \leqslant H_{q+1,m,l} \ (q = 1, 2, \ldots, m-1;\ k, l = 1, 2, \ldots;\ k \neq l) \\ \qquad\qquad \text{for } \lambda = m. \end{cases}$$

Definition 2. The matrix $\boldsymbol{T}(\bar{X})$ has the *property* $\boldsymbol{M}_1$ if there exists a number λ ($\lambda = 1, 2, \ldots, m$) such that the matrix $\boldsymbol{T}(\bar{X})$ has the property $\boldsymbol{M}_1(\lambda)$.

It is evident that if the matrix $T(\bar{X})$ has the property M_1, then the matrix $T(X)$ has the property M_1 for each permutation X.

The proofs of the following Lemma 2 and Theorem 2 are similar to those of Lemma 1 and Theorem 1, respectively.

LEMMA 2. *If the matrix $T(\bar{X})$ has the property C_1, then it also has the property M_1 for the same value of λ.*

THEOREM 2. *If the matrix $T(\bar{X})$ has the property M_1, then $t(X) = L_2(X)$ for each permutation X.*

Example 2. The following matrix has the property M_1 ($\lambda = 3$), however, it has not the property C_1:

$$\begin{bmatrix} 76 & 70 & 52 & 85 & 65 & 87 & 72 & 53 \\ 65 & 72 & 61 & 56 & 75 & 60 & 70 & 68 \\ 86 & 91 & 76 & 95 & 72 & 82 & 88 & 98 \\ 58 & 65 & 68 & 71 & 74 & 70 & 64 & 65 \\ 86 & 77 & 75 & 73 & 70 & 73 & 80 & 76 \end{bmatrix}.$$

Thus, if the matrix $T(\bar{X})$ has the property M_1, then the optimal solution can be obtained by Algorithm 2 from paper [2].

3. Proof of Lemma 1. Let, for a given matrix $T(\bar{X})$, two different numbers i_1 and i_2 ($i_1, i_2 = 1, 2, \ldots, m$) satisfy the relation $\vdash$ if and only if

$$T_{i_1 k}(\bar{X}) \leqslant T_{i_2 l}(\bar{X}) \quad \text{for } k, l = 1, 2, \ldots, n;\ k \neq l.$$

It is easy to see that if $i_1 \vdash i_2$ in the matrix $T(\bar{X})$, then also $i_1 \vdash i_2$ in the matrix $T(X)$ for each permutation X. Let us denote by $C(\lambda)$ (similarly as in [2]) the following property of the matrix $T(\bar{X})$:

$$C(\lambda):\begin{cases} 2 \vdash 3 \vdash \ldots \vdash m & \text{for } \lambda = 1, \\ \lambda \vdash (\lambda-1) \vdash \ldots \vdash 1 \text{ and } (\lambda+1) \vdash (\lambda+2) \vdash \ldots \vdash m & \\ & \text{for } 2 \leqslant \lambda \leqslant m-2, \\ (m-1) \vdash (m-2) \vdash \ldots \vdash 1 & \text{for } \lambda = m-1. \end{cases}$$

Definition 3. The matrix $T(\bar{X})$ has the *property* C if there exists a number λ ($\lambda = 1, 2, \ldots, m-1$) such that the matrix $T(\bar{X})$ has the property $C(\lambda)$.

It is evident that if the matrix $T(\bar{X})$ has the property C, then the matrix $T(X)$ has the property C for each permutation X.

For $\lambda = 1$ we prove the lemma inductively with decreasing p.

(i) For $p = m-1$ and $k, l = 1, 2, \ldots, n$, $k \neq l$, from (3) we have $D_{m,k} = T_{m,k}(\bar{X})$ and $d_{m-1,l} = T_{m-1,l}(\bar{X})$, and from the property $C(1)$ of the matrix $T(\bar{X})$ we have $T_{m,k}(\bar{X}) \geqslant T_{m-1,l}(\bar{X})$; hence $D_{m,k} \geqslant d_{m-1,l}$.

(ii) Assume that $D_{p+2,k} \geqslant d_{p+1,l}$. We must prove that $D_{p+1,k} \geqslant d_{p,l}$ $(k, l = 1, 2, \ldots, n;\ k \neq l)$. From the property $C(1)$ of the matrix $\boldsymbol{T}(\bar{X})$ we have $T_{p+1,k} \geqslant T_{p,l}$, therefore

$$D_{p+2,k} + T_{p+1,k} \geqslant d_{p+1,l} + T_{p,l};$$

hence from (3) we have $D_{p+1,k} \geqslant d_{p,l}$. For $\lambda = m-1$ the proof is analogous (with increasing q). For $\lambda = 2, 3, \ldots, m-2$ the proof is a connection of the cases above for the corresponding numbers p and q.

Proof of Theorem 1. Let $L(X)$ denote the value of the sum of elements of the matrix $\boldsymbol{T}(X)$ calculated in the following way: begin from the element $T_{11}(X)$ and end at the element $T_{mn}(X)$ and either move along rows to the right or move along columns to the bottom; then we have

$$L(X) = \sum_{j=1}^{u_1} T_{1j}(X) + \ldots + \sum_{j=u_{\lambda-1}}^{u_\lambda} T_{\lambda j}(X) + \sum_{j=u_\lambda}^{u_{\lambda+1}} T_{\lambda+1,j}(X) + \ldots + \sum_{j=u_{m-1}}^{n} T_{mj}(X),$$

where $1 \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_\lambda \leqslant u_{\lambda+1} \leqslant \ldots \leqslant u_{m-1} \leqslant n$.

Let us write (similarly as in [2])

$$L_1(X) = \sum_{j=1}^{u_\lambda} T_{1j}(X) + \sum_{i=2}^{m-1} T_{iu_\lambda}(X) + \sum_{j=u_\lambda}^{n} T_{mj}(X).$$

It follows from (1) that $t(X)$ is equal to the maximum of $L(X)$ and, therefore, $Q(X) \leqslant t(X)$, since $q_u(X)$ is one of the sums $L(X)$. Let the matrix $\boldsymbol{T}(\bar{X})$ have the property $\boldsymbol{M}$ for the fixed number λ $(\lambda = 1, 2, \ldots, m-1)$. We prove that $L_1(X) \geqslant L(X)$.

Let us introduce the expressions

$$F(X) = \sum_{j=u_1}^{u_2-1} r_{2j} + \sum_{j=u_2}^{u_3-1} r_{3j} + \ldots + \sum_{j=u_{\lambda-1}}^{u_\lambda-1} r_{\lambda j} + \sum_{j=u_\lambda+1}^{u_{\lambda+1}} d_{\lambda+1,j} + \sum_{j=u_{\lambda+1}+1}^{u_{\lambda+2}} d_{\lambda+2,j} + \ldots + \sum_{j=u_{m-2}+1}^{u_{m-1}} d_{m-1,j},$$

$$G(X) = \sum_{j=u_1+1}^{u_2} R_{1j} + \sum_{j=u_2+1}^{u_3} R_{2j} + \ldots + \sum_{j=u_{\lambda-1}+1}^{u_\lambda} R_{\lambda-1,j} + \sum_{j=u_\lambda}^{u_{\lambda+1}-1} D_{\lambda+2,j} + \sum_{j=u_{\lambda+1}}^{u_{\lambda+2}-1} D_{\lambda+3,j} + \ldots + \sum_{j=u_{m-2}}^{u_{m-1}-1} D_{mj},$$

where $1 \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_{\lambda-1} \leqslant u_\lambda \leqslant u_{\lambda+1} \leqslant \ldots \leqslant u_{m-1} \leqslant n$.

From the property $\boldsymbol{M}$ of the matrix $\boldsymbol{T}(\bar{X})$ we have the inequalities

$$R_{q,j+1} \geqslant r_{q+1,j} \quad \text{for } j = u_1, \ldots, u_\lambda - 1,$$
$$D_{p+1,j} \geqslant d_{p,j+1} \quad \text{for } j = u_\lambda, \ldots, u_{m-1} - 1,$$

where $q = 1, 2, \ldots, \lambda-1$ and $p = \lambda+1, \lambda+2, \ldots, m-1$. Hence $G(X) \geqslant F(X)$. Let us add to both sides of the inequality above the expression

$$\sum_{j=1}^{u_1} T_{1j}(X) + \sum_{i=\lambda}^{\lambda+1} T_{iu_\lambda}(X) + \sum_{j=u_{m-1}}^{n} T_{mj}(X)$$

and let us perform some transformations of the left (L) and the right (R) sides of it as follows:

$$\begin{aligned}
\mathrm{L} &= \sum_{j=1}^{u_1} T_{1j}(X) + \sum_{i=\lambda}^{\lambda+1} T_{iu_\lambda}(X) + \sum_{j=u_{m-1}}^{n} T_{mj}(X) + \sum_{j=u_1+1}^{u_2} T_{1j}(X) + \\
&\quad + \sum_{j=u_2+1}^{u_3} \sum_{i=1}^{2} T_{ij}(X) + \ldots + \sum_{j=u_{\lambda-1}+1}^{u_\lambda} \sum_{i=1}^{\lambda-1} T_{ij}(X) + \\
&\quad + \sum_{j=u_\lambda}^{u_{\lambda+1}-1} \sum_{i=\lambda+2}^{m} T_{ij}(X) + \sum_{j=u_{\lambda+1}}^{u_{\lambda+2}-1} \sum_{i=\lambda+3}^{m} T_{ij}(X) + \ldots + \sum_{j=u_{m-2}}^{u_{m-1}-1} T_{mj}(X) \\
&= \sum_{j=1}^{u_\lambda} T_{1j}(X) + \sum_{i=2}^{m-1} T_{iu_\lambda}(X) + \sum_{j=u_\lambda}^{n} T_{mj}(X) + \sum_{j=u_2+1}^{u_3} T_{2j}(X) + \\
&\quad + \sum_{j=u_3+1}^{u_4} \sum_{i=2}^{3} T_{ij}(X) + \ldots + \sum_{j=u_{\lambda-1}+1}^{u_\lambda-1} \sum_{i=2}^{\lambda-1} T_{ij}(X) + \\
&\quad + \sum_{j=u_\lambda+1}^{u_{\lambda+1}-1} \sum_{i=\lambda+2}^{m-1} T_{ij}(X) + \sum_{j=u_{\lambda+1}}^{u_{\lambda+2}-1} \sum_{i=\lambda+3}^{m-1} T_{ij}(X) + \ldots + \sum_{j=u_{m-3}+1}^{u_{m-2}-1} T_{m-1,j}(X) \\
&= L_1(X) + \sum_{j=u_2+1}^{u_3} T_{2j}(X) + \sum_{j=u_3+1}^{u_4} \sum_{i=2}^{3} T_{ij}(X) + \ldots + \\
&\quad + \sum_{j=u_{\lambda-1}+1}^{u_\lambda-1} \sum_{i=2}^{\lambda-1} T_{ij}(X) + \sum_{j=u_\lambda+1}^{u_{\lambda+1}-1} \sum_{i=\lambda+2}^{m-1} T_{ij}(X) + \\
&\quad + \sum_{j=u_{\lambda+1}}^{u_{\lambda+2}-1} \sum_{i=\lambda+3}^{m-1} T_{ij}(X) + \ldots + \sum_{j=u_{m-3}+1}^{u_{m-2}-1} T_{m-1,j}(X),
\end{aligned}$$

$$\mathrm{R} = \sum_{j=1}^{u_1} T_{1j}(X) + \sum_{i=\lambda}^{\lambda+1} T_{iu_\lambda}(X) + \sum_{j=u_{m-1}}^{n} T_{mj}(X) + \sum_{j=u_1}^{u_2-1} T_{2j}(X) +$$

$$+ \sum_{j=u_2}^{u_3-1} \sum_{i=2}^{3} T_{ij}(X) + \ldots + \sum_{j=u_{\lambda-1}}^{u_\lambda-1} \sum_{i=2}^{\lambda} T_{ij}(X) + \sum_{j=u_\lambda+1}^{u_{\lambda+1}} \sum_{i=\lambda+1}^{m-1} T_{ij}(X) +$$

$$+ \sum_{j=u_{\lambda+1}+1}^{u_{\lambda+2}} \sum_{i=\lambda+2}^{m-1} T_{ij}(X) + \ldots + \sum_{j=u_{m-2}+1}^{u_{m-1}} T_{m-1,j}(X)$$

$$= \sum_{j=1}^{u_1} T_{1j}(X) + \sum_{j=u_1}^{u_2} T_{2j}(X) + \ldots + \sum_{j=u_{\lambda-1}}^{u_\lambda} T_{\lambda j}(X) + \ldots +$$

$$+ \sum_{j=u_{m-2}}^{u_{m-1}} T_{m-1,j}(X) + \sum_{j=u_{m-1}}^{n} T_{mj}(X) + \sum_{j=u_2+1}^{u_3-1} T_{2j}(X) + T_{2u_3}(X) +$$

$$+ \sum_{j=u_3+1}^{u_4-1} \sum_{i=2}^{3} T_{ij}(X) + \sum_{i=2}^{3} T_{iu_4}(X) + \ldots + \sum_{j=u_{\lambda-1}+1}^{u_\lambda-1} \sum_{i=2}^{\lambda-1} T_{ij}(X) +$$

$$+ \sum_{j=u_\lambda+1}^{u_{\lambda+1}-1} \sum_{i=\lambda+2}^{m-1} T_{ij}(X) + \sum_{i=\lambda+3}^{m-1} T_{iu_{\lambda+1}}(X) + \sum_{j=u_{\lambda+1}+1}^{u_{\lambda+2}-1} \sum_{i=\lambda+3}^{m-1} T_{ij}(X) +$$

$$+ \sum_{i=\lambda+4}^{m-1} T_{iu_{\lambda+2}}(X) + \ldots + \sum_{j=u_{m-3}+1}^{u_{m-2}+1} T_{m-1,j}(X)$$

$$= L(X) + \sum_{j=u_2+1}^{u_3} T_{2j}(X) + \sum_{j=u_3+1}^{u_4} \sum_{i=2}^{3} T_{ij}(X) + \ldots +$$

$$+ \sum_{j=u_{\lambda-1}+1}^{u_\lambda-1} \sum_{i=2}^{\lambda-1} T_{ij}(X) + \sum_{j=u_\lambda+1}^{u_{\lambda+1}-1} \sum_{i=\lambda+2}^{m-1} T_{ij}(X) +$$

$$+ \sum_{j=u_{\lambda+1}}^{u_{\lambda+2}-1} \sum_{i=\lambda+3}^{m-1} T_{ij}(X) + \ldots + \sum_{j=u_{m-3}+1}^{u_{m-2}-1} T_{m-1,j}(X).$$

From the final forms of expressions L and R we have at once the inequality $L_1(X) \geqslant L(X)$. As $L_1(X) = q_{u_\lambda}(X)$, therefore, from (1) we have the inequality $Q(X) \geqslant L(X)$ and, in particular, $Q(X) \geqslant t(X)$, and thus the equality holds.

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DALSZE PRZYPADKI ŁATWO ROZWIĄZYWALNYCH ZAGADNIEŃ KOLEJNOŚCIOWYCH

STRESZCZENIE

Praca zawiera uogólnienie warunków przedstawionych w [2]. Przy spełnieniu tych warunków m-operacyjny proces obróbki może być sprowadzony do 2-operacyjnego, dla którego istnieje efektywna metoda otrzymywania rozwiązania optymalnego (patrz np. [3]).