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J. NIEMINEN (Helsinki)

A LINEAR PSEUDO-BOOLEAN VIEWPOINT ON MATCHING AND OTHER CENTRAL CONCEPTS IN GRAPH THEORY

1. INTRODUCTION

There are several ways of solving a bivalent programming scheme, e.g. pseudo-Boolean [4], branch and bound [1], Boolean branch and bound [3], and others (for these, see the reference list in [4]). Independently of the choice of the method, the greatest efficiency is reached if the programming scheme in question contains only linear pseudo-Boolean equations and inequalities. The purpose of this review paper is to construct a set of linear programming schemes for the maximum independent set, minimum covering set and covering edge family, maximum matching and minimum dominating set of a given undirected graph. In these schemes the number of variables equals with those of the schemes given in [4], and whence they can offer an alternative for determining the above-mentioned sets.

2. NOTATIONS AND DEFINITIONS

In what follows, we shall consider finite, connected and undirected graphs $G = (V, E)$ without loops and multiple edges. V is the set of vertices of G, and E the set of edges. We shall follow the notations and definitions of Ore given in [7].

A set I of vertices of G is called *independent* if there are no connecting edges between any of its vertices. A *maximum independent set* I_0 of G is an independent set such that $|I_0| \geqslant |I|$ for any independent set I of G, where $|I|$ denotes the number of elements in the set I. The set of *independent edges* is defined analogously.

A set K of vertices of G will be called a *covering set* if each edge of G has at least one end point in K. A covering set K_0 is a *minimum covering set* if $|K_0| \leqslant |K|$ for any covering set K of G. An edge family $D(e_i)$ is a *covering edge family* of G if it has at least one edge e_t at each vertex of G. The definition of the *minimum covering edge family* is obvious.

A subset D of V is a *dominating set* for G if every vertex not in D is the end point of some edge from a vertex in D. A set D_0 is a minimum dominating set of G if $|D_0| \leqslant |D|$ for any dominating set D of G.

A maximum independent edge set of G is called a *maximum matching* of G.

The terminology and notations concerning functions of bivalent variables used in this paper follow those of Hammer and Rudeanu in [4].

3. BIVALENT PROGRAMMING SCHEMES

Let $N = \{1, 2, \ldots, n\}$ be a given set, and S a subset of N. With each subset S of N we associate the characteristic vector $(z_1, \ldots, z_n)$, where $z_i = 1$ if $i \in S$, and $z_i = 0$ if $i \notin S$ for each $i = 1, \ldots, n$. Thus the problem is to determine the characteristic vector of the set under interest.

By $M = [m_{ij}]$ we denote the vertex-edge incidence matrix of the given graph G.

3.1. Minimum covering set and maximum independent set. Let $|V| = p$ and let $(x_1, \ldots, x_p)$ be the characteristic vector of a set $K \subset V$. K is a minimum covering set of G if its characteristic vector is an absolutely minimizing point of the function

$$f(x_1, \ldots, x_p) = x_1 + x_2 + \ldots + x_p \tag{1}$$

with subject to the conditions

$$\sum_i m_{ij} x_i \geqslant 1 \tag{2}$$

for any value of j, $j = 1, \ldots, r$, i.e. $|E| = r$. Indeed, the conditions in (2) ensure that for any edge j of G at least one of its end points belongs to K, and thus K is a covering set.

A set K is a covering set if and only if its complement $\bar{K}$ in V is an independent set of G. Moreover, if K_0 is a minimum covering set and I_0 a maximum independent set of G, then $|K_0| + |I_0| = p$ (see, e.g., [7], Theorem 13.3.4). Hence, $\bar{K}_0$ is a maximum independent set, and thus (1) and (2) offer a linear bivalent programming scheme for determining all sets I_0 of G.

3.2. Maximum matching and minimum covering edge family. According to the definition of independence, an edge set $C \subset E$ is a *matching* of G if and only if no two edges of C have a common end point in G. This condition is obviously equivalent to the linear pseudo-Boolean expressions in (4). Thus, a set $C \subset E$ is a maximum matching of G if its characteristic vector $(y_1, \ldots, y_r)$ is an absolutely maximizing point of the function

$$f(y_1, \ldots, y_r) = y_1 + y_2 + \ldots + y_r \tag{3}$$

with subject to the conditions

$$\sum_j m_{ij} y_j \leqslant 1 \tag{4}$$

for any value of i, $i = 1, \ldots, p$.

Let $D(e_t)$ be an edge set of G. If, for any vertex x_i of G,

$$\sum_j m_{ij} y_j \geqslant 1$$

is valid for the edges of $D(e_t)$, there is at least one edge of $D(e_t)$ incident at each vertex of G, i.e. $D(e_t)$ is a covering edge family of G. Thus, a set $D(e_t) \subset E$ is a minimum covering edge family of G if its characteristic vector $(y_1, \ldots, y_r)$ is an absolutely minimizing point of the function (3) with subject to the conditions

$$\sum_j m_{ij} y_j \geqslant 1 \tag{5}$$

for any value of i, $i = 1, \ldots, p$.

As shown by Norman and Rabin [6] (or see Berge [2], p. 124), the maximum matchings of G determine the minimum covering edge families of G and conversely, and whence only one of schemes (3), (4) and (3), (5) is necessary to solve in oder to obtain the sets C_0 and $D_0(e_t)$.

3.3. Minimum dominating set. Let G be a given graph. Add a loop at each vertex of G and denote by $A = [a_{is}]$ the vertex incidence (adjacency) matrix of the garaph G^0 thus obtained.

According to the definition of a dominating set, D is a dominating set of G if $D \subset V$ and any vertex of G^0 is connected by an edge to at least one of the vertices of D. This condition is equivalent to the set of expressions in (6). Thus, a set $D \subset V$ is a minimum dominating set of G if its characteristic vector $(x_1, \ldots, x_p)$ is an absolutely minimizing point of the function (1) with subject to the conditions (see [4], p. 220)

$$\sum_i a_{is} x_i \geqslant 1 \tag{6}$$

for any value of s, $s = 1, \ldots, p$. Note that, in A, $a_{ii} = 1$ for each value of i.

4. SOME REMARKS

The remarks of this section concern the case of undirected graphs and some well-known problems to which the above-given schemes apply.

The definition of D remains valid in case of directed graphs. Hence, schemes (1) and (6) give also the minimum dominating sets of a directed graph.

Let $\vec{G}$ be a given directed graph, and G the corresponding undirected graph obtained from $\vec{G}$ by omitting the directions of the edges. If I_0 is a maximum independent set of G, it obviously is an independent set of $\vec{G}$ as well. If I_0' is a maximum independent set of $\vec{G}$ such that $|I_0'| > |I_0|$, then in G at least two vertices of I_0' are adjacent according to the maximality of I_0, from which it follows that I_0' is not an independent set of $\vec{G}$. Thus the maximum independent sets of G are obtained by determining those of G and, analogously, the maximum matching of $\vec{G}$ are given by those of G.

As shown in the monography of Hammer and Rudeanu (see [4], p. 252-254, and also [5]), the solution of a system of distinct representatives leads to the maximum matching, and whence this problem can be handled by the linear schemes (1) and (4). A Boolean way (which is more efficient than the integer linear programming methods) of solving this problem is recently proposed by Sysło [8].

The separating set of a graph G (see [4], p. 242) is equivalent to the covering set of G. In [4] (p. 248-252), the most repeated step of the transportation problem is reduced to the determination of a separating set minimizing a given function. According to (1) and (2), this step can be linearized.

Finally, the assignement problem with profit weights on the edges of the graph G describing the problem has, as its solution, a matching of G with the maximum total profit. Thus the solution is obtained by maximizing the edge function

$$f(y_1, \ldots, y_r) = c_1 y_1 + c_2 y_2 + \ldots + c_r y_r \tag{7}$$

with subject to the conditions in (4). The coefficients c_j are the profit weights of the edges y_j of G.

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DEPARTMENT OF TECHNICAL SCIENCES, FINNISH ACADEMY
00200 HELSINKI 20, FINLAND

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J. NIEMINEN (Helsinki)

ZAGADNIENIE SKOJARZENIA I INNE PROBLEMY EKSTREMALNE TEORII GRAFÓW JAKO ZAGADNIENIA LINIOWEGO PROGRAMOWANIA PSEUDOBOOLOWSKIEGO

STRESZCZENIE

W pracy przedstawiono liniowe programy zero-jedynkowe dla następujących zagadnień teorii grafów: wyznaczanie maksymalnego skojarzenia, maksymalnego zbioru elementów niezależnych, minimalnego zbioru elementów dominujących oraz minimalnego pokrycia wierzchołków i łuków grafu.
