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SOME REMARKS ON THE EXTENSION OF A FINITE SET OF STOCHASTIC AUTOMATA

1. Introduction. The recent wide-spread interest in deterministic and stochastic automata concentrates not only on their abstract theory but also on practical purposes. It appeared that deterministic and stochastic automata are convenient tools in investigations of many real processes. Thus, deterministic automata have been used in reliable systems, deterministic simulation and random processes, computer sequencing and timing schemes, and communication, programming and economic systems, etc. (see [3] and [11]). In the terms of stochastic automata one can describe not only the working (running) some type of reliable deterministic systems but also a great many processes of computing, stochastic regulations, learning, stochastic optimalization, genetics and economics [14].

There are many kinds of deterministic and stochastic automata. The well known are, among others, Turing machines [12], the automata of Mealy [5], Moore [6] and Rabin-Scott [9]. A different model of a deterministic automaton was introduced and investigated by Pawlak [7].

Let U be a finite non-empty set called the *alphabet* or the *set of states* of the object M. Suppose that the object M is at some fixed state u_0 which is subjected to the acting a transition function f (a control function). As the result of that acting we obtain a new state u_1 of M. The function f acts again at u_1 if only $u_1 \in D_f$, and so on. The domain of D_f is the proper non-empty subset of U. An ordered pair $M = (U, f)$ is a *deterministic automaton of Pawlak* [7]. Such automata generate finite and infinite sequences $u_0, u_1, \ldots$ with $u_{i+1} = f(u_i)$ for $i = 1, 2, \ldots$ Each such sequence is called the *word* or the *computation* of M.

The fundamental problems concerning deterministic automata are, for example, the problems of synthesis analysis, computing, complexity, simplification and minimizing (see [4], [5] and [11]).

Analogously to deterministic automata, one could divide stochastic ones into several groups. Some of them include probabilistic automata of Rabin [8], multistochastic automata [13], probabilistic Turing machines

[10] and their generalizations. To the other groups we attest stochastic sequential machines [4] and stochastic automata of Bartoszyński's type [1].

It is worth noting that certain problems concerning deterministic automata have not their counterparts in the group of stochastic automata. The typical problems concerning stochastic automata are, for instance, the problems of stability, generalization, reduction, approximation, estimation, definability of the events, equivalence, shrinkage and extension (see [1], [2], [4] and [8]).

The last two problems are also dealt with in this note.

The concept of Bartoszyński's stochastic automaton is a generalization of the concept of a deterministic automaton defined in [7]. Bartoszyński has considered problems of extension and shrinkage for a pair of stochastic automata.

This paper concerns the two above-mentioned problems for a finite number of stochastic automata. The obtained results are natural, nevertheless non-trivial, generalizations of the questions asked by Bartoszyński [1].

By *Bartoszyński's stochastic automaton* (shortly, s.a.) we mean an ordered triple $M = (U, a, \pi)$, where $U = (u_1, u_2, \ldots, u_n)$ is a finite non-empty set called the alphabet of s.a. M, and a and π are functions such that

$$a: U \to [0,1] \quad \text{and} \quad \sum_{i=1}^{n} a(u_i) = 1,$$

$$\pi: U \times U \to [0,1] \quad \text{and} \quad \sum_{i=1}^{n} \pi(u_j, u_i) = 1 \quad \text{for every } j = 1, 2, \ldots, n.$$

On the set $U^N = U \times U \times \ldots \times U$, where $N \geqslant 1$, we define a probability measure (p.m.) by the following formula:

$$P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) = a(u_{i_1})\pi(u_{i_1}, u_{i_2})\pi(u_{i_2}, u_{i_3}) \ldots \pi(u_{i_{N-1}}, u_{i_N}).$$

In what follows we use the notations $a(u_i) = a_i$ and $\pi(u_i, u_j) = \pi_i(u_j) = \pi_{i,j}$.

The sets $L(N)$ and L such that

$$L(N) = \{w = (u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in U^N : P_N(w) > 0\}$$

and

$$L = \Big\{(u_{i_1}, u_{i_2}, \ldots) \in U^\infty : \bigwedge_{N \geqslant 1} P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) > 0\Big\}$$

are called the set of *N-words* and the *language* of s.a. M, respectively.

Let $M_i = (U, a^{(i)}, \pi^{(i)})$ $(i = 1, 2, \ldots, m,\ m \geqslant 2)$ be s.a. We put $L_i(N)$ and L_i $(i = 1, 2, \ldots, m,\ m \geqslant 2)$ for the set of N-words and the language of s.a. M_i, respectively.

We deal with the following two problems:

(a) The existence of s.a. $M = (U, a, \pi)$ such that, for every $N \geqslant 1$,

$$L(N) = \bigcap_{i=1}^{m} L_i(N), \tag{1.1}$$

$$\bigwedge_{1\leqslant i\leqslant m} \bigvee_{s_N^{(i)}>0} \bigwedge_{w\in L(N)} P_N(w) = s_N^{(i)} P_N^{(i)}(w). \tag{1.2}$$

(b) The existence of s.a. $M = (U, a, \pi)$ such that, for every $N \geqslant 1$,

$$L(N) \supset \bigcup_{i=1}^{m} L_i(N), \tag{1.3}$$

$$\bigwedge_{1\leqslant i\leqslant m} \bigvee_{e_N^{(i)}>0} \bigwedge_{w\in L_i(N)} P_N(w) = e_N^{(i)} P_N^{(i)}(w). \tag{1.4}$$

2. Definitions and lemmas. Let $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ be p.m. on X. The set $C_i = \{x\in X : P_i(x) > 0\}$ for $i = 1, 2, \ldots, m$ is called a *carrier* of p.m. P_i for $i = 1, 2, \ldots, m$.

Definition 1. Probability measures $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ are said to be *concordant* if there exist positive constants $c_1, c_2, \ldots, c_m$ such that

$$\bigwedge_{x\in C} c_1 P_1(x) = c_2 P_2(x) = \ldots = c_m P_m(x), \quad \text{where } C = \bigcap_{i=1}^{m} C_i.$$

If, moreover, $C \neq \emptyset$, then p.m. $P_1, P_2, \ldots, P_m$ are said to be *truly concordant.*

Without loss of generality we can put $c_1 = 1$.

Definition 2. Probability measures $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ are said to be *concordant* (*truly concordant*) *in pairs* if every pair P_i, P_j for $i, j = 1, 2, \ldots, m$, $m \geqslant 2$, is concordant (truly concordant).

The coefficient of concordance of the pair P_i, P_j will be denoted by c_j^i (for every $x\in C_i \cap C_j$, we have $P_i(x) = c_j^i P_j(x)$).

Definition 3. A probability measure P with the carrier

$$C = \bigcap_{i=1}^{m} C_i$$

is said to be a *shrinkage* of p.m. $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ if there exist positive constants $s_1, s_2, \ldots, s_m$ such that

$$\bigwedge_{x\in C} P(x) = s_1 P_1(x) = s_2 P_2(x) = \ldots = s_m P_m(x).$$

Definition 4. A probability measure P is said to be an *extension* of p.m. $P_1, P_2, \ldots, P_m$ if there exist positive constants $e_1, e_2, \ldots, e_m$ such that

$$\bigwedge_{1 \leqslant i \leqslant m} \bigwedge_{x \in C_i} P(x) = e_i P_i(x).$$

LEMMA 1. *Let $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ be p.m. on X. A shrinkage P of p.m. $P_1, P_2, \ldots, P_m$ exists if and only if $P_1, P_2, \ldots, P_m$ are truly concordant.*

Proof. The necessity follows immediately from Definitions 1 and 3.

Sufficiency. Let us assume that p.m. $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ are truly concordant with coefficients $c_1 = 1, c_2, \ldots, c_m$ and let us define p.m. P by the formula

$$P(x) = \begin{cases} v^{-1}P_1(x) = v^{-1}c_2P_2(x) = \ldots = v^{-1}c_mP_m(x) & \text{for } x \in C, \\ 0 & \text{for } x \notin C, \end{cases}$$

where

$$v = P_1(C) = \sum_{x \in C} P_1(x).$$

It is easy to see that P is the shrinkage of p.m. $P_1, P_2, \ldots, P_m$ ($m \geqslant 2$) with coefficients $s_1 = v^{-1}$ and $s_i = v^{-1}c_i$ for $i = 2, 3, \ldots, m$.

LEMMA 2. *Let $P_1, P_2, \ldots, P_m$ for $m \geqslant 2$ be p.m. on X. An extension P of p.m. $P_1, P_2, \ldots, P_m$ exists if and only if $P_1, P_2, \ldots, P_m$ are concordant in pairs and if, for $m \geqslant 3$, the following condition is satisfied*:

$$\begin{aligned} (2.1)\qquad \bigwedge_{3 \leqslant t \leqslant m} \bigwedge_{(i_1, \ldots, i_t) \subset (1, \ldots, m)} \Big\{\Big[\Big(\bigwedge_{(j_1, j_2, j_3) \subset (i_1, \ldots, i_t)} \bigcap_{l=1}^{3} C_{i_l} = \emptyset\Big) \wedge \\ \wedge (C_{i_1} \cap C_{i_t} \neq \emptyset) \wedge \Big(\bigwedge_{1 \leqslant l \leqslant t-1} C_{i_l} \cap C_{i_{l+1}} \neq \emptyset\Big) \wedge \\ \wedge \big((t \geqslant 4) \wedge (2 \leqslant |k-l| \leqslant t-2) \Rightarrow (C_{i_k} \cap C_{i_l}) = \emptyset\big)\Big] \\ \Rightarrow (c_{i_t}^{i_1} = c_{i_2}^{i_1} c_{i_3}^{i_2} \ldots c_{i_t}^{i_{t-1}})\Big\}. \end{aligned}$$

The extension P is unique if and only if

$$(2.2)\qquad \bigcup_{i=1}^{m} C_i = X,$$

$$(2.3)\qquad \bigvee_{(i_1, \ldots, i_m) \subset (1, \ldots, m)} \bigwedge_{2 \leqslant j \leqslant m} C_{i_j} \cap \Big(\bigcup_{l=1}^{j-1} C_{i_l}\Big) \neq \emptyset.$$

Proof. The necessity of the existence of P. Let P be an extension of p.m. $P_1, P_2, \ldots, P_m$. The concordance of the pairs P_i, P_j for $(i, j) \subset (1, 2, \ldots, m)$ follows immediately from Definition 4.

Now, let $m \geqslant 3$ and let $(i_1, i_2, \ldots, i_t) \subset (1, 2, \ldots, m)$ for $3 \leqslant t \leqslant m$ satisfy the premise of implication (2.1). Without restriction of the generality of our considerations we can put $i_l = l$ for $l = 1, 2, \ldots, t$.

If p.m. P is an extension of $P_1, P_2, \ldots, P_m$, then P is an extension of $P_1, P_2, \ldots, P_t$ for $3 \leqslant t \leqslant m$. This fact and Definition 4 imply

$$\bigwedge_{x \in C_1 \cap C_t} P_1(x) = c_t^1 P_t(x), \qquad \text{where } c_t^1 = e_t/e_1,$$

and

$$\bigwedge_{1 \leqslant i \leqslant t-1} \; \bigwedge_{x \in C_i \cap C_{i+1}} P_i(x) = c_{i+1}^i P_{i+1}(x), \qquad \text{where } c_{i+1}^i = e_{i+1}/e_i.$$

Therefore,

$$c_t^1 e_1 = e_t = c_t^{t-1} e_{t-1} = \ldots = c_t^{t-1} c_{t-1}^{t-2} \ldots c_3^2 c_2^1 e_1,$$

which implies $c_t^1 = c_2^1 c_3^2 \ldots c_t^{t-1}$.

The sufficiency of the existence of P. First, let us observe that if p.m. P is an extension of $P_1, P_2, \ldots, P_m$, then p.m. P' defined by the formula

$$P'(x) = \begin{cases} \dfrac{P(x)}{v} & \text{for } x \in \bigcup_{i=1}^{m} C_i, \\ 0 & \text{for } x \notin \bigcup_{i=1}^{m} C_i, \end{cases}$$

where

$$v = P\Big(\bigcup_{i=1}^{m} C_i\Big),$$

is also an extension of $P_1, P_2, \ldots, P_m$ with the carrier

$$C' = \bigcup_{i=1}^{m} C_i.$$

Therefore, it is enough to prove the existence of the extension P'.

Let P_{i_1} and P_{i_2} be any pair of $P_1, P_2, \ldots, P_m$. Without loss of generality we can put $i_1 = 1$ and $i_2 = 2$. By the assumption of Lemma 2, p.m. P_1 and P_2 are concordant. Hence, by Lemma 1 of [1], there exists p.m. Q_1 with the carrier $C^1 = C_1 \cup C_2$ which is an extension of P_1 and P_2, i.e.

$$(2.4) \qquad Q_1(x) = e_i^1 P_i(x) \quad \text{for } x \in C_i,\ i = 1, 2.$$

Now, let P_{i_1}, P_{i_2} and P_{i_3} be any triple of $P_1, P_2, \ldots, P_m$. As previously, we put $i_1 = 1$, $i_2 = 2$ and $i_3 = 3$. To prove the existence of an extension of P_1, P_2 and P_3 we need only consider the following five relations between the carriers C_1, C_2 and C_3 of P_1, P_2 and P_3, respectively:

(a) $C_1 \cap C_3 = \emptyset$,

(b) $C_2 \cap C_3 = \emptyset$,

(c) $C_1 \cap C_3 \neq \emptyset \wedge C_2 \cap C_3 \neq \emptyset \wedge C_1 \cap C_2 = \emptyset$,

(d) $C_1 \cap C_2 \cap C_3 \neq \emptyset$,

(e) $C_1 \cap C_3 \neq \emptyset \wedge C_2 \cap C_3 \neq \emptyset \wedge C_1 \cap C_2 \neq \emptyset \wedge C_1 \cap C_2 \cap C_3 = \emptyset$.

Let Q_1 denote an extension of P_1 and P_2 in cases (a) and (b). Then we have

$$Q_1(x) = \begin{cases} e_2^1 P_2(x) = e_2^1 c_3^2 P_3(x) & \text{for } x \in C_2 \cap C_3 \text{ (in case (a))}, \\ e_1^1 P_1(x) = e_1^1 c_3^1 P_3(x) & \text{for } x \in C_1 \cap C_3 \text{ (in case (b))}, \end{cases}$$

which proves the concordance of Q_1 and P_3. The last fact and Lemma 1 [1] imply the existence of an extension of Q_1 and P_3 and, at the same time, P_1, P_2 and P_3.

Now, in case (c), let Q_1 denote an extension of P_1 and P_3. Then we have

$$Q_1(x) = e_3^1 P_3(x) = e_3^1 c_2^3 P_2(x) \quad \text{for } x \in C_2 \cap C_3,$$

which proves, similarly as previously, the existence of an extension of P_1, P_2 and P_3.

In case (d) there exists an extension Q_1 of P_1 and P_2. Thus, in view of (2.4) and the concordance in pairs, we have

$$\bigwedge_{x \in C_i \cap C_3} Q_1(x) = e_i^1 P_i(x) = e_i^1 c_3^i P_i(x) \quad \text{for } i = 1, 2 \tag{2.5}$$

and

$$\bigwedge_{x \in C_1 \cap C_2 \cap C_3} Q_1(x) = e_1^1 P_1(x) = e_1^1 c_3^1 P_3(x) = e_2^1 P_2(x) = e_2^1 c_3^2 P_3(x).$$

Hence there exists the constant $c_1 = e_1^1 c_3^1 = e_2^1 c_3^2$ such that $Q_1(x) = c_1 P_3(x)$ for $x \in C^1 \cap C_3$, i.e. Q_1 and P_3 are concordant and, therefore, there exists an extension of P_1, P_2 and P_3.

Now we can see that, for Q_1 being an extension of P_1 and P_2 in case (e), condition (2.5) holds. By (2.4) and (2.1), we have $e_2^1 = e_1^1 c_2^1$ and $c_2^1 c_3^2 = c_3^1$ which together with (2.5) imply the existence of the constant $c_1 = e_1^1 c_2^1 c_3^2$ such that $Q_1(x) = c_1 P_3(x)$ for $x \in C^1 \cap C_3$. Therefore, there exists an extension of P_1, P_2 and P_3.

Thus we have proved that there exists an extension of P_1, P_2 and P_3, say Q_2, with the carrier $C^2 = C^1 \cup C_3 = C_1 \cup C_2 \cup C_3$. The coefficients of the extension Q_2 are $e_1^2 = e^1 e_1^1$, $e_2^2 = e^1 e_2^1$ and e_3^2, where e^1 is the coefficient of the extension Q_2 of the measure Q_1.

Now, let $P_{i_1}, P_{i_2}, \ldots, P_{i_k}, P_{i_{k+1}}$ for $3 \leqslant k \leqslant m-1$ be any $(k+1)$-tuple of $P_1, P_2, \ldots, P_m$. Without loss of generality we can put $i_1 = 1$, $i_2 = 2, \ldots, i_k = k$, $i_{k+1} = k+1$.

Let Q_{k-1} with the carrier

$$C^{k-1} = \bigcup_{l=1}^{k} C_l$$

denote an extension of $P_1, P_2, \ldots, P_k$. At first we take into account the pair Q_{k-1}, P_{k+1}. Obviously,

$$C^{k-1} \cap C_{k+1} = \bigcup_{l=1}^{k} C_l \cap C_{k+1}.$$

If at most one of the sets $C_l \cap C_{k+1}$ for $l = 1, 2, \ldots, k$ is non-empty, then it is easy to see that the pair Q_{k-1}, P_{k+1} is concordant.

Now let us assume that there exist indices $j_1, j_2, \ldots, j_r$, where $(j_1, j_2, \ldots, j_r) \subset (1, 2, \ldots, k)$ and $2 \leqslant r \leqslant k$, such that

$$C_{k+1} \cap \Big(\bigcap_{l=1}^{r} C_{j_l}\Big) \neq \emptyset.$$

As previously, we can put $j_l = l$ for $l = 1, 2, \ldots, r$. Since p.m. $P_1, P_2, \ldots, P_r$ are truly concordant in pairs, we have

$$\begin{aligned} Q_{k-1}(x) &= e_l^{k-1} P_l(x) = e_l^{k-1} c_{k+1}^{l} P_{k+1}(x) \\ &= e_{l-1}^{k-1} c_l^{l-1} c_{k+1}^{l} P_{k+1}(x) = \ldots \\ &= e_1^{k-1} c_2^1 c_3^2 \ldots c_l^{l-1} c_{k+1}^{l} P_{k+1}(x) \quad \text{for } x \in C_{k+1} \cap C_l,\ l = 1, 2, \ldots, r. \end{aligned}$$

Since

$$C_{k+1} \cap \Big(\bigcap_{l=1}^{r} C_l\Big) \neq \emptyset,$$

we obtain

$$c_{k+1}^{1} = c_2^1 c_{k+1}^{2} = \ldots = c_2^1 c_3^2 \ldots c_r^{r-1} c_{k+1}^{r}.$$

Thus there exists a constant $c_{k-1} = e_1^{k-1} c_{k+1}^{1}$ such that

$$Q_{k-1}(x) = c_{k-1} P_{k+1}(x) \quad \text{for } x \in C^{k-1} \cap C_{k+1}. \tag{2.6}$$

Thus the pair Q_{k-1}, P_{k+1} is concordant.

Moreover, let us assume that there exist indices $i_1, i_2, \ldots, i_s$, where $(i_1, i_2, \ldots, i_s) \subset (1, 2, \ldots, k)$ and $2 \leqslant s \leqslant k$, such that the sequence $i_1, i_2, \ldots, i_s, k+1$ satisfies the premise of implication (2.1). As above we can put $i_l = l$ for $l = 1, 2, \ldots, s$. We can observe that in this case

$$C^{k-1} \cap C_{k+1} = C_1 \cap C_{k+1} \cup C_s \cap C_{k+1}.$$

Hence

$$Q_{k-1}(x) = e_i^{k-1} P_i(x) = e_i^{k-1} c_{k+1}^{i} P_{k+1}(x) \quad \text{for } x \in C_i \cap C_{k+1},\ i = 1, s.$$

We want to prove that $e_1^{k-1}c_{k+1}^1 = e_s^{k-1}c_{k+1}^s$. By the concordance in pairs of $P_1, P_2, \ldots, P_s$, the equality

$$e_s^{k-1} = e_1^{k-1}c_2^1 c_3^2 \ldots c_s^{s-1}$$

holds. That fact and assumption (2.1) imply the equality $e_1^{k-1}c_{k+1}^1 = e_s^{k-1}c_{k+1}^s$. Therefore, there exists a constant $c_{k-1} = e_1^{k-1}c_{k+1}^1$ such that (2.6) holds, and this proves that the pair Q_{k-1}, P_{k+1} is concordant.

At last let us assume that there exist two sets of indices $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_s$ $(r+s \leqslant k+1)$ such that

$$C_{k+1} \cap \Big(\bigcap_{l=1}^{r} C_{i_l}\Big) \neq \emptyset$$

and that the sequence $j_1, j_2, \ldots, j_s, k+1$ satisfies the premise of implication (2.1) and, moreover, there exist l $(1 \leqslant l \leqslant r)$ such that $i_l \in (j_1, j_2, \ldots, j_s)$. Without loss of generality we can put $i_1 = j_1$. The above-given facts imply

$$c_{i_2}^{i_1} c_{i_3}^{i_2} \ldots c_{i_r}^{i_{r-1}} c_{k+1}^{i_r} = c_{k+1}^{i_1} = c_{k+1}^{j_1} = c_{j_2}^{j_1} c_{j_3}^{j_2} \ldots c_{j_s}^{j_{s-1}} c_{k+1}^{j_s}.$$

Therefore, the pair Q_{k-1}, P_{k+1} is concordant.

Finally, the above-given considerations allow us to observe that any system of carriers of $P_{i_1}, P_{i_2}, \ldots, P_{i_{k+1}}$ can be investigated in the same way as one of the considered four cases.

We have asserted that the pair Q_{k-1}, P_{k+1} is, under the conditions of Lemma 2, concordant in each case. Therefore, by Lemma 1 of [1], there exists p.m. Q_k with the carrier $C^k = C^{k-1} \cup C_{k+1}$ being an extension of p.m. Q_{k-1} and P_{k+1} with coefficients e^{k-1} and e_{k+1}^k, respectively. It is obvious that p.m. Q_k is also an extension of p.m. $P_1, P_2, \ldots, P_k$ with coefficients

$$e_1^k = e^{k-1}e^{k-2} \ldots e^1 e_1^1, \quad e_2^k = e^{k-1}e^{k-2} \ldots e^1 e_2^1,$$
$$e_3^k = e^{k-1}e^{k-2} \ldots e^2 e_3^2, \ \ldots, \ e_k^k = e^{k-1}e_k^{k-1}, \ e_{k+1}^k,$$

respectively.

Thus there exists p.m. P being an extension of p.m. $P_1, P_2, \ldots, P_m$.

The necessity of the uniqueness of P. Let us assume that (2.2) does not holds and let Q be p.m. with the carrier

$$C_Q = X \setminus \Big(\bigcup_{i=1}^{m} C_i\Big).$$

Then, for an arbitrary b $(0 < b < 1)$, p.m. $R(x) = bP(x) + (1-b)Q(x)$ is also an extension of $P_1, P_2, \ldots, P_m$. Thus p.m. P is not a unique extension of $P_1, P_2, \ldots, P_m$.

Now, let us assume that condition (2.3) is not satisfied. Then, for each subset $(i_1, i_2, \ldots, i_m) \subset (1, 2, \ldots, m)$, there exists a j $(2 \leqslant j \leqslant m)$ such that

$$C_{i_j} \cap \left(\bigcup_{k=1}^{j-1} C_{i_k}\right) = \emptyset .$$

This implies that, for any subset $(i_1, i_2, \ldots, i_m) \subset (1, 2, \ldots, m)$, there exist a number k $(1 \leqslant k \leqslant m)$ and sets

$$C^1 = \bigcup_{j=1}^{k} C_{i_j} \quad \text{and} \quad C^2 = \bigcup_{j=k+1}^{m} C_{i_j}$$

such that $C^1 \cap C^2 = \emptyset$.

Let $P^{(1)}$ and $P^{(2)}$ be two extensions of p.m. $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ and $P_{i_{k+1}}, P_{i_{k+2}}, \ldots, P_{i_m}$, respectively. Then $P(x) = bP^{(1)}(x) + (1-b)P^{(2)}(x)$, where $0 < b < 1$, is also an extension of $P_1, P_2, \ldots, P_m$ and that proves the necessity of (2.3).

The sufficiency of the uniqueness of P. Without loss of generality we can assume that (2.3) is satisfied by the sequence $(1, 2, \ldots, m)$.

Let us take the notation $h = i^j_{l_j}$. By the assumptions of Lemma 2 and the proof of its first part for every j $(1 \leqslant j \leqslant m)$, there exist k_j $(1 \leqslant k_j \leqslant j)$ and a subsequence $(i^j_1, i^j_2, \ldots, i^j_{k_j}) \subset (1, 2, \ldots, j-1)$ such that, for every l_j $(1 \leqslant l_j \leqslant k_j)$, we have

$$C_j \cap C_h \neq \emptyset ,$$

$$P(x) = e_h P_h(x) = e_j P_j(x) = e_h c^h_j P_j(x) \quad \text{for } x \in C_j \cap C_h ,$$

$$e_j = c^h_j e_h .$$

Hence, we obtain the system of independent equations

$$e_2 = c^1_2 e_1 , \quad e_j = c_j^{i^j_1} e_{i^j_1} \quad (j = 3, 4, \ldots, m), \tag{2.7}$$

where $C_1 \cap C_2 \neq \emptyset$ and $C_j \cap C_{i^j_1} \neq \emptyset$ $(j = 3, 4, \ldots, m)$.

To determine uniquely m unknown coefficients $e_1, e_2, \ldots, e_m$ we must get one equation more. For this purpose let us define the following sets:

$$D_1 = C_1 \quad \text{and} \quad D_i = C_i \setminus \left(\bigcup_{j=1}^{i-1} D_j\right) \quad (i = 2, 3, \ldots, m).$$

We can see that

$$X = \bigcup_{i=1}^{m} D_i \quad \text{and} \quad D_i \cap D_j = \emptyset \quad \text{for } (i, j) \subset (1, 2, \ldots, m),\ i \neq j ,$$

and

$$(2.8)\qquad 1 = P(X) = \sum_{i=1}^{m} \sum_{x \in D_i} P(x) = \sum_{i=1}^{m} e_i \sum_{x \in D_i} P_i(x)$$
$$= e_1 + v_2 e_2 + \ldots + v_m e_m,$$

where

$$v_i = \sum_{x \in D_i} P_i(x) \quad (i = 2, 3, \ldots, m).$$

The system (2.7) together with (2.8) gives

$$(2.9)\qquad \begin{cases} e_1 + v_2 e_2 + v_3 e_3 + \ldots + v_m e_m = 1, \\ e_2 = c_2^1 e_1, \\ e_j = c_j^{i_1^j} e_{i_1^j} \quad (j = 3, 4, \ldots, m). \end{cases}$$

One can observe that the system of equations (2.9) has the unique solution. This statement completes the proof of Lemma 2.

Let $P_1, P_2, \ldots, P_m$ be probability measures satisfying (2.3) and let P^* denote the class of all their extensions. For $P \in P^*$, let $e_1 = t_P, e_2, e_3, \ldots, e_m$ be the proportionality constants for $P_1, P_2, \ldots, P_m$, respectively.

Now we prove the following

LEMMA 3. *A function t_P attains its maximum for the extension $P \in P^*$ with the carrier*

$$C_P = \bigcup_{i=1}^{m} C_i.$$

Moreover, if

$$a = \max_{P \in P^*} t_P \quad \text{and} \quad C_P \neq X,$$

then, for every β $(0 < \beta < a)$, there exists at least one measure $P_\beta \in P^*$ with $t_{P_\beta} = \beta$.

Proof. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ be a permutation of $P_1, P_2, \ldots, P_m$ $(m \geqslant 2)$ satisfying (2.3) and let Q_{j-1} be an extension of $P_{i_1}, P_{i_2}, \ldots, P_{i_j}$. Without loss of generality we can put $i_l = l$ $(l = 1, 2, \ldots, m)$.

Now, putting $t_{Q_1} = e_1^1$ and $t_{Q_i} = e^{i-1}$ $(i = 2, 3, \ldots, m-1)$, we can see, by Lemma 2 of [1], that t_{Q_1} and t_{Q_i} $(i = 2, 3, \ldots, m-1)$ attain their maxima for extensions Q_1 and Q_i with the carriers $C^1 = C_1 \cup C_2$ and $C^i = C^{i-1} \cup C_{i+1}$ $(i = 2, 3, \ldots, m-1)$, respectively. In view of the proof of Lemma 2 we have

$$t_P = e_1 = e^{m-2} e^{m-3} \ldots e^1 e_1^1 = t_{Q_{m-1}} t_{Q_{m-2}} \ldots t_{Q_1}.$$

Hence t_P attains its maximum together with all coefficients t_{Q_i} $(i = 1, 2, \ldots, m-1)$, i.e. if $C_P = C_1 \cup C_2 \cup \ldots \cup C_m$.

Let Q_1^* and Q_i^* $(i = 2, 3, \ldots, m-1)$ be the classes of all extensions of pairs P_1, P_2 and Q_{i-1}, P_{i+1} $(i = 2, 3, \ldots, m-1)$, respectively. Moreover, let

$$a_i = \max_{Q \epsilon Q_i^*} t_Q \quad \text{and} \quad \beta_i \epsilon (0, a_i) \quad \text{for } i = 1, 2, \ldots, m.$$

By Lemma 2 of [1] for $i = 1, 2, \ldots, m-1$ there exists a $Q_{\beta_i} \epsilon Q_i^*$ such that $t_{Q_i} = \beta_i$. We can see that

$$a = a_1 a_2 \ldots a_{m-1} \quad \text{and} \quad t_P = t_{Q_{m-1}} t_{Q_{m-2}} \ldots t_{Q_1}.$$

Hence, for any $\beta = \beta_1 \beta_2 \ldots \beta_{m-1}$ $(0 < \beta < a)$, there exists p.m. $P_\beta \epsilon P^*$ such that $t_{P_\beta} = \beta$, and this completes the proof of Lemma 3.

3. Shrinkage of stochastic automata. Let $M_i = (U, a^{(i)}, \pi^{(i)})$ for $i = 1, 2, \ldots, m$, $m \geqslant 2$, be s.a., and let C_{i0}, C_{ij} for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ be carriers of p.m. $a^{(i)}$, $\pi_j^{(i)}$, respectively.

Definition 5. Stochastic automata $M_1, M_2, \ldots, M_m$ $(m \geqslant 2)$ are said to be *concordant* (*truly concordant*) if p.m. $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ and $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$ $(j = 1, 2, \ldots, n)$ are concordant (truly concordant).

We denote by s_{i0} (e_{i0}) and s_{ij} (e_{ij}), where $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, the coefficients of shrinkages (extensions) a and π_j of p.m. $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ and $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$, respectively.

THEOREM 1. *If s.a.* $M_i = (U, a^{(i)}, \pi^{(i)})$ *for* $i = 1, 2, \ldots, m$, $m \geqslant 2$, *are truly concordant and*

$$s_{i1} = s_{i2} = \ldots = s_{in} \quad \text{for } i = 1, 2, \ldots, m, \tag{3.1}$$

then there exists s.a. $M = (U, a, \pi)$ *such that*

$$L(N) = \bigcap_{i=1}^{m} L_i(N) \quad \text{for } N \geqslant 1, \tag{3.2}$$

(3.3) P_N *is a shrinkage of p.m.* $P_N^{(1)}, P_N^{(2)}, \ldots, P_N^{(m)}$ *for* $N \geqslant 1$.

Proof. By Lemma 1 there exist p.m. a and π_j $(j = 1, 2, \ldots, n)$ with carriers

$$C_0 = \bigcap_{i=1}^{m} C_{i0} \quad \text{and} \quad C_j = \bigcap_{i=1}^{m} C_{ij}$$

being shrinkages of $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ and $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$ $(j = 1, 2, \ldots, n)$, respectively. P.m. a and π_j $(j = 1, 2, \ldots, n)$ define s.a. $M = (U, a, \pi)$.

First we prove that s.a. $M = (U, a, \pi)$ satisfies (3.2).

For every N-word

$$(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \epsilon \bigcap_{r=1}^{m} L_r(N)$$

we have

$$(3.4)\qquad \Big\{(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in \bigcap_{r=1}^{m} L_r(N)\Big\}$$
$$\Leftrightarrow \Big\{\prod_{r=1}^{m} a_{i_1}^{(r)} > 0 \text{ and } \prod_{r=1}^{m} \pi_{i_j i_{j+1}}^{(r)} > 0 \text{ for } j = 1, 2, \ldots, N-1\Big\}$$
$$\Leftrightarrow \Big\{u_{i_1} \in \bigcap_{r=1}^{m} C_{r0} \text{ and } u_{i_{j+1}} \in \bigcap_{r=1}^{m} C_{ri_j} \text{ for } j = 1, 2, \ldots, N-1\Big\}$$
$$\Leftrightarrow \{a_{i_1} > 0 \text{ and } \pi_{i_j i_{j+1}} > 0 \text{ for } j = 1, 2, \ldots, N-1\}$$
$$\Leftrightarrow \{P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) > 0\} \Leftrightarrow \{(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in L(N)\}.$$

Hence we get (3.2).

Now we are going to establish (3.3). Obviously, the carriers of P_N and $P_N^{(i)}$ $(i = 1, 2, \ldots, m)$ are the sets $L(N)$ and $L_i(N)$ $(i = 1, 2, \ldots, m)$, respectively.

By (3.4), for every N-word $(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in L(N)$, we have

$$\begin{aligned} P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) &= a_{i_1}\pi_{i_1 i_2}\pi_{i_2 i_3} \ldots \pi_{i_{N-1} i_N} \\ &= s_{r0}\, a_{i_1}^{(r)} s_{ri_1} \pi_{i_1 i_2}^{(r)} s_{ri_2} \pi_{i_2 i_3}^{(r)} \ldots s_{ri_{N-1}} \pi_{i_{N-1} i_N}^{(r)} \\ &= s_N^{(r)} P_N^{(r)}(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \quad \text{for } r = 1, 2, \ldots, m. \end{aligned}$$

Taking into account (3.1) it can be seen that the coefficients $s_N^{(r)}$ $(r = 1, 2, \ldots, m)$ are independent of N-word from $L(N)$, and that completes the proof of Theorem 1.

4. Extension of stochastic automata. Let A_{1k}, A_{2k}, B_{1k} and B_{2k} $(k = 2, 3, \ldots, m)$ be subsets of a set U defined as follows:

$$A_{1k} = \Big\{u_j \in U\colon \text{there exists a subsequence } (i_1, i_2, \ldots, i_k) \subset (1, 2, \ldots, m) \text{ such that } C_{i_r j} \cap \Big(\bigcup_{l=1}^{r-1} C_{i_l j}\Big) \neq \emptyset \text{ for } r = 2, \ldots, k\Big\},$$

$$A_{2k} = U \setminus A_{1k},$$

$$B_{1k} = \Big\{u_j \in U\colon \text{there exists a subsequence } (i_1, i_2, \ldots, i_k) \subset (1, 2, \ldots, m) \text{ such that } \bigcup_{l=1}^{m} C_{i_l j} = U\Big\},$$

$$B_{2k} = U \setminus B_{1k}.$$

We can see that

$$A_{12} \supset A_{13} \supset \ldots \supset A_{1m} \quad \text{and} \quad B_{12} \subset B_{13} \subset \ldots \subset B_{1m}.$$

Let c_{l0}^k and c_{lj}^k for $(k, l) \subset (1, 2, \ldots, m)$, $j = 1, 2, \ldots, n$, be the coefficients of concordance of the pairs $a^{(k)}$, $a^{(l)}$ and $\pi_j^{(k)}$, $\pi_j^{(l)}$, respectively.

THEOREM 2. *If s.a.* $M_i = (U, a^{(i)}, \pi^{(i)})$ *for* $i = 1, 2, \ldots, m$, $m \geqslant 2$, *are such that p.m.* $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ *and* $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$ *for* $j = 1, 2, \ldots, n$ *satisfy the assumptions of Lemma 2 concerning the existence of extension of p.m. and either* $A_{22} = U$ *or*

$$\bigwedge_{1\leqslant k,l\leqslant m}\ \bigwedge_{1\leqslant j\leqslant n} [(C_{kj}\cap C_{lj} \neq \emptyset) \Rightarrow (c_{lj}^k = c_l^k)], \tag{4.1}$$

$$\bigwedge_{2\leqslant l\leqslant m} A_{2l}\cap B_{1l} = \emptyset, \tag{4.2}$$

$$\bigwedge_{2\leqslant l\leqslant m}\ \bigvee_{e_{i_1},\ldots,e_{i_l}>0}\ \bigwedge_{1\leqslant j\leqslant n} [(u_j\in A_{1l}\cap B_{1l}) \Rightarrow (e_{i_1 j} = e_{i_1}, e_{i_2 j} = e_{i_2}, \ldots, e_{i_l j} = e_{i_l})], \tag{4.3}$$

$$\bigwedge_{2\leqslant l\leqslant m}\ \bigwedge_{1\leqslant j\leqslant n} [(u_j\in A_{1l}\cap B_{2l}) \Rightarrow (e_{i_1 j} \geqslant e_{i_1}, e_{i_2 j} \geqslant e_{i_2}, \ldots, e_{i_l j} \geqslant e_{i_l})], \tag{4.4}$$

where $e_{i_1}, e_{i_2}, \ldots, e_{i_l}$ *are constants of* (4.3), *then there exists s.a.* $M = (U, a, \pi)$ *such that*

$$L(N) \supset \bigcup_{i=1}^{m} L_i(N) \quad \textit{for } N \geqslant 1 \tag{4.5}$$

and

(4.6) P_N *is an extension of p.m.* $P_N^{(1)}, P_N^{(2)}, \ldots, P_N^{(m)}$ *for* $N \geqslant 1$.

Proof. By the assumptions and Lemma 2, there exist p.m. a and π_j $(j = 1, 2, \ldots, n)$ being extensions of $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ and $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$ $(j = 1, 2, \ldots, n)$, respectively. P.m. a and π_j $(j = 1, 2, \ldots, n)$ define s.a. $M = (U, a, \pi)$.

First we prove that s.a. $M = (U, a, \pi)$ satisfies (4.5).

For every N-word

$$(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in \bigcup_{r=1}^{m} L_r(N),$$

we have

$$\begin{aligned}
&\Big\{(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in \bigcup_{r=1}^{m} L_r(N)\Big\} \\
&\Leftrightarrow \Big\{\sum_{r=1}^{m} P_N^{(r)}(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) > 0\Big\} \\
&\Rightarrow \Big\{\sum_{r=1}^{m} a_{i_1}^{(r)} > 0 \text{ and } \sum_{r=1}^{m} \pi_{i_j i_{j+1}}^{(r)} > 0 \text{ for } j = 1, 2, \ldots, N-1\Big\} \\
&\Leftrightarrow \Big\{u_{i_1} \in \bigcup_{r=1}^{m} C_{r0} \text{ and } u_{i_{j+1}} \in \bigcup_{r=1}^{m} C_{ri_j} \text{ for } j = 1, 2, \ldots, N-1\Big\} \\
&\Rightarrow \{a_{i_1} > 0 \text{ and } \pi_{i_j i_{j+1}} > 0 \text{ for } j = 1, 2, \ldots, N-1\} \\
&\Leftrightarrow \{P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) > 0\} \Leftrightarrow \{(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in L(N)\}.
\end{aligned} \tag{4.7}$$

Hence we get (4.5).

Now we are going to establish (4.6).

By (4.7), for every N-word $(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \in L(N)$, we have

$$(4.8)\quad \begin{aligned} P_N(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) &= a_{i_1}\pi_{i_1i_2}\pi_{i_2i_3}\ldots\pi_{i_{N-1}i_N} \\ &= e_{r0}a_{i_1}^{(r)}e_{ri_1}\pi_{i_1i_2}^{(r)}\ldots e_{ri_{N-1}}\pi_{i_{N-1}i_N}^{(r)} \\ &= e_{r0}e_{ri_1}e_{ri_2}\ldots e_{ri_{N-1}}P_N^{(r)}(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \\ &= e_N^{(r)}P_N^{(r)}(u_{i_1}, u_{i_2}, \ldots, u_{i_N}) \\ &\qquad \text{for } r = 1, 2, \ldots, m. \end{aligned}$$

To prove that P is an extension of $P_N^{(1)}, P_N^{(2)}, \ldots, P_N^{(m)}$ we must show that

$$(4.9)\qquad e_{r1} = e_{r2} = \ldots = e_{rm} \quad \text{for } r = 1, 2, \ldots, m,$$

i.e. that a coefficient $e_N^{(r)}$ is independent of N-word from $L_r(N)$ for $r = 1, 2, \ldots, m$.

First let us consider the case where $A_{22} = U$. Then p.m.

$$\pi_j(u) = e_{1j}\pi_j^{(1)}(u) + e_{2j}\pi_j^{(2)}(u) + \ldots + e_{mj}\pi_j^{(m)}(u),$$

$$\text{where } \sum_{i=1}^{m} e_{ij} = 1,\ j = 1, 2, \ldots, n,$$

is an extension of $\pi_j^{(1)}, \pi_j^{(2)}, \ldots, \pi_j^{(m)}$. Obviously, we can choose coefficients e_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$e_{i1} = e_{i2} = \ldots = e_{in} = e_i \quad \text{for } i = 1, 2, \ldots, m \text{ and } \sum_{i=1}^{m} e_i = 1,$$

which proves equality (4.9) in the case where $A_{22} = U$.

Now let us consider the case where $A_{22} \neq U$. We take into account the pair $\pi^{(1)}, \pi^{(2)}$. For $i = 1, 2$ the equalities $e_{i1} = e_{i2} = \ldots = e_{in} = e_i$ hold by the proof of Theorem 2 in [1].

Considering the triple $\pi^{(1)}, \pi^{(2)}, \pi^{(3)}$, we need first observe that U can be divided into four disjoint subsets: $A_{13}\cap B_{13}$, $A_{13}\cap B_{23}$, $A_{23}\cap B_{13}$ and $A_{23}\cap B_{23}$. By assumption (4.2) we have $A_{23}\cap B_{13} = \emptyset$.

If $u_j \in A_{13}\cap B_{13}$ for $j \in (1, 2, \ldots, n)$, then, by (4.3), there exists a constant e_3 such that $e_{3j} = e_3$. By (4.1), (4.3), (4.4) and Lemma 3, for $j \in (1, 2, \ldots, n)$ such that $u_j \in A_{13}\cap B_{23}$, we have $e_{3j} = c_{1j}^3 e_{1j} = c_1^3 e_1 = e_3$. Now, let $u_j \in A_{23}\cap B_{23}$ for some $j \in (1, 2, \ldots, n)$. If there exists an $i \in (1, 2)$ such that $C_{ij}\cap C_{3j} \neq \emptyset$, then, by (4.1), (4.3) and (4.4), we have $e_{3j} = c_{ij}^3 e_{ij}$

$= c_i^3 e_i = e_3$. If $C_{ij} \cap C_{3j} = \emptyset$ for $i = 1, 2$, then the proof of Lemma 2 allows us to choose e_{3j} such that $e_{3j} = e_3$. Hence,

$$e_{i1} = e_{i2} = \ldots = e_{in} = e_i \quad \text{for } i = 1, 2, 3.$$

Now let us assume that, for p.m. $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}$ $(2 \leqslant k \leqslant m-1)$,

$$e_{i1} = e_{i2} = \ldots = e_{in} = e_i \quad \text{for } i = 1, 2, \ldots, k.$$

We prove that $e_{k+1,j} = e_{k+1}$ for $j = 1, 2, \ldots, n$. For this purpose let us divide U into four disjoint subsets: $A_{1,k+1} \cap B_{1,k+1}$, $A_{1,k+1} \cap B_{2,k+1}$, $A_{2,k+1} \cap B_{1,k+1}$ and $A_{2,k+1} \cap B_{2,k+1}$. By assumption (4.2), we then obtain $A_{2,k+1} \cap B_{1,k+1} = \emptyset$.

If $u_j \in A_{1,k+1} \cap A_{1,k+1}$ for $j \in (1, 2, \ldots, n)$, then, by (4.3), there exists a constant e_{k+1} such that $e_{k+1,j} = e_{k+1}$. Moreover, by (4.1), (4.3), (4.4) and Lemma 3, for $j \in (1, 2, \ldots, n)$ such that $u_j \in A_{1,k+1} \cap B_{2,k+1}$, we have

$$e_{k+1,j} = c_{1j}^{k+1} e_{1j} = c_1^{k+1} e_1 = e_{k+1}.$$

Now let $u_j \in A_{2,k+1} \cap B_{2,k+1}$ for some $j \in (1, 2, \ldots, n)$. If there exists an $i \in (1, 2, \ldots, k)$ such that $C_{ij} \cap C_{k+1,j} \neq \emptyset$, then, by (4.1), (4.3) and (4.4), we get $e_{k+1,j} = e_{k+1}$. If $C_{ij} \cap C_{k+1,j} = \emptyset$ for $j = 1, 2, \ldots, k$, then the proof of Lemma 2 allows us to choose $e_{k+1,j}$ such that $e_{k+1,j} = e_{k+1}$. Hence

$$e_{i1} = e_{i2} = \ldots = e_{in} = e_i \quad \text{for } i = 1, 2, \ldots, k+1,$$

which completes the proof of Theorem 2.

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UWAGI O ROZSZERZANIU SKOŃCZONEGO ZBIORU MASZYN STOCHASTYCZNYCH

STRESZCZENIE

W pracy zajmujemy się problemem zawężania i rozszerzania zbioru maszyn stochastycznych. Otrzymane twierdzenia są uogólnieniem wyników Bartoszyńskiego [1].