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ON THE INEQUALITY OF CRAMÉR-RAO TYPE IN SEQUENTIAL ESTIMATION THEORY

1. Introduction. Let $\xi_\vartheta(t)$ be a stochastic process with continuous time parameter, defined on a probability space $(\Omega, \mathscr{F}, P)$, and with probability distributions entirely defined by a given parameter $\vartheta \in \Theta$. The time τ up to which the process is realized is a Markov stopping time. Knowing values τ and $\xi_\vartheta(\tau)$ we wish to estimate the value of a function $g(\vartheta)$ of the parameter ϑ. Thus the problem is to find Markov stopping times τ and estimators $f(\tau, \xi_\vartheta(\tau))$ having some optimal properties. A sequential plan will be defined by a Markov stopping time τ and the unbiased estimator f of the function $g(\vartheta)$. We assume the quadratic loss function and the notion of optimality of the plan will refer to the lower variance bound for the estimator f. This lower bound will be determined by the inequality which is the analogue of the Cramér-Rao inequality in a classical case. It is worthwhile to remark that the problem of the optimal unbiased sequential estimation even in the classical Bernoulli scheme for various types of loss functions is not fully solved at present.

In the continuation of this paper we shall consider the problem of optimality of sequential plans for the exponential class of processes (Definition 4) and for this class we shall generalize the results obtained by Trybuła in [4], concerning efficiently estimable functions.

2. Sudakov lemma (1). Let

$$\{\xi_\vartheta\} = \{\xi_\vartheta(t) = x(t, \omega), \omega \in (\Omega, \mu_\vartheta), \vartheta \in \Theta, t \in T\}$$

be a family of stochastically continuous processes whose probability distributions are entirely defined by a given parameter ϑ. We denote by X $(X \subseteq R^n)$ the space of values of the processes of $\{\xi_\vartheta\}$ and we mean by T the half-line $\langle 0, \infty)$. Moreover, let $\mathscr{F}_t$, $t \in T$, denote the σ-algebra generated

(1) See [2], p. 55-59, and [3].

by the functions $x(s, \omega)$, $s \leqslant t$. We suppose that, for any $t < \infty$, all measures μ_ϑ are absolutely continuous with respect to a certain measure μ_{ϑ_0} on the σ-algebra $\mathscr{F}_t$.

The function $\tau(\omega)$ measurable on Ω with values in $T \cup \{\infty\}$ is said to be a *Markov stopping time* if $\{\omega: \tau(\omega) \leqslant t\} \in \mathscr{F}_t$ for every $t \in T$.

If the process $\xi_\vartheta(t) = x(t, \omega)$ is measurable, then the function $x\big(\tau(\omega), \omega\big)$ is measurable on Ω. After Sudakov, the functions $\tau(\omega)$ and $x\big(\tau(\omega), \omega\big)$, both measurable on Ω, generate, for every ϑ, the measure m on the product space $T \times X$ according to the mapping $\omega \to \big(\tau(\omega), x(\tau(\omega), \omega)\big) = Z(\omega)$ in the following manner: for a Borel set $C \subset T \times X$, we put $m(C) = \mu\big(Z^{-1}(C)\big)$.

If the measure μ_ϑ corresponding to the process $\xi_\vartheta(t)$ is absolutely continuous with respect to a certain measure corresponding to a particular value $\vartheta = \vartheta_0$, then the measure m_ϑ has the same property. Further, under certain additional assumptions which are essential in the sequential estimation theory, the Sudakov lemma permits to determine the Radon-Nikodym derivative $dm_\vartheta/dm_{\vartheta_0}$.

Let $p_{\vartheta_0}(t, x; \vartheta)$ be the distribution density of values of the process $\xi_\vartheta(t)$ relative to the distribution of values of the process $\xi_{\vartheta_0}(t)$ at the time t.

SUDAKOV LEMMA. *Let the functions $x(t, \omega)$ of the variable t be right-continuous for almost every $\omega \in \Omega$ (with respect to μ_{ϑ_0}). Suppose that the function $p_{\vartheta_0}(t, x; \vartheta)$ is continuous with respect to both arguments t and x on $T \times X$. If the family $\{\xi_\vartheta\}$ is such that the end time t determines a sufficient statistic* (2)*, then the measures m_ϑ are absolutely continuous with respect to the measure m_{ϑ_0} and $dm_\vartheta/dm_{\vartheta_0} = p_{\vartheta_0}(t, x; \vartheta)$.*

In some particular cases the densities $dm_\vartheta/dm_{\vartheta_0}$ were obtained first by Trybuła in [4].

If the paths of the process $\xi_\vartheta(t)$ are right-continuous and the set $C \subset T \times X$ is closed, then, following Sudakov [3], the Markov stopping time

$$\tau(\omega) = \sup\big\{t': \big(t, x(t, \omega)\big) \notin C,\ 0 \leqslant t \leqslant t'\big\}$$

is the time of the first attainment of the set C. If $\tau(\omega)$ is the time of the first attainment of some closed set $C \subset T \times X$, then the support of the measure m is contained in this set.

The Sudakov lemma, published in 1969 [3], goes a long way towards the further research in the sequential estimation theory for stochastic processes (see [2], Chapter 12, and [5], where the Sudakov lemma is used).

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(2) For the definition, see [2], p. 56, or [3]. If we denote by $x_t(\omega)$ the state of the process $\xi_\vartheta(t)$ at a time t, then instead of "the end time t determines a sufficient statistic" we also say "the random variable $x_t(\omega)$ is a sufficient statistic for the parameter ϑ".

Taking into account the Sudakov result of the Radon-Nikodym derivative $dm_\vartheta/dm_{\vartheta_0}$ determination, one can give (by a similar method as in the classical case) the lower variance bound for the parameter estimator in a class of processes for which the Sudakov lemma is valid.

3. Lower variance bound for the parameter estimator. Let $U = T\times X$, and let $\mathscr{B}$ be the σ-algebra of Borel subsets of U. By $t = t(u)$ and $x = x(u)$ are meant the components of the point $u \in U$.

In the sequel we assume that values of the parameter ϑ belong to some closed interval $\langle a, b\rangle$. Let $g(\vartheta)$ be a function of the variable ϑ. The real function f defined on U and $\mathscr{B}$-measurable is called an *estimator of the parameter* $Q = g(\vartheta)$.

Definition 1. By a *sequential plan* we mean any triplet (τ, g, f), consisting of a Markov stopping time τ, the function $g(\vartheta)$ and its estimator f, if the condition $\mathrm{P}(\{\omega\colon 0<\tau(\omega)<\infty\}) = 1$ is valid for all $\vartheta\in\langle a, b\rangle$ and f is an unbiased estimator of the function $g(\vartheta)$, i.e.

$$\mathrm{E}_\vartheta(f) = \int\limits_U f(u)p_{\vartheta_0}(t, x;\ \vartheta)m_{\vartheta_0}(du) = g(\vartheta) \quad \text{for every } \vartheta\in\langle a, b\rangle. \tag{1}$$

Moreover, we consider only such estimators for which $\mathrm{E}_\vartheta(f^2)$ exists and is finite, i.e.

$$\mathrm{E}_\vartheta(f^2) = \int\limits_U f^2(u)p_{\vartheta_0}(t, x;\ \vartheta)m_{\vartheta_0}(du) < \infty \tag{2}$$

in the underlying interval of values of ϑ.

Further, suppose that the functions $g(\vartheta)$ and $p_{\vartheta_0}(t, x;\ \vartheta)$ satisfy certain regularity conditions. Namely, let $g(\vartheta)$ be differentiable in the interval $\langle a, b\rangle$ and such that its derivative $g'(\vartheta)$ is not equal to zero for any $\vartheta\in\langle a, b\rangle$. Moreover, assume that, for any fixed ϑ, there exist a function $G(t, x)$ and an $\varepsilon > 0$ such that

$$\left|\frac{p_{\vartheta_0}(t, x;\ \vartheta') - p_{\vartheta_0}(t, x;\ \vartheta)}{(\vartheta'-\vartheta)p_{\vartheta_0}(t, x;\ \vartheta)}\right| < G(t, x), \qquad |\vartheta'-\vartheta| < \varepsilon, \tag{3}$$

where

$$\int\limits_U G^2(t, x)p_{\vartheta_0}(t, x;\ \vartheta)m_{\vartheta_0}(du) < \infty. \tag{4}$$

Let, for a given sequential plan (τ, g, f) and ϑ, $\mathrm{D}^2_\vartheta(f)$ denote the variance of the estimator f.

The following theorem holds:

THEOREM. *Under the above-given assumptions, for each sequential plan* (τ, g, f), *we have*

$$\mathrm{D}^2_\vartheta(f) \geqslant \frac{[g'(\vartheta)]^2}{\int_U [\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)/\partial\vartheta]^2 p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du)}; \tag{5}$$

the equality holds at a particular value of the parameter ϑ *if and only if*

$$f(u) - g(\vartheta) = h(\vartheta)\frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} \tag{6}$$

almost everywhere with respect to (a.e.w.r.t.) the measure m_{ϑ_0}, *where* $h(\vartheta) \neq 0$.

Proof. Since the estimator f of the function $g(\vartheta)$ is unbiased, we can write

$$\int_U f(u)\frac{p_{\vartheta_0}(t, x;\ \vartheta') - p_{\vartheta_0}(t, x;\ \vartheta)}{(\vartheta' - \vartheta) p_{\vartheta_0}(t, x;\ \vartheta)} p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du) = \frac{g(\vartheta') - g(\vartheta)}{\vartheta' - \vartheta}. \tag{7}$$

Condition (3) implies that

$$\left| f(u)\frac{p_{\vartheta_0}(t, x;\ \vartheta') - p_{\vartheta_0}(t, x;\ \vartheta)}{(\vartheta' - \vartheta) p_{\vartheta_0}(t, x;\ \vartheta)} \right| < |f(u)G(t, x)|,$$

where $\mathrm{E}_\vartheta(|fG|) < \infty$ for $\mathrm{E}_\vartheta(f^2) < \infty$ and $\mathrm{E}_\vartheta(G^2) < \infty$, according to assumptions (2) and (4). Then from the Lebesgue bounded convergence theorem, after evaluating the limits of both sides of (7) as ϑ' tends to ϑ we have

$$\int_U f(u)\frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du) = g'(\vartheta). \tag{8}$$

In an analogous way, taking into account assumptions (3), (4) and the condition

$$\int_U p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du) = 1 \quad \text{for each } \vartheta \in \langle a, b\rangle,$$

we obtain

$$\int_U \frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du) = 0. \tag{9}$$

Therefore,

$$\begin{aligned}\int_U f(u)\frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du) \\ = \int_U [f(u) - g(\vartheta)]\frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} p_{\vartheta_0}(t, x;\ \vartheta) m_{\vartheta_0}(du).\end{aligned} \tag{10}$$

From equalities (8), (10) and the Schwartz inequality it follows that

$$(11)\quad [g'(\vartheta)]^2 \leqslant \int_U [f(u)-g(\vartheta)]^2 p_{\vartheta_0}(t,x;\vartheta)\, m_{\vartheta_0}(du) \int_U \left[\frac{\partial \ln p_{\vartheta_0}(t,x;\vartheta)}{\partial\vartheta}\right]^2 p_{\vartheta_0}(t,x;\vartheta)\, m_{\vartheta_0}(du).$$

Hence and in view of condition (1) we have

$$[g'(\vartheta)]^2 \leqslant \mathrm{D}^2_\vartheta(f) \int_U \left[\frac{\partial \ln p_{\vartheta_0}(t,x;\;\vartheta)}{\partial\vartheta}\right]^2 p_{\vartheta_0}(t,x;\;\vartheta)\, m_{\vartheta_0}(du)$$

which yields inequality (5).

Inequality (5) becomes the equality at a particular value of ϑ if and only if the equality in the Schwartz inequality holds, i.e. if there exist constants c_1 and c_2 (which can both depend on ϑ) not both zero and such that

$$(12)\quad c_1(\vartheta)[f(u)-g(\vartheta)] = c_2(\vartheta)\frac{\partial \ln p_{\vartheta_0}(t,x;\;\vartheta)}{\partial\vartheta} \quad \text{a.e.w.r.t. } m_\vartheta.$$

Since the density $p_{\vartheta_0}(t,x;\;\vartheta)$ should be strictly positive, it follows that the equality holds a.e.w.r.t. m_{ϑ_0}. Moreover, the derivative $g'(\vartheta)$ has no zeros on the interval $\langle a, b\rangle$, and, consequently, inequality (11) implies that neither $f(u)-g(\vartheta)$ nor $\partial \ln p_{\vartheta_0}(t,x;\;\vartheta)/\partial\vartheta$ can be equal to zero a.e.w.r.t. m_{ϑ_0}. Then, neither $c_1(\vartheta)$ nor $c_2(\vartheta)$ can be equal to zero. Therefore, equality (12) holds if and only if equality (6) holds a.e.w.r.t. m_{ϑ_0}, where $h(\vartheta) \neq 0$. Thus the proof of the theorem is complete.

Definition 2. A sequential plan (τ, g, f) is said to be *efficient* for a given value ϑ if inequality (5) becomes the equality at ϑ. The estimator f is then called *efficient* at this value ϑ, and the function g is *efficiently estimable* at the point ϑ.

It follows from the Theorem that *an estimator f is efficient at a given value ϑ if and only if it is of the form*

$$(13)\quad f(u) = h(\vartheta)\frac{\partial \ln p_{\vartheta_0}(t,x;\;\vartheta)}{\partial\vartheta} + g(\vartheta) \quad \textit{a.e.w.r.t. } m_{\vartheta_0}.$$

Definition 3. A sequential plan (τ, g, f) is said to be *efficient* if it is efficient for each $\vartheta \in \langle a, b\rangle$. The estimator f is then called *efficient*, and the function g is *efficiently estimable.*

4. Optimal sequential estimation for the exponential class of processes. We take into consideration some class of processes for which the Sudakov lemma is valid. We suppose that, for this class, the requirements of the Theorem hold. We denote by $x_t(\omega)$ the state of the process $\xi_\vartheta(t)$ at a time t, and the random variable $x(\tau(\omega), \omega)$ will be denoted, more simply, by $x_\tau(\omega)$.

A. We introduce the following

Definition 4. By the *exponential class of processes* we mean the class of homogeneous processes with independent increments satisfying $P(\{\omega: x_0(\omega) = 0\}) = 1$ and defined by the exponential family of probability distribution densities (with respect to the Lebesgue or the counting measure)

$$(14)\qquad p(t, x;\ \vartheta) = r(t, x)\exp[w_1(\vartheta)t + w_2(\vartheta)x]$$
$$\text{for } t > 0,\ x \in X \subseteq R,\ \vartheta \in \langle a, b\rangle,$$

where the function $r(t, x)$ is strictly positive.

The right-continuity of paths is assumed and, for this class of processes, the random variable $x_t(\omega)$ is a sufficient statistic for the parameter ϑ. In the sequential estimation theory, the exponential class of processes was also treated by Franz and Winkler in [1].

Let us assume that the functions $w_1(\vartheta)$ and $w_2(\vartheta)$ are differentiable in the interval $\langle a, b\rangle$ and their derivatives $w_1'(\vartheta)$ and $w_2'(\vartheta)$ do not vanish for any $\vartheta \in \langle a, b\rangle$.

Let us remark that the Poisson, negative-binomial, gamma and Wiener (with linear drift) processes belong to the underlying class.

By (14) we have

$$(15)\qquad p_{\vartheta_0}(t, x;\ \vartheta) = \exp[w_1(\vartheta)t + w_2(\vartheta)x]\exp[-w_1(\vartheta_0)t - w_2(\vartheta_0)x].$$

It follows from the Sudakov lemma that there exists a countably additive measure ϱ_τ on $\mathcal{B}$ independent of ϑ and such that, for each $\vartheta \in \langle a, b\rangle$,

$$(16)\qquad \begin{aligned} &P\big(\{\omega: (\tau(\omega), x_\tau(\omega)) \in C\}\big) \\ &= \int_C \exp[w_1(\vartheta)t + w_2(\vartheta)x]\exp[-w_1(\vartheta_0)t - w_2(\vartheta_0)x]\, m_{\vartheta_0}(du) \\ &\overset{\text{df}}{=} \int_C \exp[w_1(\vartheta)t + w_2(\vartheta)x]\,\varrho_\tau(du) \qquad \text{for } C \in \mathcal{B}. \end{aligned}$$

From (15) we obtain

$$(17)\qquad \frac{\partial \ln p_{\vartheta_0}(t, x;\ \vartheta)}{\partial\vartheta} = w_1'(\vartheta)t + w_2'(\vartheta)x.$$

Thus inequality (5) takes the form

$$D_\vartheta^2[f(Z)] \geqslant \frac{[g'(\vartheta)]^2}{E_\vartheta[w_1'(\vartheta)\tau + w_2'(\vartheta)x_\tau]^2}\ .$$

Taking into account (13) and (17) we state that *an estimator f of the function $g(\vartheta)$ is efficient at a given value ϑ if and only if it is of the form*

$$(18)\qquad f(u) = h(\vartheta)[w_1'(\vartheta)t(u) + w_2'(\vartheta)x(u)] + g(\vartheta) \qquad \textit{a.e.w.r.t. } \varrho_\tau.$$

B. Suppose that the plan (τ, g, f) is efficient. Then we can choose the values ϑ_1 and ϑ_2 $(\vartheta_1 \neq \vartheta_2)$ belonging to the interval $\langle a, b\rangle$ and we can write equation (18) for them. Thus we have

$$f(u) = h(\vartheta_1)[w_1'(\vartheta_1)t(u)+w_2'(\vartheta_1)x(u)]+g(\vartheta_1) \quad \text{a.e.w.r.t. } \varrho_\tau$$

and

$$f(u) = h(\vartheta_2)[w_1'(\vartheta_2)t(u)+w_2'(\vartheta_2)x(u)]+g(\vartheta_2) \quad \text{a.e.w.r.t. } \varrho_\tau.$$

Following the paper [4] we can subtract one equality from the other. We then obtain

$$[h(\vartheta_1)w_1'(\vartheta_1)-h(\vartheta_2)w_1'(\vartheta_2)]t(u)+[h(\vartheta_1)w_2'(\vartheta_1)-h(\vartheta_2)w_2'(\vartheta_2)]x(u)+ \\ +g(\vartheta_1)-g(\vartheta_2) = 0 \quad \text{a.e.w.r.t. } \varrho_\tau.$$

Thus we come to the following statement: *if the plan (τ, g, f) is efficient, then $a_1t(u)+a_2x(u)+a_3 = 0$ a.e.w.r.t. ϱ_τ, where $a_3 \neq 0$.*

It follows from this result that selecting efficient plans one should consider such plans for which the measure ϱ_τ is accumulated on some line.

C. The results concerning efficiently estimable functions, obtained by Trybuła in [4], we can generalize on the class of processes to be considered.

In view of (15), (16) and (17), formula (9) takes in this case the form

$$\int_U [w_1'(\vartheta)t+w_2'(\vartheta)x]\exp[w_1(\vartheta)t+w_2(\vartheta)x]\,\varrho_\tau(du) = 0$$

which yields the following form of the Wald first identity:

$$\mathrm{E}_\vartheta(x_\tau) = -\frac{w_1'(\vartheta)}{w_2'(\vartheta)}\mathrm{E}_\vartheta(\tau). \tag{19}$$

Let (τ, g, f) be an efficient plan for $\vartheta = \vartheta^0$. Thus, by the formula

$$f(Z) = h(\vartheta^0)[w_1'(\vartheta^0)\tau+w_2'(\vartheta^0)x_\tau]+g(\vartheta^0),$$

almost surely, we have

$$\mathrm{E}_\vartheta(f) = h(\vartheta^0)[w_1'(\vartheta^0)\mathrm{E}_\vartheta(\tau)+w_2'(\vartheta^0)\mathrm{E}_\vartheta(x_\tau)]+g(\vartheta^0).$$

Hence, taking into account (19), we get

$$\mathrm{E}_\vartheta(f) = \frac{h(\vartheta^0)}{w_2'(\vartheta)}[w_1'(\vartheta^0)w_2'(\vartheta)-w_1'(\vartheta)w_2'(\vartheta^0)]\mathrm{E}_\vartheta(\tau)+g(\vartheta^0). \tag{20}$$

On the other hand, we have $\mathrm{E}_\vartheta(f) = g(\vartheta)$ for every $\vartheta \in \langle a, b \rangle$. Thus, comparing this formula with (20), it is seen that *the function* $g(\vartheta)$ *is efficiently estimable at* $\vartheta = \vartheta^0$ *if and only if it is of the form*

$$g(\vartheta) = \frac{h(\vartheta^0)}{w_2'(\vartheta)}[w_1'(\vartheta^0)w_2'(\vartheta) - w_1'(\vartheta)w_2'(\vartheta^0)]\mathrm{E}_\vartheta(\tau) + g(\vartheta^0). \tag{21}$$

Suppose now that, for a given $\tau(\omega)$, the function $g(\vartheta)$ is efficiently estimable (by two perhaps distinct estimators) at points ϑ_1 and ϑ_2, $\vartheta_1 \neq \vartheta_2$. Then, by condition (21), the following relations must hold:

$$g(\vartheta) = \frac{h(\vartheta_1)}{w_2'(\vartheta)}[w_1'(\vartheta_1)w_2'(\vartheta) - w_1'(\vartheta)w_2'(\vartheta_1)]\mathrm{E}_\vartheta(\tau) + g(\vartheta_1),$$

$$g(\vartheta) = \frac{h(\vartheta_2)}{w_2'(\vartheta)}[w_1'(\vartheta_2)w_2'(\vartheta) - w_1'(\vartheta)w_2'(\vartheta_2)]\mathrm{E}_\vartheta(\tau) + g(\vartheta_2).$$

If we eliminate $\mathrm{E}_\vartheta(\tau)$ from these equations, we obtain

$$\begin{aligned}\{[h(\vartheta_1)w_2'(\vartheta_1) - h(\vartheta_2)w_2'(\vartheta_2)]w_1'(\vartheta) + [h(\vartheta_2)w_1'(\vartheta_2) - h(\vartheta_1)w_1'(\vartheta_1)]w_2'(\vartheta)\}g(\vartheta) \\ = [h(\vartheta_1)g(\vartheta_2)w_2'(\vartheta_1) - h(\vartheta_2)g(\vartheta_1)w_2'(\vartheta_2)]w_1'(\vartheta) + \\ + [h(\vartheta_2)g(\vartheta_1)w_1'(\vartheta_2) - h(\vartheta_1)g(\vartheta_2)w_1'(\vartheta_1)]w_2'(\vartheta).\end{aligned}$$

It follows from the consideration carried out in Section B that the constants $h(\vartheta_1)w_1'(\vartheta_1) - h(\vartheta_2)w_1'(\vartheta_2)$ and $h(\vartheta_1)w_2'(\vartheta_1) - h(\vartheta_2)w_2'(\vartheta_2)$ cannot vanish simultaneously. Therefore, by the assumed properties of the functions $w_1(\vartheta)$ and $w_2(\vartheta)$, the function standing by $g(\vartheta)$ in the above-given equation cannot vanish. Thus, *if the function* $g(\vartheta)$ *is efficiently estimable at two distinct values of* ϑ, *then it is of the form*

$$g(\vartheta) = \frac{k_1 w_1'(\vartheta) + k_2 w_2'(\vartheta)}{k_3 w_1'(\vartheta) + k_4 w_2'(\vartheta)}. \tag{22}$$

This result implies, in particular, that *if a plan* (τ, g, f) *is efficient, then the function* $g(\vartheta)$ *must be of form* (22).

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O NIERÓWNOŚCI TYPU RAO-CRAMÉRA W TEORII ESTYMACJI SEKWENCYJNEJ

STRESZCZENIE

W pracy podano ograniczenie od dołu na wariancję estymatora parametru $Q = g(\vartheta)$ w estymacji sekwencyjnej dla klasy procesów stochastycznych, dla której obowiązuje twierdzenie Sudakowa [3] ($g(\vartheta)$ oznacza pewną funkcję parametru ϑ rozkładów prawdopodobieństwa rozważanej klasy procesów). To ograniczenie od dołu jest typu Rao-Craméra i może służyć jako kryterium przy wyborze optymalnych planów sekwencyjnych. Uwzględniając ten fakt, uogólniono wyniki dotyczące funkcji efektywnie estymowalnych [4] na przypadek wykładniczej klasy procesów (definicja 4).