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COUPLED FIELDS GENERATED BY LINEAR DIFFERENTIAL OPERATORS: ELECTRODYNAMICS OF DEFORMABLE CONTINUA

1. Introduction. In the last twenty years one can observe intensive investigations of the so-called coupled fields in the theory of deformable media. Such new sections of continuum mechanics as thermoelasticity, piezoelectricity and magnetoelasticity have been developed recently. Many authors dealt with this subject and approached it by various methods in various directions (see, e.g., [3], [7], [9], [10]).

In this paper a uniform formalism generating mathematical models of interactions of coupled fields in deformable media is presented. The advantage of the presented method lies in the fact that it can be used regardless of the physical interpretation of the considered coupled fields. The formulation proposed here has its origin in the paper [1], where a method of describing purely mechanical interactions for the statical problem of a deformable Cosserat continuum is given.

In this paper, interactions of coupled mechanics and electromagnetic fields in deformable media embedded in a four-dimensional space-time $T\times E^3$ are described with the aid of some linear differential operators [4]. The Maxwell macroscopic equations have been written in the invariant form. The non-homogeneous Maxwell equations are interpreted as the equations of equilibrium for the electromagnetic stress tensor. The homogeneous ones have been conveyed to the form of the so-called compatibility equations. In order to interpret the boundary conditions obtained on the ground of the presented method some generalization of the classical stress-principle of Euler–Cauchy has been here admitted for the case of local electromagnetic and mechanic interactions in a deformable continuum embedded in space-time. The special cases of piezoelectricity and magnetoelasticity are discussed.

2. Coupled fields in a deformable continuum. Basic assumptions of a formalism. Assume that interactions in a material deformable continuum are described by two four-vectors $u_\mu(a)$ and $A_\mu(a)$ defined on some region $\mathscr{M}$ of the space-time $T\times E^3$ and on its boundary $\partial\mathscr{M}$, respectively. Here, the region

$\mathscr{M}$ is given by the Cartesian product $T\times\Omega$, where the bounded region Ω denotes a region of variability of the so-called Lagrange coordinates of a material continuum. The vector $u_\mu(a)$ denotes the four-vector of displacements of a deformable medium [4], however the four-vector $A_\mu(a)$ describes some non-mechanical phenomena (electromagnetical phenomena).

Physical properties of the material continuum in a space-time are defined by the linear differential operator Λ^μ which assigns to each of the vector fields $u_\mu(a)$ and $A_\mu(a)$ defined in the region $\mathscr{M}$ and on the boundary $\partial\mathscr{M}$, respectively, a scalar-valued function

$$\varphi(a) = \Lambda^\mu\big(u_\mu(a),\, A_\mu(a)\big)$$

belonging to a unitary space $L^2(\mathscr{M}\times\partial\mathscr{M})$ with a given inner product $\langle\cdot,\cdot\rangle$.

Next we assume that the differential operator Λ^μ is of the form

$$\Lambda^\mu(u_\mu, A_\mu) = \Lambda_1^\mu(u_\mu)+\Lambda_2^\mu(A_\mu),$$

where the operator $\Lambda_1^\mu(u_\mu)$ is given by the formula

$$\Lambda_1^\mu(u_\mu) = \begin{cases} F^\mu u_\mu + F^{\mu\nu}\nabla_\nu u_\mu + \ldots, & a\in\mathscr{M},\\ f^\mu u_\mu + \ldots, & a\in\partial\mathscr{M},\end{cases} \tag{1}$$

and the operator $\Lambda_2^\mu(A_\mu)$ is of the following form:

$$\Lambda_2^\mu(A_\mu) = \begin{cases} \mathscr{F}^\mu A_\mu + \mathscr{F}^{\mu\nu}\nabla_\nu A_\mu + \ldots, & a\in\mathscr{M},\\ j^\mu A_\mu + \ldots, & a\in\partial\mathscr{M}.\end{cases} \tag{2}$$

The Greek indices range from 0 to 3 and the Latin ones from 1 to 3. In formulae (1) and (2) the quantities $F^\mu, F^{\mu\nu}, \ldots, f^\mu, \ldots$ and $\mathscr{F}^\mu, \mathscr{F}^{\mu\nu}, \ldots, j^\mu, \ldots$ are tensor-valued functions given in the region $\mathscr{M}$ and on its boundary $\partial\mathscr{M}$, respectively, and $\nabla_\nu(\ldots)$ denotes the covariant derivative in $\mathscr{M}$.

The inner product $\langle\cdot,\cdot\rangle$ in the space $L^2(\mathscr{M}\times\partial\mathscr{M})$ is defined as follows:

$$\langle\varphi, \psi\rangle = \int_{\mathscr{M}} \varphi\cdot\psi\, dv + \int_{\partial\mathscr{M}} \varphi\cdot\psi\, d\sigma. \tag{3}$$

Let us define the formally adjoint operator $^*\Lambda^\mu$ with respect to the inner product (3) in the following form:

$$^*\Lambda^\mu\psi = {}^*\Lambda_1^\mu\psi + {}^*\Lambda_2^\mu\psi,$$

where

$$\langle\Lambda_1^\mu u_\mu, \psi\rangle = \langle u_\mu, {}^*\Lambda_1^\mu\psi\rangle, \qquad \langle\Lambda_2^\mu A_\mu, \psi\rangle = \langle A_\mu, {}^*\Lambda_2^\mu\psi\rangle.$$

Next we assume (see [1] and [4]) that the equilibrium equations of deformable continuum result from the following postulate: to say that the deformable continuum represented by the operator Λ^μ is *in equilibrium* (*motion* is understood as an equilibrium in space-time) means that

(4) $$^*\Lambda^{\mu}\psi = 0 \quad \text{if } \psi = \text{const}.$$

Condition (4) is equivalent to the system of equations

(5) $$\begin{aligned} ^*\Lambda_1^{\mu}\psi &= 0 \quad \text{if } \psi = \text{const}, \\ ^*\Lambda_2^{\mu}\psi &= 0 \quad \text{if } \psi = \text{const}, \end{aligned}$$

which represent the stress equations for the unknown generalized stresses F^{μ}, $F^{\mu\nu}$, f^{μ}, ... and $\mathscr{F}^{\mu}$, $\mathscr{F}^{\mu\nu}$, j^{μ}, ...

The fields u_{μ} and A_{μ} which describe interactions in a deformable continuum are called *stress coupled fields* if there exist some relationships between the generalized stresses F^{μ}, $F^{\mu\nu}$, ... and $\mathscr{F}^{\mu}$, $\mathscr{F}^{\mu\nu}$, ... The form of these relations results from physical and geometrical considerations.

Later we give an example of mechanical and electromagnetical coupled fields where the relations among the stresses are of the form

$$F^{\mu} = f(\mathscr{F}^{\mu}, \mathscr{F}^{\mu\nu}).$$

As it was shown in [6], for the Cosserat continuum we have some contrary relations, i.e.,

$$\mathscr{F}^{\mu} = f(F^{\mu}, F^{\mu\nu}).$$

We assume here also that the derivatives $\nabla_{\mu}u_{\nu}$ and $\nabla_{\mu}A_{\nu}$ of the fields u_{μ} and A_{μ}, respectively, define some measure of deformations. We call them here *generalized deformations* and use the following notation:

$$\varepsilon_{\mu\nu} = \nabla_{\mu}u_{\nu}, \qquad \mathscr{G}_{\mu\nu} = \nabla_{\mu}A_{\nu}.$$

Note that fields which are not stress coupled may still be coupled by the so-called constitutive equations if there exist appropriate coupled material constants. We return to this point later and give an example of fields which are coupled only by constitutive equations of the form

$$F^{\mu\nu} = f(\varepsilon^{\mu\nu}, \mathscr{G}^{\mu\nu}), \qquad \mathscr{F}^{\mu\nu} = f(\varepsilon^{\mu\nu}, \mathscr{G}^{\mu\nu})$$

for questions of piezoelectricity.

3. Material continuum in an exterior electromagnetic field. Let us consider first the case of non-deformable continuum in an exterior electromagnetic field. The electric field is described by the vector fields E_i and D^i which are called a *vector of electrical field intensity* and a *vector of electrical induction* (*electric displacement*), respectively. The magnetic field in a material is described by vector fields B_i and H^i, which are called here *magnetic field intensity* and a *magnetic induction vector*, respectively. Note that triaditionally the vector fields B_i and H^i are called inversely. Here we point out an analogy between the vectors E_i and B_i and also between D^i and H^i. For this reason the above notation is assumed. The electric and magnetic inductions D^i and H^i represent a reaction of a material continuum on the acting vectors of electric and magnetic intensity E_i and B_i, respectively. Functional relationships between the fields D^i, H^i and E_i, B_i, called *equations of polarization*, are constitutive equations which characterize the material continuum.

For materials called *dielectrics*, in the case of a small exterior electric field E_i, the relations of polarization are linear:

$$D^i = c^{ij} E_j, \tag{6}$$

where the material constants c^{ij} denote the tensor of electrical permittivity.

In a similar way, for materials called *diamagnetics*, in the case of a small magnetic field B_i, the polarization equations take the linear form

$$H^i = k^{ij} B_j, \tag{7}$$

where the material constants k^{ij} denote the tensor of magnetical permittivity.

For the so-called ferroelectrics and ferromagnetics constitutive relationships take a more complicated functional form because of the existence of electric or magnetic hysteresis.

The electromagnetic field in a material continuum is described by Maxwell equations, which hold in the region $\mathcal{M} = T \times \Omega$ of space-time, and in the rationalized MKSA system of physical units they can be written in the form (see [2])

$$\bar{V} \cdot \bar{D} = \varrho, \qquad \bar{V} \times \bar{H} - \partial \bar{D}/\partial t = \bar{I}, \tag{8}$$

$$\bar{V} \times \bar{E} + \partial \bar{B}/\partial t = 0, \qquad \bar{V} \cdot \bar{B} = 0, \tag{9}$$

where the operator $V_i(\ldots)$ denotes the three-vector of derivative with respect to space variables, ϱ and $\bar{I}$ are sources of the electrical field $\bar{E}$ and of the magnetical field $\bar{B}$, respectively, i.e., ϱ denotes a charge density per a unit volume element, and $\bar{I}$ denotes an electric current density.

To equations (8) and (9) we must still add constitutive relations (polarization laws)

$$\bar{D} = \bar{D}(\bar{E}, \bar{B}), \qquad \bar{H} = \bar{H}(\bar{E}, \bar{B}),$$

as well as boundary conditions on the boundary $\partial \mathcal{M}$ of a region $\mathcal{M}$.

Now we write (8) and (9) in a convenient four-tensor form. With the aid of the electric field intensity E_i and the magnetic field intensity B_i one can build an antisymmetric four-tensor $\mathcal{G}_{\alpha\beta}$ which is called here a *deformation tensor* of the electromagnetic field and is defined in the following matrix form:

$$
(10) \qquad \mathscr{G}_{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.
$$

In a similar way, with the aid of three-vectors of electrical and magnetical inductions D^i and H^i, one can build an antisymmetric four-tensor $\mathscr{F}^{\alpha\beta}$ called a *stress tensor* of the electromagnetic field and defined in the following matrix form:

$$
(11) \qquad \mathscr{F}^{\alpha\beta} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{pmatrix}.
$$

It should be remarked that $\mathscr{G}_{\alpha\beta}$ and $\mathscr{F}^{\alpha\beta}$ are four-tensors under Lorentz transformations in space-time. The assumption that the velocities are small leads to Galilean transformations in space-time and to the Galilean approximation of the electromagnetic field theory. As it is well known, the Galilean transformations are commonly used in mechanic of a deformable continuum. This subject has been discussed in [5] and [10]. Here we shall not elaborate that subject but only indicate these appearing problems.

The raising and lowering of tensorial indices go on the following metrix tensor:

$$
g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
$$

In this way the space-time is treated here as a pseudo-Euclidean one with the index 3.

Let us introduce the four-vector of current

$$
(12) \qquad I^\mu = (\varrho, I^i),
$$

where, as it was defined above, the scalar field ϱ is a volume density of charge, and I^i denotes the three-vector of volume density of the electric current.

Using the deformation tensor $\mathscr{G}_{\alpha\beta}$ defined by (10) and the stress-tensor $\mathscr{F}^{\alpha\beta}$ defined by (11), we can write the Maxwell equations in the media (8) and (9) in the form

$$\nabla_\alpha \mathscr{F}^{\alpha\beta} = I^\beta, \tag{13}$$

$$\varepsilon^{\alpha\beta\gamma\delta} \nabla_\alpha \mathscr{G}_{\gamma\delta} = 0, \tag{14}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ denotes the usual four-dimensional permutation symbol.

Using the introduced deformation and stress tensor of the electromagnetic field, we can write (6) and (7) in a uniform way as follows:

$$\mathscr{F}^{\alpha\beta} = c^{\alpha\beta\gamma\delta} \mathscr{G}_{\gamma\delta}.$$

Now let us concern ourselves with the interpretation of equations (13) and (14). It is easy to notice an analogy between equation (13) for the electromagnetic stress $\mathscr{F}^{\alpha\beta}$ and the equations of equilibrium of a deformable material continuum in space-time [4]. There are however also differences. The electromagnetic stress tensor $\mathscr{F}^{\alpha\beta}$ is antisymmetric while the stress four-tensor of a material medium is traditionally assumed to be symmetric.

Let us have another look at the homogeneous Maxwell equations (14) which can be written in an equivalent form as follows:

$$\nabla_\alpha \mathscr{G}_{\gamma\delta} + \nabla_\gamma \mathscr{G}_{\delta\alpha} + \nabla_\delta \mathscr{G}_{\alpha\gamma} = 0, \tag{15}$$

where α, γ, δ denote any three indices from the set $\{0, 1, 2, 3\}$.

Increasing the order of equations (15) (making use of the identities $\nabla_{\alpha\beta}(\ldots) = \nabla_{\beta\alpha}(\ldots)$, and $\mathscr{G}_{\alpha\beta} = -\mathscr{G}_{\beta\alpha}$), we can write them in the following form:

$$\nabla_{\alpha\beta} \mathscr{G}_{\gamma\delta} + \nabla_{\gamma\delta} \mathscr{G}_{\beta\alpha} - \nabla_{\gamma\alpha} \mathscr{G}_{\beta\delta} - \nabla_{\delta\beta} \mathscr{G}_{\gamma\alpha} = 0. \tag{16}$$

In this way it is easy to notice that equations (14) for the electromagnetic deformation tensor $\mathscr{G}_{\alpha\beta}$ have the same form as the so-called compatibility relations for the deformation tensor of a material deformable continuum in the case of infinitesimal deformations.

As it is well known, the compatibility equations for a deformation tensor of a continuum material result immediately from the assumption that an embedding space is flat. This fact seems to open a new geometrical interpretation of the homogeneous Maxwell equations.

It is easy to see that if the electromagnetic deformation tensor $\mathscr{G}_{\alpha\beta}$ is built up from the derivatives of the four-vector of the electromagnetic potential A_μ as

$$\mathscr{G}_{\alpha\beta} = \nabla_{[\alpha} A_{\beta]}, \tag{17}$$

then $\mathscr{G}_{\alpha\beta}$ satisfies equations identical to (14), and therefore also equations (16). For this reason, if we take the tensor $\mathscr{G}_{\alpha\beta}$ in the form (17), we may understand the non-homogeneous equations (13) only as the Maxwell equations in a medium.

The relations between the components of the deformable tensor $\mathscr{G}_{\alpha\beta}$ and the components of the four-potential $A_\beta = (\phi, A_i)$ are of the form

$$\bar{B} = \bar{V} \times \bar{A}, \quad \bar{E} = -\bar{V}\phi + \partial\bar{A}/\partial t.$$

Let us now return to the description of interactions in a deformable material with the aid of the linear differential operator Λ^μ presented in Section 2.

Assume that the electromagnetic field in a medium is described by the differential operator Λ^μ which assigns to each four-vector A_μ of the electromagnetic potential a scalar-valued function $\varphi(a) = \Lambda^\mu(A_\mu(a))$ as follows:

$$\Lambda^\mu(A_\mu) = \begin{cases} I^\mu A_\mu - \mathscr{F}^{\mu\nu} V_\nu A_\mu, & a \in \mathscr{M}, \\ j^\mu A_\mu, & a \in \partial\mathscr{M}, \end{cases}$$

where the current vector I^μ is given by (12), the stress tensor $\mathscr{F}^{\mu\nu}$ is defined by (11), and the electromagnetic deformations $\mathscr{G}_{\alpha\beta}$ determined by (10) are of the form (17).

The four-vector j^μ defined on the boundary $\partial\mathscr{M}$ of the region $\mathscr{M}$ is

$$j^\mu = (\sigma, j^k), \tag{18}$$

where σ denotes the surface density of charge and j^k denotes the surface density of electric current.

The assumed condition $*\Lambda^\mu \psi = 0$ if $\psi = \text{const}$, defining the continuum equilibrium, leads us now to the non-homogeneous Maxwell equations (13). Equations (14) are satisfied, for we assume here condition (17).

The boundary conditions obtained on the ground of the presented method are discussed in the next section.

4. Deformable continuum in an exterior electromagnetic field. We now discuss the problem of describing the interactions of mechanical and electromagnetical fields in a deformable medium embedded in space-time. Let us assume that those interactions are described by two differential operators Λ_1^μ and Λ_2^μ defined on the four-vector u_μ of the continuum displacement [4] and on the four-potential A_μ of the electromagnetic field, respectively. According to the assumptions which have been adopted in Section 2 we write

$$\Lambda^\mu(u_\mu, A_\mu) = \Lambda_1^\mu(u_\mu) + \Lambda_2^\mu(A_\mu). \tag{19}$$

Mechanical interactions are described by the operator of the form

$$\Lambda_1^\mu(u_\mu) = \begin{cases} \tilde{F}^\mu u_\mu - F^{\mu\nu} V_\nu u_\mu, & a \in \mathscr{M}, \\ f^\mu u_\mu, & a \in \partial\mathscr{M}. \end{cases} \tag{20}$$

The stress four-tensor $F^{\mu\nu}$ appearing in (20) is symmetrical, i.e., $F^{\mu\nu} = F^{\nu\mu}$. The four-vector f^μ defined on the boundary $\partial\mathscr{M}$ is of the form

$$f^\mu = (f, f^i), \tag{21}$$

where the scalar field f denotes the surface density of thermal power and f^i denotes the surface density of mechanical forces.

Electromagnetical interactions are described by the operator

$$\Lambda_2^{\mu}(A_\mu) = \begin{cases} I^\mu A_\mu - \mathscr{F}^{\mu\nu} \nabla_\nu A_\mu, & a \in \mathscr{M}, \\ j^\mu A_\mu, & a \in \partial\mathscr{M}. \end{cases} \tag{22}$$

The stress four-tensor $\mathscr{F}^{\mu\nu}$ of electromagnetical field in formula (22) is antisymmetrical: $\mathscr{F}^{\mu\nu} = -\mathscr{F}^{\nu\mu}$. The four-vector j^μ acting on the boundary $\partial\mathscr{M}$ is of the form (18).

Here the mechanical and electromagnetical fields are coupled by the relation

$$\tilde{F}^\mu = F^\mu + F^\mu_{\text{e-m}}, \tag{24}$$

where F^μ denotes the four-vector of volume density of mechanical mass-forces, and $F^\mu_{\text{e-m}}$ denotes the volume density of the so-called Lorentz force, i.e., the force with which the electromagnetic field acts on a material medium and defined by

$$F^\mu_{\text{e-m}} = (\bar{I} \cdot \bar{D},\ \varrho\bar{D} + \bar{I} \times \bar{H}) \tag{25}$$

with the notation as in formulae (11) and (12).

Note that relations (24) and (25) can be written in the general form

$$\tilde{F}^\mu = f(I^\mu, \mathscr{F}^{\mu\nu}),$$

which is an example of the stress coupled relation in the sense of the definition adopted in Section 2.

According to the postulates admitted in Section 2 the equations of equilibrium of a deformable continuum can be written in the general form

$$^*\Lambda_1 \psi = 0 \text{ if } \psi = \text{const}, \quad ^*\Lambda_2 \psi = 0 \text{ if } \psi = \text{const}. \tag{26}$$

If the operators Λ_1^μ and Λ_2^μ take the form (20) and (22), respectively, then conditions (26) lead us to the following coupled equations of a deformable continuum in the exterior electromagnetic field:

$$\nabla_\mu \mathscr{F}^{\mu\nu} = I^\nu, \tag{27}$$

$$\nabla_\mu F^{\mu\nu} = F^\nu + F^\nu_{\text{e-m}}(\mathscr{F}^{\mu\nu}, I^\nu), \tag{28}$$

where the Lorentz force is defined by (25) and the stress tensor of the electromagnetic field $\mathscr{F}^{\mu\nu}$ is of the matrix form (11).

Note that equations (27) are non-homogeneous Maxwell equations. The homogeneous ones are satisfied, for we assumed here that condition (17) holds. Formulae (28) are equations of the continuum equilibrium in space-time [4]. They are coupled with equations (27) with the aid of (25) which defines the Lorentz force $F^\mu_{\text{e-m}}$. In the case where the Lorentz force vanishes, equations (27) and (28) are no more coupled in the stresses $\mathscr{F}^{\mu\nu}$ and $F^{\mu\nu}$ (the system of equations is separated).

Note also that equation (28) which one can obtain for the case $\mu = 0$ is the first principle of thermodynamics as it has been shown in [4]. Here this principle has added the term $F^0_{\text{e-m}} = \bar{I} \cdot \bar{D}$, which describes the variation of electromagnetic field energy in time.

Equations (27) and (28) which are valid in the region $\mathscr{M}$ of space-time are supplemented by the corresponding boundary conditions resulting from the assumed form of the operator Λ^μ on the boundary $\partial\mathscr{M}$. We get

(29) $$f^\mu - n_\nu F^{\mu\nu} = 0, \quad a \in \partial\mathscr{M},$$

(30) $$j^\mu - n_\nu \mathscr{F}^{\mu\nu} = 0, \quad a \in \partial\mathscr{M},$$

where n_ν denotes the four-vector normal to the boundary $\partial\mathscr{M}$ and directed outside of $\mathscr{M}$.

The physical and geometrical interpretation of conditions (29) has been discussed in [4]. For small velocity motions, equations (29) take the form

(31) $$f^i - n_j F^{ij} = 0, \quad a \in S,$$

(32) $$f - n_j q^j = 0, \quad a \in S,$$

where $\partial\mathscr{M} = S \cup S_0 \cup S_1$ and the conditions on S_0 are the initial conditions.

From (31) and (32) we obtain immediately the interpretation of the tensors F^{ij} and q^j as the stress tensor in a material deformable continuum and the heat flux vector, respectively, since f^i and f denote the given surface density of mechanical forces and surface density of thermal power, respectively.

Substituting (11) and (18) into (30) we get the following boundary conditions for the electromagnetic field:

(33) $$j^k - (\bar{n} \cdot \bar{v}) D^k - (\bar{n} \times \bar{H})^k = 0, \quad a \in S,$$

(34) $$\sigma - n_i D^i = 0, \quad a \in S.$$

The form of the boundary conditions on S is the same as that in [8], where those conditions have been obtained in a different way.

For the case of electrostatic fields ($\bar{H} = 0$) and under the assumption that the continuum velocity $\bar{v}$ vanishes, conditions (33) and (34) take the form

(35) $$\sigma - n_i D^i = 0, \quad a \in S.$$

For the case of magnetostatics ($\bar{D} = 0$) we get

(36) $$j^k - (\bar{n} \times \bar{H})^k = 0, \quad a \in S.$$

From (35) and (36) we obtain immediately the interpretation of the fields D^i and H^i as the vectors of electric and magnetic induction since the fields σ

and j^k given on the boundary S are the surface density of the charge and electric current, respectively.

Let us choose inside the region $\mathcal{M} \subset T\times E^3$ any region $\mathcal{A}$ with boundary $\partial\mathcal{A}$ having a unit normal vector n_μ directed outside $\mathcal{A}$. We assume here that the material outside $\mathcal{A}$ acts partly inside $\mathcal{A}$ through the four-vector fields $f^\mu = (f, f^i)$ and $j^\mu = (\sigma, j^i)$ (see (21) and (23)) on the oriented surface element ds of the boundary $\partial\mathcal{A}$.

Note that in this way we adopt here a generalized stress principle of Euler–Cauchy.

Equations (27) and (28) should be obviously completed with the constitutive relations connecting the mechanical stresses $F^{\mu\nu}$ and electromagnetical stresses $\mathcal{F}^{\mu\nu}$ to the suitable deformations $\varepsilon_{\mu\nu} = \nabla_{(\mu} u_{\nu)}$ and $\mathcal{G}_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$. These topics will be discussed in a separate paper. This problem in the case of a traditional three-dimensional approach is widely described in the literature. Here we indicate only that the advantage of the presented formalism lies also in that it shows evidently the deformations corresponding to the generalized stresses $F^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$. In the theories of deformable media the definitions of the deformations have an element of arbitrariness.

Let us take now into consideration two simple peculiar cases of electromagnetic and mechanic interactions in a deformable medium, namely the case of piezoelectricity and magnetoelasticity.

The *piezoelectricity* is understood here as the interactions of the mechanical and electrostatical fields ($\bar{H} = 0$). The stress tensor $\mathcal{F}^{\mu\nu}$ takes now the simple form

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & & & \\ D^2 & & 0 & \\ D^3 & & & \end{pmatrix}.$$

In this case one takes into consideration [9] non-conductive material media ($\varrho = 0$, $\bar{I} = 0$). As it results from (25) we have, consequently, $F^\mu_{\text{e-m}} = 0$, and so the equations of equilibrium (27) and (28) are no more coupled.

The fields u_μ and A_μ are coupled in this case by constitutive relations which have been assumed in the linear form (see [9])

$$F^{ij} = c^{ijkl}\varepsilon_{kl} - e^{ijk}E_k, \qquad D^i = e^{ikl}\varepsilon_{kl} + d^{ik}E_k,$$

where c^{ijkl}, e^{ijk}, d^{ik} denote the tensors of material constants.

Magnetoelasticity is understood here as a connection of mechanical statics and magnetostatics ($\bar{D} = 0$). We have then

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -H^3 & H^2 \\ 0 & H^3 & 0 & -H^1 \\ 0 & -H^2 & H^1 & 0 \end{pmatrix}.$$

In this case conductive materials have been considered, so the Lorentz force (25) equals

$$F^{\mu}_{\text{e-m}} = (0, \bar{I} \times \bar{H}),$$

and does not vanish. Therefore, equations (27) and (28) are stress coupled. Constitutive relations in this case have been admitted in the separable form (see [9])

$$H^i = k^{ij} B_j, \qquad F^{ij} = c^{ijkl} \varepsilon_{kl},$$

where k^{ij} and c^{ijkl} denote the tensors of material constants.

Magnetoelasticity in the case of dynamical problems of a deformable medium is understood as follows [9]: the conductive continuum is found in a strong exterior magnetic field. For small velocity continuum motions, the induced electromagnetic field can be described by a small fluctuation of the electric and magnetic fields. Such an approach permits us to separate the Maxwell equations in the electric field $\bar{E}$ and in the magnetic field $\bar{B}$, and allows us to assume the linearization of magnetoelasticity relations.

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