ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 4 (1985), p.597-608

J. BARTOSZEWICZ (Wrocław)

MEAN-SQUARE-ERROR-ROBUSTNESS OF LINEAR ESTIMATES IN THE EXPONENTIAL MODEL

1. Introduction and summary. Let us consider the statistical model $M_1 = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,1}, \lambda > 0\})$, where R_1^+ is the real positive half-line, $\mathscr{B}_1^+$ is the family of Borel subsets of R_1^+ and $P_{\lambda,1}$ is the exponential distribution with probability density function (pdf)

$$f_{\lambda,1}(x) = \frac{1}{\lambda}\exp(-x/\lambda), \qquad x > 0. \tag{1}$$

As in Zieliński [13] (see also Bartoszewicz [3] and Zieliński and Zieliński [14]) suppose that the exponential model is violated in such way that the random variable under consideration has the exponential power distribution $P_{\lambda,p}$ with pdf

$$f_{\lambda,p}(x) = \frac{\exp(-(x/\lambda)^p)}{\lambda\Gamma(1+1/p)}, \qquad x > 0, \tag{2}$$

rather than $P_{\lambda,1}$, the shape parameter p being unknown. Formally we consider the exponential power extension M_{p_1,p_2} of M_1 defined as follows

$$M_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,p}, \lambda > 0, p_1 \leqslant p \leqslant p_2\}), \tag{3}$$

where $0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16$.

Let $X_1, X_2, \ldots, X_n$ be a sample from the distribution $P_{\lambda,p}$ and $\mathscr{T}^+$ be the following class of statistics

$$\mathscr{T}^+ = \{T = \sum_{j=1}^{n} a_j X_{j:n}, a_j \geqslant 0, j = 1, 2, \ldots, n, \mathrm{E}_{1,1} T = 1\}, \tag{4}$$

where $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are order statistics, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)' \in R_n^+$ and $\mathrm{E}_{1,1} T$ denotes the expected value of T under the distribution $P_{1,1}$. Thus $\mathscr{T}^+$

is the class of linear estimates of the scale parameter λ which are unbiased in the original model M_1. On the basis of a general concept of robustness presented in [12], Zieliński [13] has studied the bias-robustness of the estimates of λ from $\mathcal{T}^+$ for the extension M_{p_1,p_2} of M_1. He has proved that for each extension M_{p_1,p_2} of M_1 the estimate $T^0 = nX_{1:n}$ is the uniformly most bias-robust estimate of λ in the class $\mathcal{T}^+$, i.e. $b_{T^0}(\lambda) \leqslant b_T(\lambda)$ for each $\lambda > 0$ and every $T \in \mathcal{T}^+$, where

$$b_T(\lambda) = \sup_{p_1 \leqslant p \leqslant p_2} (\mathrm{E}_{\lambda,p} T - \lambda) - \inf_{p_1 \leqslant p \leqslant p_2} (\mathrm{E}_{\lambda,p} T - \lambda)$$

is the function of bias-robustness of the estimate T.

Bartoszewicz [3] has considered the gamma extension M_{p_1,p_2} of the model M_1 defined as follows

$$M^*_{p_1,p_2} = (R_1^+, \mathcal{B}_1^+, \{P^*_{\lambda,p}, \lambda > 0, p_1 \leqslant p \leqslant p_2\}), \tag{5}$$

where $0 < p_1 \leqslant 1 \leqslant p_2 < \infty$ and $P^*_{\lambda,p}$ is the gamma distribution with pdf

$$f^*_{\lambda,p}(x) = \frac{x^{p-1} \exp(-x/\lambda)}{\lambda^p \Gamma(p)}, \quad x > 0. \tag{6}$$

He has proved that for each extension $M^*_{p_1,p_2}$ of M_1 the estimate $T^* = X_{n:n}/\mathrm{E}_{1,1} X_{n:n} = X_{n:n}/(1 + 1/2 + \ldots + 1/n)$ is the uniformly most bias-robust estimate of λ in the class $\mathcal{T}^+$.

Zieliński and Zieliński [14] have presented another approach to the robustness of linear estimates of λ in the extensions (3) and (4) of the model M_1. They have studied the so-called infinitesimal bias-robustness and the infinitesimal mean-square-error-robustness as well.

In this paper we consider the problem of the mean-square-error-robustness (v-robustness to be short, see [14]) of linear estimates of λ in some gamma and exponential power extensions of the model M_1. Two theorems concerning the existence of the uniformly most v-robust estimate (UMVRE) of λ are given. It is also proved that the sample mean $\bar{X}$ is not there the UMVRE of λ. For the gamma extension the case $n = 2$ is completely solved.

2. Preliminaries. We present definitions and lemmas which are used in the sequel.

Let F and G be distribution functions. To avoid technical complications in the statements of the results and in their proofs, we shall assume throughout that $F(0) = 0 = G(0)$, $\operatorname{supp} F = \operatorname{supp} G = [0, \infty)$ and that these distributions have no atoms. Thus, these distributions have strictly increasing and continuous inverses on the interval (0, 1). Also we assume that all expectations in consideration exist and are finite.

2.1. Dispersive ordering of distributions. Saunders and Moran [10] have introduced the notion of ordering in dispersion. We say that the distribution G has a smaller dispersion than F ($G \overset{\text{disp}}{<} F$) if $G^{-1}(\beta)-G^{-1}(\alpha) \leqslant F^{-1}(\beta)-F^{-1}(\alpha)$ whenever $0<\alpha<\beta<1$. Shaked [11] has studied this notion in detail. We recall some of his results in the following lemma.

LEMMA 1 (Shaked [11]). (a) *If $G \overset{\text{disp}}{<} F$, then $G \overset{\text{st}}{\leqslant} F$, i.e. $G(x) \geqslant F(x)$ for each $x \geqslant 0$.*

(b) *$G \overset{\text{disp}}{<} F$ if and only if the function $x-G^{-1}F(x)$ is nondecreasing in $x>0$.*

(c) *Assume that there exist densities f and g of the distributions F and G respectively. Then $G \overset{\text{disp}}{<} F$ if and only if $f(F^{-1}(u)) \leqslant g(G^{-1}(u))$ for each $u \in (0, 1)$.*

If the densities f and g exist, then $\varphi_F = f/(1-F)$ and $\varphi_G = g/(1-G)$ are the failure rate functions of the distributions F and G respectively. We can state the following lemma.

LEMMA 2. *If $\varphi_F(x) \leqslant \varphi_G(x)$ for each $x>0$ and F or G is a decreasing failure rate (DFR) distribution, then $G \overset{\text{disp}}{<} F$.*

Proof. If $\varphi_F(x) \leqslant \varphi_G(x)$, then $F(x) \leqslant G(x)$ for each $x>0$. Therefore

$$G^{-1}F(x) \leqslant x \leqslant F^{-1}G(x), \quad x>0. \tag{7}$$

Suppose now that F is a DFR distribution. Then from (7) we obtain

$$\frac{f(F^{-1}G(x))}{1-F(F^{-1}G(x))} \leqslant \frac{f(x)}{1-F(x)} \leqslant \frac{g(x)}{1-G(x)}, \quad x>0,$$

which gives $f(F^{-1}G(x)) \leqslant g(x)$ for each $x>0$. Putting $x=G^{-1}(u)$, $u \in (0, 1)$, we have $f(F^{-1}(u)) \leqslant g(G^{-1}(u))$ for each $u \in (0, 1)$, i.e. $G \overset{\text{disp}}{<} F$ by Lemma 1 (c).

The proof is quite similar if G is assumed to be a DFR distribution.

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ and $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ be order statistics of samples of size n from the distributions F and G respectively. Let $F_{jn}(x) = P(X_{j:n} \leqslant x)$ and $G_{jn}(x) = P(Y_{j:n} \leqslant x)$, $j=1, 2, \ldots, n$. Since $F_{jn}(x) = B_{jn}F(x)$ and $G_{jn}(x) = B_{jn}G(x)$, where B_{jn} is the beta distribution function with parameters j, $n-j+1$, then we obtain immediately from Lemma 1 (b) the following result.

LEMMA 3. *If $G \overset{\text{disp}}{<} F$, then $G_{jn} \overset{\text{disp}}{<} F_{jn}$, $j=1, 2, \ldots, n$.*

2.2. Inequalities for moments of order statistics. Some lemmas concerning the moments of the order statistics from the distributions F and G are used in the next sections. Lemma 3 implies the following result.

LEMMA 4 (Bartoszewicz [4]). *If $G \overset{\text{disp}}{<} F$, then for all nondecreasing convex function ξ and η and each $i, j = 1, 2, \ldots, n$*

$$\operatorname{Cov}[\xi(X_{i:n}), \eta(X_{j:n})] \geqslant \operatorname{Cov}[\xi(Y_{i:n}), \eta(Y_{j:n})].$$

LEMMA 5 (Barlow and Proschan [1]). *If $G^{-1}F$ is convex, then $\mathrm{E}X_{j:n}/\mathrm{E}Y_{j:n}$ is decreasing in $j = 1, 2, \ldots, n$.*

LEMMA 6. *If $F^{-1}G$ is convex, then $\mathrm{E}X_{j:n}^{\alpha}/\mathrm{E}Y_{j:n}$ is increasing in $j = 1, 2, \ldots, n$ for each $\alpha \geqslant 1$.*

Proof. The proof is based on Lemma 3.5 of Barlow and Proschan [1] and is quite similar to that of Theorem 3.6 ibidem. It is obvious that $Y_{j:n} \overset{\mathrm{st}}{=} G^{-1}F(X_{j:n})$. To prove this lemma we ought to show that for every $c > 0$ the function $h(x) = x^{\alpha} - cG^{-1}F(x)$, $x > 0$, changes sign at most once and from negative to positive values if at all. It is easy to notice that h changes sign at the same points and in the same way as $v(x) = F^{-1}G(x^{\alpha}/c) - x$, $x > 0$. Since x^{α}/c is increasing convex for $\alpha \geqslant 1$, then $F^{-1}G(x^{\alpha}/c)$ is also increasing convex and $v(0) = 0$. Hence for arbitrary $c > 0$ $v(x)$ changes sign at most once and from negative to positive values if at all.

2.3. Properties of the gamma distribution. Let us denote by $F_{\lambda,p}^{*}$ the distribution function of the gamma distribution $P_{\lambda,p}^{*}$ with pdf (6). It is well known that if $p_1 < p_2$, then $F_{\lambda,p_1}^{*} \overset{\mathrm{st}}{\leqslant} F_{\lambda,p_2}^{*}$ for every $\lambda > 0$. Moreover $F_{\lambda,p}^{*}$ is a DFR distribution for $p \in (0, 1]$ and an increasing failure rate (IFR) distribution for $p \geqslant 1$, i.e. $F_{1,1}^{-1}F_{1,p}$ is concave for $p \in (0, 1]$ and convex for $p \geqslant 1$ (see [2], Chapter 4). Saunders and Moran [10] (see also [11]) have proved that if $p_1 < p_2$ then $F_{1,p_1} \overset{\mathrm{disp}}{<} F_{1,p_2}$. Hence from Lemma 4 it follows that the variances and covariances of all order statistics from the gamma distribution $F_{1,p}^{*}$ are nondecreasing in p.

The following lemma is applied in the next sections.

LEMMA 7. *If $X_1, X_2, \ldots, X_n$ are independent identically distributed random variables with the gamma distribution function $F_{1,p}^{*}$, then for each $i = 1, 2, \ldots, n$ and every $k = 1, 2, \ldots$*

$$(8)\quad \mathrm{E}_{1,p}\left(X_{i:n}^{k}\frac{1}{n}\sum_{j=1}^{n} X_j^{k}\right) = \frac{\Gamma(p+k)}{\Gamma(p)}\left[\mathrm{E}_{1,p}X_{i:n}^{k} + \frac{k}{n}\sum_{j=0}^{k-1}\frac{\Gamma(p)}{\Gamma(p+1+j)}\mathrm{E}_{1,p}X_{i:n}^{k+j}\right].$$

In particular

$$\mathrm{E}_{1,p}(X_{i:n}\bar{X}) = \mathrm{E}_{1,p}X_{i:n}\left(p + \frac{1}{n}\right) = \mathrm{E}_{1,p}X_{i:n}\left(\mathrm{E}_{1,p}\bar{X} + \frac{1}{n}\right),$$

i.e.

$$\mathrm{Cov}_{1,p}(X_{i:n}, \bar{X}) = \frac{1}{n}\mathrm{E}_{1,p}X_{i:n}.$$

Proof. Let us denote by $\gamma(p, z)$ the incomplete gamma function, i.e.

$$\gamma(p, z) = \int_0^z t^{p-1}e^{-t}\,dt.$$

Then $F^*_{1,p}(z) = \gamma(p, z)/\Gamma(p)$. It is easy to verify, integrating by parts, that

$$(9)\quad \frac{\gamma(p, z)}{\Gamma(p)} - \frac{\gamma(p+k, z)}{\Gamma(p+k)} = e^{-z}\sum_{j=0}^{k-1}\frac{z^{p+j}}{\Gamma(p+1+j)}, \qquad z > 0,\ p > 0,\ k = 1, 2, \ldots$$

Denoting for brevity $F^*_p \equiv F^*_{1,p}$, $\mathrm{E}_p X \equiv \mathrm{E}_{1,p} X$ and $S(m, r) = \sum_{j=m}^{r} X_j^k$ we have

$$(10)\quad \frac{1}{n}\mathrm{E}_p(X^k_{i:n}S(1, n)) = \frac{1}{n}\mathrm{E}_p\{X^k_{i:n}\mathrm{E}_p[S(1, n)|X_{i:n}]\}$$
$$= \frac{1}{n}\mathrm{E}_p\{X^k_{i:n}\mathrm{E}_p[S(1, i-1)|X_{i:n}] + X^{2k}_{i:n} + X^k_{i:n}\mathrm{E}_p[S(i+1, n)|X_{i:n}]\}.$$

It is easy to see that for $j < i$ the conditional distribution of $X_{j:n}$ given $X_{i:n} = z$ is the same as the distribution of the order statistic $V_{j:i-1}$ of a sample of size $i-1$ from the right-truncated gamma distribution with the distribution function $\gamma(p, v)/\gamma(p, z)$, $0 \leqslant v \leqslant z$. Similarly, if $j > i$, then the conditional distribution of $X_{j:n}$ given $X_{i:n} = z$ is the same as the distribution of the order statistic $W_{j-i,n-i}$ of a sample of size $n-i$ from the left-truncated gamma distribution with the distribution function

$$[\gamma(p, w) - \gamma(p, z)]/[\Gamma(p) - \gamma(p, z)], \qquad z \leqslant w < \infty.$$

It is easy to verify that

$$\mathrm{E}_p[S(1, i-1)|X_{i:n} = z] = (i-1)\frac{\gamma(p+k, z)}{\gamma(p, z)}$$

and

$$\mathrm{E}_p[S(i+1, n)|X_{i:n} = z] = (n-i)\frac{\Gamma(p+k) - \gamma(p+k, z)}{\Gamma(p) - \gamma(p, z)}.$$

Hence after some complicated calculations we obtain

$$(11)\quad \mathrm{E}_p[X^k_{i:n}S(1, i-1)] + \mathrm{E}_p[X^k_{i:n}S(i+1, n)]$$
$$= \frac{\Gamma(p+k)}{\Gamma(p)}\Bigg\{(n-1)\mathrm{E}_p X^k_{i:n} + C\int_0^\infty \left[\frac{\gamma(p, z)}{\Gamma(p)} - \frac{\gamma(p+k, z)}{\Gamma(p+k)}\right] \times$$
$$\times \{(n-i)F_p^{i-1}(z)[1-F_p(z)]^{n-i-1} -$$
$$-(i-1)F_p^{i-2}(z)[1-F_p(z)]^{n-i}\}\frac{z^{p+k-1}}{\Gamma(p)}e^{-z}\,dz\Bigg\},$$

where $C = n!/[(i-1)!(n-i)!]$. Let us denote by $I(p)$ the integral in (11).

Using (9) we have

$$I(p) = -\int_0^\infty z^k \left[\frac{\gamma(p, z)}{\Gamma(p)} - \frac{\gamma(p+k, z)}{\Gamma(p+k)}\right] d\{F_p^{i-1}(z)[1-F_p(z)]^{n-i}\}$$

$$= \int_0^\infty F_p^{i-1}(z)[1-F_p(z)]^{n-i}\Bigg\{kz^{k-1}\left[\frac{\gamma(p, z)}{\Gamma(p)} - \frac{\gamma(p+k, z)}{\Gamma(p+k)}\right] +$$

$$+ z^k\left[\frac{z^{p-1}e^{-z}}{\Gamma(p)} - \frac{z^{p+k-1}e^{-z}}{\Gamma(p+k)}\right]\Bigg\} dz$$

$$= \frac{k}{C}\sum_{j=0}^{k-1} \frac{\Gamma(p)}{\Gamma(p+1+j)} \mathrm{E}_p X_{i:n}^{k+j} + \frac{1}{C}\mathrm{E}_p X_{i:n}^k - \frac{\Gamma(p)}{C\Gamma(p+k)}\mathrm{E}_p X_{i:n}^{2k}.$$

Putting this in (10) we obtain (8).

2.4. Properties of the exponential power distribution. Let us denote by $F_{\lambda,p}$ the distribution function of the exponential power distribution $P_{\lambda,p}$ with pdf (2). It is easy to prove, that if a random variable X has the gamma distribution $P^*_{1,1/p}$ then the random variable $Z = X^{1/p}$ has the exponential power distribution $P_{1,p}$. Therefore the vector of order statistics $(Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n})$ from the distribution $P_{1,p}$ has the same distribution that the vector $(X_{1:n}^{1/p}, X_{2:n}^{1/p}, \ldots, X_{n:n}^{1/p})$, where $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are order statistics from the distribution $P^*_{1,1/p}$. Zieliński [13] has shown that if $0 < p_1 < p_2 \leqslant 2.16$ then $F_{\lambda,p_2} \overset{\mathrm{st}}{\lessgtr} F_{\lambda,p_1}$ for every $\lambda > 0$. Moreover $F_{1,p}$ is a DFR distribution for $p \in (0, 1]$ and an IFR distribution for $p \geqslant 1$ (see [3]).

We prove the following lemma.

LEMMA 8. *If* $0 < p_1 \leqslant p_2 \leqslant 1$, *then* $F_{1,p_2} \overset{\mathrm{disp}}{<} F_{1,p_1}$.

Proof. The failure rate function of the distribution $P_{1,p}$ is of the form

$$\varphi_p(x) = \frac{f_{1,p}(x)}{1-F_{1,p}(x)} = e^{-x^p}\Big/\int_x^\infty e^{-u^p}du.$$

Let $x \in (0, 1]$ be fixed. Since for $0 < p_1 < p_2 \leqslant 1$ the inequalities

$$\frac{1}{\Gamma(1+1/p_1)} < \frac{1}{\Gamma(1+1/p_2)}, \quad \frac{1}{1-F_{1,p_1}(x)} < \frac{1}{1-F_{1,p_2}(x)}$$

and

$$e^{-x^{p_1}} < e^{-x^{p_2}}$$

hold, we have $\varphi_{p_1}(x) < \varphi_{p_2}(x)$, $x \in (0, 1]$.

Now let $x > 1$ be fixed. Then

$$\operatorname{sign}\frac{\partial}{\partial p}\varphi_p(x) = \operatorname{sign} e^{-x^p}\left[\int_x^\infty u^p e^{-u^p}\log u\,du - x^p\log x\int_x^\infty e^{-u^p}du\right] \geqslant 0,$$

since $u^p \log u$ is increasing in $u > 1$. Hence $\varphi_p(x)$ is nondecreasing in $p \in (0, 1]$ for each $x > 0$. Now from Lemma 2 it follows that $F_{1,p_2} \overset{\text{disp}}{<} F_{1,p_1}$ for $0 < p_1 < p_2 \leqslant 1$.

From Lemmas 3, 4 and 8 it follows that the variances and covariances of all order statistics from the exponential power distribution $F_{1,p}$ are nonincreasing for $0 < p \leqslant 1$.

3. v-robustness for the gamma extension. Let us consider the exponential model M_1 and its gamma extension M^*_{1,p_0}, $1 < p_0 < \infty$, defined by (5) and (6). Let $X_1, X_2, \ldots, X_n$ be a sample from a distribution $P^*_{\lambda,p}$, $1 \leqslant p \leqslant p_0$. We study the problem of the existence of the uniformly most v-robust estimate (UMVRE) of λ for the model M_1 with respect to the extension M^*_{1,p_0} in the class $\mathscr{T}^+$ of estimators defined by (4). Following the general concept presented in [12] the estimate $T_0 \in \mathscr{T}^+$ is called the UMVRE of λ if $v_{T_0}(\lambda) \leqslant v_T(\lambda)$ for each $\lambda > 0$ and every $T \in \mathscr{T}^+$, where

$$v_T(\lambda) = \sup_{1 \leqslant p \leqslant p_0} \mathrm{E}_{\lambda,p}(T-\lambda)^2 - \inf_{1 \leqslant p \leqslant p_0} \mathrm{E}_{\lambda,p}(T-\lambda)^2 \tag{12}$$

is the v-robustness function of the estimate T.

We can state the following theorem.

THEOREM 1. *In the class $\mathscr{T}^+$ there exists the UMVRE of λ with respect to the extension M^*_{1,p_0} of the model M_1, $1 < p_0 < \infty$, provided that the matrix $A(p_0) = \|\mathrm{E}_{1,p_0}(X_{i:n}X_{j:n}) - \mathrm{E}_{1,1}(X_{i:n}X_{j:n})\|$ is positive definite.*

Proof. Let $T \in \mathscr{T}^+$, i.e. $T = \sum_{i=1}^n a_i X_{i:n}$, $a_i \geqslant 0$, $i = 1, 2, \ldots, n$ and $\mathrm{E}_{\lambda,1} T = \lambda$. Since $\mathrm{E}_{\lambda,p}(T-\lambda)^2 = \lambda^2 \mathrm{E}_{1,p}(T-1)^2$, we have $v_T(\lambda) = \lambda^2 v_T(1)$ and the problem of seeking T_0 which minimizes $v_T(\lambda)$ uniformly in λ in the class $\mathscr{T}^+$ reduces to minimizing $v_T(1)$ with respect to $T \in \mathscr{T}^+$. It is easy to notice that

$$\begin{aligned} &\mathrm{E}_{1,p}\Big(\sum_{j=1}^n a_j X_{j:n} - 1\Big)^2 \\ &= \sum_{j=1}^n a_j^2 \operatorname{Var}_{1,p} X_{j:n} + \sum\sum_{i \neq j} a_i a_j \operatorname{Cov}_{1,p}(X_{i:n}, X_{j:n}) + \Big[\mathrm{E}_{1,p}\Big(\sum_{j=1}^n a_j X_{j:n}\Big) - 1\Big]^2. \end{aligned} \tag{13}$$

From Preliminaries it follows that the expectations, variances and covariances of order statistics from the gamma distribution $P^*_{1,p}$ are increasing in p and hence for $T \in \mathscr{T}^+$ the expected value (13) is increasing in $p \geqslant 1$. Therefore we have

$$(14) \quad v_T(1) = \sum_{i=1}^{n} \sum_{j=1}^{n} [\mathrm{E}_{1,p_0}(X_{i:n} X_{j:n}) - \mathrm{E}_{1,1}(X_{i:n} X_{j:n})] a_i a_j - \\ -2 \sum_{j=1}^{n} (\mathrm{E}_{1,p_0} X_{j:n} - \mathrm{E}_{1,1} X_{j:n}) a_j, \qquad T \in \mathscr{T}^+.$$

Now our problem is a problem of quadratic programming: seeking the minimum of quadratic function (14) with respect to $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)'$ under conditions:

$$\sum_{j=1}^{n} a_j \mathrm{E}_{1,1} X_{j:n} = 1 \quad \text{and} \quad a_j \geqslant 0, \quad j = 1, 2, \ldots, n.$$

The theorem follows from the well-known result of the theory of quadratic programming (see [5] or [7]).

It seems difficult to find a solution of the problem in a general form. In the sequel we consider the particular case $n = 2$, but first we prove the following theorem.

THEOREM 2. *The statistic* $\bar{X} = \frac{1}{n} \sum_{j=1}^{n} X_{j:n}$, *i.e. the uniformly minimum variance unbiased estimate of* λ *in the model* M_1, *is not the UMVRE of* λ *with respect to any extension* M^*_{1,p_0} *of the model* M_1, $1 < p_0 < \infty$, *in the class* $\mathscr{T}^+$.

Proof. Let us consider a problem of quadratic programming: minimizing the function $\Psi(\boldsymbol{a}) = \boldsymbol{a}' \boldsymbol{C} \boldsymbol{a} + \boldsymbol{q}' \boldsymbol{a}$ under conditions $\boldsymbol{c}' \cdot \boldsymbol{a} = 1$ and $\boldsymbol{a} \geqslant \boldsymbol{0}$, where $\boldsymbol{C}$ is a known quadratic symetric matrix, $\boldsymbol{q}$ and $\boldsymbol{c}$ are known column-vectors. One can give the necessary conditions for a solution of this problem, the so-called Kuhn–Tucker differential conditions (see [5] or [7]):

$$(15) \quad \begin{gathered} \boldsymbol{q} + 2\boldsymbol{C}\boldsymbol{a} + \varkappa \boldsymbol{c} \geqslant \boldsymbol{0}, \\ (\boldsymbol{q} + 2\boldsymbol{C}\boldsymbol{a} + \varkappa \boldsymbol{c})' \cdot \boldsymbol{a} = 0, \\ \boldsymbol{c}' \cdot \boldsymbol{a} = 1, \quad \boldsymbol{a} \geqslant \boldsymbol{0}, \end{gathered}$$

where $\varkappa$ is Lagrange's multiplier. In our v-robustness problem we have $\boldsymbol{C} = \boldsymbol{A}(p_0)$, $\boldsymbol{q} = -2(\mathrm{E}_{1,p_0} X_{1:n} - \mathrm{E}_{1,1} X_{1:n}, \ldots, \mathrm{E}_{1,p_0} - \mathrm{E}_{1,1} X_{n:n})'$, $\boldsymbol{c} = (\mathrm{E}_{1,1} X_{1:n}, \ldots, \mathrm{E}_{1,1} X_{n:n})'$.

Suppose that $\bar{X}$ is the UMVRE of λ with respect to an extension M^*_{1,p_0}. Thus the vector $\boldsymbol{a}_0 = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)'$ satisfies (15). Substituting $\boldsymbol{a}_0$ in (15) we obtain the system of n inequalities:

$$(16) \quad \mathrm{E}_{1,p_0}(X_{i:n} \bar{X}) - \mathrm{E}_{1,1}(X_{i:n} \bar{X}) - \mathrm{E}_{1,p_0} X_{i:n} + \mathrm{E}_{1,1} X_{i:n}(1 + \varkappa) \geqslant 0, \\ i = 1, 2, \ldots, n,$$

and

$$(17)\quad \sum_{i=1}^{n} [\mathrm{E}_{1,p_0}(X_{i:n}\bar{X})-\mathrm{E}_{1,1}(X_{i:n}\bar{X})-\mathrm{E}_{1,p_0}X_{i:n}+\mathrm{E}_{1,1}X_{i:n}(1+\varkappa)] = 0.$$

Since the left-hand side of (17) is the sum of the nonnegative left-hand sides of the inequalities (16), it follows that in (16) there must be equalities. Dividing the ith equality in (16) by $\mathrm{E}_{1,1}X_{i:n}$ and taking into account Lemma 7 we obtain

$$\frac{\mathrm{E}_{1,p_0}X_{i:n}}{\mathrm{E}_{1,1}X_{i:n}} = \frac{1/n-\varkappa}{p+1/n-1} \quad \text{for every } i = 1, 2, \ldots, n, \quad p_0 > 1,$$

which contradicts Lemma 5. Thus the statistic $\bar{X}$ is not the UMVRE of λ with respect to any extension M^*_{1,p_0}, $p_0 > 1$, in the class $\mathscr{T}^+$.

EXAMPLE. Let us consider the case $n = 2$. It is easy to calculate that $\mathrm{E}_{1,p}X_{1:2} = p-\frac{1}{B(p,\,1/2)}$, $\mathrm{E}_{1,p}X_{2:2} = p+\frac{1}{B(p,\,1/2)}$, $\mathrm{E}_{1,p}X^2_{1:2} = p(p+1)-\frac{2p+1}{B(p,\,1/2)}$, $\mathrm{E}_{1,p}X^2_{2:2} = p(p+1)+\frac{2p+1}{B(p,\,1/2)}$, $\mathrm{E}_{1,p}(X_{1:2}X_{2:2}) = p^2$, where

$$B(p,\,1/2) = \frac{\Gamma(p)\Gamma(1/2)}{\Gamma(p+1/2)}.$$

In particular we have $\mathrm{E}_{1,1}X_{1:2} = 1/2$, $\mathrm{E}_{1,1}X_{2:2} = 3/2$, $\mathrm{E}_{1,1}X^2_{1:2} = 1/2$, $\mathrm{E}_{1,1}X^2_{2:2} = 7/2$, $\mathrm{E}_{1,1}(X_{1:2}X_{2:2}) = 1$. Thus

$$A(p) = \begin{bmatrix} p(p+1)-\frac{2p+1}{B(p,\,1/2)}-\frac{1}{2} & p^2-1 \\ p^2-1 & p(p+1)+\frac{2p+1}{B(p,\,1/2)}-\frac{7}{2} \end{bmatrix}.$$

This matrix is positive definite for $p > 1$, because of the monotonicity of $\mathrm{E}_{1,p}X^2_{1:2}$ and the fact that

$$\det A(p) = \frac{2p+1}{B(p,\,1/2)}\left[3-\frac{2p+1}{B(p,\,1/2)}\right]-\left(p^3+2p^2-4p+\frac{3}{4}\right)$$

is a convex function with the minimum equal to zero at $p = 1$. We show that $a = (0,\, 1/\mathrm{E}_{1,1}X_{2:2})'$ satisfies the Kuhn–Tucker conditions and thus the statistic $X_{2:2}/\mathrm{E}_{1,1}X_{2:2} = X_{2:2}/(1+1/2)$ is the UMVRE of λ with respect to each extension $M^*_{1,p}$, $p > 1$, of the model M_1 in the class $\mathscr{T}^+$. In this case the Kuhn–Tucker conditions are as follows

$$(18)\quad \frac{\mathrm{E}_{1,p}(X_{i:2}X_{2:2})-\mathrm{E}_{1,1}(X_{i:2}X_{2:2})}{\mathrm{E}_{1,1}X_{2:2}}-(\mathrm{E}_{1,p}X_{i:2}-\mathrm{E}_{1,1}X_{i:2}) \geqslant -\varkappa\mathrm{E}_{1,1}X_{i:2}, \quad i = 1, 2,$$

and

$$\frac{\mathrm{E}_{1,p}X^2_{2:2}-\mathrm{E}_{1,1}X^2_{2:2}}{(\mathrm{E}_{1,1}X_{2:2})^2}-\frac{\mathrm{E}_{1,p}X_{2:2}-\mathrm{E}_{1,1}X_{2:2}}{\mathrm{E}_{1,1}X_{2:2}} = -\varkappa. \tag{19}$$

It is easy to notice that (18) and (19) are equivalent to the inequality

$$\frac{\mathrm{E}_{1,p}(X_{1:2}X_{2:2})-\mathrm{E}_{1,1}(X_{1:2}X_{2:2})}{\mathrm{E}_{1,1}X_{1:2}\,\mathrm{E}_{1,1}X_{2:2}}-\frac{\mathrm{E}_{1,p}X_{1:2}}{\mathrm{E}_{1,1}X_{1:2}} \geqslant \frac{\mathrm{E}_{1,p}X^2_{2:2}-\mathrm{E}_{1,1}X^2_{2:2}}{(\mathrm{E}_{1,1}X_{2:2})^2}-\frac{\mathrm{E}_{1,p}X_{2:2}}{\mathrm{E}_{1,1}X_{1:2}}$$

which after substituting suitable values takes the form

$$W(p) = 2p^2-p-\frac{11}{8}+\frac{5-2p}{4B(p,\,1/2)} \geqslant 0. \tag{20}$$

It is not difficult to prove that $W(1) = 0$ and $\frac{d}{dp}W(p) \geqslant 0$ for $p \geqslant 1$ and thus the inequality (20) holds.

Some numerical results. Using the tables of moments of order statistics from the gamma distribution [6], [8], [9] we have computed the matrix $A(p)$ for $n = 3, 4, 5$ and $p = 1.5, 2, 3, 4$. For all these parameters the matrix $A(p)$ is positive definite. Next we have applied Wolfe's algorithm of quadratic programming (see [7]) for solving the problem. In all considered cases the solution is $T^* = X_{n:n}/\mathrm{E}_{1,1}X_{n:n}$.

The values of $v_{\bar{X}}(1)$ and $v_{T^*}(1)$ for considered n and some extensions M^*_{1,p_0} are given in Table 1. For comparison the variances of $\bar{X}$ and T^* in the model M_1 are also given there.

TABLE 1

p_0	$v_T(1)$									
	$n=2$		$n=3$		$n=4$		$n=5$		$n=\infty$	
	$\bar{X}$	T^*	$\bar{X}$	T^*	$\bar{X}$	T^*	$\bar{X}$	T^*	$\bar{X}$	T^*
1.1	0.060	0.052	0.043		0.035		0.030		0.010	
1.2	0.140	0.109	0.107		0.090		0.080		0.040	
1.5	0.500	0.397	0.417	0.284	0.375	0.231	0.350	0.198	0.250	
2.0	1.500	1.112	1.333	0.827	1.250	0.684	1.200	0.599	1.000	
3.0	5.000	3.442	4.667	2.605	4.500	2.178	4.400	1.914	4.000	
4.0	10.50	6.915	10.00	5.238	9.750	4.385	9.600	3.857	9.000	
$\mathrm{Var}_{1,1}$	0.500	0.556	0.333	0.405	0.250	0.328	0.200	0.281	0.000	0.000

From Table 1 we can conclude that the v-robustness of the estimates T^* and $\bar{X}$ is decreasing in n. Moreover even for small values of n the v-robustness of T^* is considerably less than the asymptotical v-robustness of $\bar{X}$.

4. v-robustness for the exponential power extension. Consider the exponential model M_1 and its exponential power extension $M_{p_0,1}$, $0 < p_0 < 1$, defined by (2) and (3). Let now $X_1, X_2, \ldots, X_n$ be a sample from a distribution $P_{\lambda,p}$, $p_0 \leqslant p \leqslant 1$. We study the problem of the existence of the UMVRE of λ for the model M_1 with respect to the extension $M_{p_0,1}$ in the class $\mathscr{T}^+$ of estimators defined by (4). In that case the v-robustness function of T is of the form

$$v_T(\lambda) = \sup_{p_0 \leqslant p \leqslant 1} \mathrm{E}_{\lambda,p}(T-\lambda)^2 - \inf_{p_0 \leqslant p \leqslant 1} \mathrm{E}_{\lambda,p}(T-\lambda)^2,$$

where now $\mathrm{E}_{\lambda,p} X$ denotes the expectation of X having the distribution $P_{\lambda,p}$. We can state the following theorem, the analogue of Theorem 1.

THEOREM 3. *In the class $\mathscr{T}^+$ there exists the UMVRE of λ with respect to the extension $M_{p_0,1}$ of the model M_1, $0 < p_0 < 1$, provided that the matrix*

$$\boldsymbol{B}(p_0) = \|\mathrm{E}_{1,p_0}(X_{i:n} X_{j:n}) - \mathrm{E}_{1,1}(X_{i:n} X_{j:n})\|$$

is positive definite.

The proof of Theorem 3 is quite similar to the proof of Theorem 1. One uses the monotonicity of the expectations, variances and covariances of order statistics from the exponential power distribution which are given in Preliminaries.

We can also formulate the analogue of Theorem 2, but in a less general form.

THEOREM 4. *The statistic $\bar{X}$ is not the UMVRE of λ with respect to any extension $M_{1/k,1}$ of the model M_1, $k = 2, 3, \ldots$, in the class $\mathscr{T}^+$.*

Proof. Suppose that $\bar{X}$ is the UMVRE of λ with respect to an extension $M_{1/k,1}$, i.e. the vector $\boldsymbol{a}_0 = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)'$ satisfies the Kuhn–Tucker conditions (15) with $\boldsymbol{C} = \boldsymbol{B}(1/k)$, $\boldsymbol{q} = -2(\mathrm{E}_{1,1/k} X_{1:n} - \mathrm{E}_{1,1} X_{1:n}, \ldots, \mathrm{E}_{1,1/k} X_{n:n} - \mathrm{E}_{1,1} X_{n:n})'$ and $\boldsymbol{c} = (\mathrm{E}_{1,1} X_{1:n}, \ldots, \mathrm{E}_{1,1} X_{n:n})'$. Thus we have the following system of n inequalities

$$\mathrm{E}_{1,1/k}(X_{i:n}\bar{X}) - \mathrm{E}_{1,1}(X_{i:n}\bar{X}) - \mathrm{E}_{1,1/k} X_{i:n} + (1+\varkappa)\mathrm{E}_{1,1} X_{i:n} \geqslant 0, \quad i = 1, 2, \ldots, n, \tag{21}$$

and

$$\sum_{i=1}^{n} [\mathrm{E}_{1,1/k}(X_{i:n}\bar{X}) - \mathrm{E}_{1,1}(X_{i:n}\bar{X}) - \mathrm{E}_{1,1/k} X_{i:n} + (1+\varkappa)\mathrm{E}_{1,1} X_{i:n}] = 0,$$

where $\varkappa$ is Lagrange's multiplier. For the same reason as in the proof of Theorem 2 the inequalities (21) are equalities. From Preliminaries it follows that $E_{1,1/k} X_{i:n} = E_{1,k} Y_{i:n}^k$ and $E_{1,1/k}(X_{i:n}\bar{X}) = E_{1,k}(Y_{i:n}^k \overline{Y^k})$, where $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ are order statistics from the gamma distribution $P_{1,k}^*$. Therefore from Lemma 7 it follows that

$$(22)\quad E_{1,1/k}(X_{i:n}\bar{X}) = \frac{\Gamma(2k)}{\Gamma(k)}\left[E_{1,1/k} X_{i:n} + \frac{k}{n}\sum_{j=0}^{k-1} \frac{\Gamma(k)}{\Gamma(k+1+j)} E_{1,1/k} X_{i:n}^{1+j/k}\right].$$

Dividing (21) by $E_{1,1} X_{i:n}$ and taking into account (22) we obtain

$$(23)\quad \left[\frac{\Gamma(2k)}{\Gamma(k)}\left(1+\frac{1}{n}\right)-1\right]\frac{E_{1,1/k} X_{i:n}}{E_{1,1} X_{i:n}} + \frac{k}{n}\sum_{j=1}^{k-1} \frac{\Gamma(2k)}{\Gamma(k+1+j)} \frac{E_{1,1/k} X_{i:n}^{1+j/k}}{E_{1,1} X_{i:n}} = \frac{1}{n} - \varkappa$$

for every $i = 1, 2, \ldots, n$ and $k > 1$.

From Lemma 5 it follows that the left-hand side of (23) is strongly increasing in $i = 1, 2, \ldots, n$. The contradiction proves the theorem.

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

Received on 31. 1. 1984