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EXTREME POINTS OF A CONVEX POLYTOPE AND EXTREME RAYS OF THE CORRESPONDING CONVEX CONE

1. Introduction. The method of finding all extreme points of a convex polytope and extreme rays generating all edges of the corresponding convex cone can be related to a linear programming problem with a parametric objective function. The general case of that problem, i.e. the case with no assumptions concerning the vector of the objective function cannot be solved without finding all extreme points and all extreme rays.

An efficient method for the case where all elements of that vector are linear affine functions of one parameter was given by Saaty and Gass [9]. This problem has also been studied by Wagner [12] and Kelley, Jr. [7].

The case where all elements of that vector are linear affine functions of two parameters was considered by Gass and Saaty [6]. Two methods proposed by the authors solve that case but the adaptation of those methods to a general case is rather doubtful.

Simons [11] considered the general case and proved that a set of the so-called *admissible vectors* (i.e. such vectors that for any of them the problem has an optimal solution) is a convex polytope.

The general case can be related to a variety of substantial problems.

One of them is the problem of finding all efficient points, posed by Charnes and Cooper [2], p. 294-321.

Another, is a convex programming problem with linear constraints whose function to be maximized is differentiable and concave. The problem can be solved by the use of the Frank-Wolfe method (see [5]) as it is suggested in [1], p. 89-91. The essential part of each step of an approximation procedure is the solution of a linear programming problem whose vector of the objective function is a gradient of $f(x)$ at an examined point x_k. Because of slow convergence of the sequence $\{f(x_k)\}$, one has to examine many points in order to find a satisfactory solution. Thus, instead of solving many linear programming problems it might be worthwhile to find all extreme points and extreme rays of a set of feasible solutions.

It should be pointed out that a linear programming problem with the right-hand side vector treated parametrically can be solved by considering the corresponding dual problem.

The method is described by introducing a linear programming problem which is, for convenience of description, considered as dual to another linear programming problem. The latter problem is transformed in Section 2 by the Fourier-Motzkin elimination method (see [4] and [8]) into a problem with one variable. As a by-product of this transformation, another proof of the duality theorem, given in Section 3, is obtained. This proof plays a crucial role in justification of the method given in Section 4. The method is illustrated by examples in Section 5. In general, the method yields also points which are neither extreme points nor extreme rays. These points can be excluded by the use of properties stated in Sections 6 and 7. One can stress that Corollary 11 states a necessary and sufficient condition for a point to be an extreme ray of a specially defined convex cone and that Propositions 13 and 14 allow to treat extreme points of a convex polytope and extreme rays of a convex cone as extreme rays of another convex cone.

Although the method seems to be now of limited computational interest, future developments in constructing computers may make it more applicable.

2. Treatment of linear programming problems by the Fourier-Motzkin elimination method. Let us state the following two linear programming problems:

PROBLEM P. Minimize

$$\sum_{j=1}^{n} c_j x_j \tag{2.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i, \qquad i = 1, \ldots, m,$$

$$x_j \geqslant 0, \qquad j = 1, \ldots, n.$$

PROBLEM D. Maximize $\sum_{i=1}^{n} y_i b_i$ subject to

$$\sum_{i=1}^{m} y_i a_{ij} \leqslant c_j, \qquad j = 1, \ldots, n,$$

$$y_i \geqslant 0, \qquad i = 1, \ldots, m.$$

Remark 1. In what follows we give a method of finding all extreme points of a convex polytope defined by a system of constraints of Problem D and all extreme rays of an associated convex cone.

Since minimizing (2.1) is equivalent to minimizing x_{n+1} subject to

$$x_{n+1} \geqslant \sum_{j=1}^{n} c_j x_j,$$

Problem P can be rewritten as follows:

PROBLEM P_0. Minimize x_{n+1} subject to $d_i x \geqslant d_{i0}$ for $i \in I_0$, with

$$I_0 = \{0, 1, \ldots, m+n\}, \quad x^T = (x_1, x_2, \ldots, x_{n+1})^T,$$

$$d_{i0} = \begin{cases} 0 & \text{for } i = 0, \\ b_i & \text{for } i = 1, \ldots, m, \\ 0 & \text{for } i = m+1, \ldots, m+n, \end{cases}$$

$$(2.2) \quad d_i = \begin{cases} (-c, 1) = (-c_1, \ldots, -c_n, 1) & \text{for } i = 0, \\ (a_i, 0) = (a_{i1}, \ldots, a_{in}, 0) & \text{for } i = 1, \ldots, m, \\ (\delta_{i-m}, 0) & \text{for } i = m+1, \ldots, m+n, \end{cases}$$

where δ_k denotes the k-th n-dimensional unit vector.

Remark 2. An optimal solution $x_1^*, x_2^*, \ldots, x_{n+1}^*$ of P_0 satisfies

$$x_{n+1}^* = \sum_{j=1}^{n} c_j x_j^*.$$

Let

$$(2.3) \quad (e_i, e_{i0}) = \begin{cases} \dfrac{1}{|d_{i1}|}(d_i, d_{i0}) & \text{if } d_{i1} \neq 0, \\ (d_i, d_{i0}) & \text{if } d_{i1} = 0, \end{cases}$$

where d_{i1} is the first element of $d_i = (d_{i1}, \ldots, d_{i,n+1})$.

Problem P_0 is, obviously, equivalent to the following problem:

PROBLEM P_0'. Minimize x_{n+1} subject to $e_i x \geqslant e_{i0}$ for $i \in I_0$.

Observe that the first elements of e_i are equal to 0 or to 1 or to -1. Let then

$$(2.4) \quad I_0^0 = \{i \mid e_{i1} = 0\}, \quad I_0^+ = \{i \mid e_{i1} = 1\}, \quad I_0^- = \{i \mid e_{i1} = -1\}$$

and state the following problem:

PROBLEM P_1. Minimize x_{n+1} subject to $e_i x \geqslant e_{i0}$ for $i \in I_0^0$,

$$(2.5) \quad (e_i + e_k)x \geqslant e_{i0} + e_{k0} \quad \text{for } (i, k) \in I_0^+ \times I_0^-.$$

Remark 3. Problem P_1 is obtained from Problem P_0' by replacing

$$(2.6) \quad e_i x \geqslant e_{i0} \quad \text{for } i \in I_0^+ \cup I_0^-$$

with (2.5).

Since $e_{i1} = 0$ for $i \in I_0^0$ and $e_{i1} + e_{k1} = 1 + (-1) = 0$ for $(i, k) \in I_0^+ \times I_0^-$ (see (2.4)), x_1 does not appear in Problem P_1.

The system of constraints of Problem P_1 can be written as $d_i^1 x \geqslant d_{i0}^1$ for $i \in I_1$, where (d_i^1, d_{i0}^1) is equal to (e_i, e_{i0}) or to $(e_i + e_k, e_{i0} + e_{k0})$. Introducing (compare with (2.3))

$$(e_i^1, e_{i0}^1) = \begin{cases} \dfrac{1}{|d_{i2}^1|}(d_i^1, d_{i0}^1) & \text{if } d_{i2}^1 \neq 0, \\ (d_i^1, d_{i0}^1) & \text{if } d_{i2}^1 = 0, \end{cases} \tag{2.7}$$

we obtain the following problem which is, obviously, equivalent to P_1.

Problem P_1'. Minimize x_{n+1} subject to $e_i^1 x \geqslant e_{i0}^1$ for $i \in I_1$.

For convenience of description, we treat P_1 and P_1' as problems with the variables $x_2, \ldots, x_{n+1}$ or as problems with the variables $x_1, x_2, \ldots, x_{n+1}$, where x_1 appears with the coefficient zero.

Proposition 1. *Any feasible solution to* P_0 *is a feasible solution to* P_1'.

Proof. Because of the obvious equivalence of P_0 and P_0' and of P_1 and P_1', it is enough to show that any feasible solution to P_0' is a feasible solution to P_1. But this is obvious since, by Remark 3, (2.6) implies (2.5).

Lemma 1. *If* $x_2^*, \ldots, x_{n+1}^*$ *satisfy* (2.5), *then there exists an* x_1^* *such that* $x_1^*, x_2^*, \ldots, x_{n+1}^*$ *satisfy* (2.6).

Proof. If at least one of the sets I_0^+, I_0^- is empty, the lemma holds, since the set of x_1's such that $x_1, x_2, \ldots, x_{n+1}$ satisfy (2.6) is unbounded. Then assume that $I_0^+ \neq \emptyset$ and $I_0^- \neq \emptyset$. By the assumption of the lemma,

$$\sum_{j=2}^{n+1} (e_{ij} + e_{kj}) x_j^* - e_{i0} - e_{k0} \geqslant 0 \quad \text{for } (i, k) \in I_0^+ \times I_0^-$$

which is equivalent to

$$\min_{k \in I_0^-} \Big(\sum_{j=2}^{n+1} e_{kj} x_j^* - e_{k0} \Big) \geqslant \max_{i \in I_0^+} \Big(- \sum_{j=2}^{n+1} e_{ij} x_j^* + e_{i0} \Big).$$

Hence, there exists an x_1^* such that

$$\min_{k \in I_0^-} \Big(\sum_{j=2}^{n+1} e_{kj} x_j^* - e_{k0} \Big) \geqslant x_1^* \geqslant \max_{i \in I_0^+} \Big(- \sum_{j=2}^{n+1} e_{ij} x_j^* + e_{i0} \Big)$$

which can be rewritten as

$$x_1^* \geqslant - \sum_{j=2}^{n+1} e_{ij} x_j^* + e_{i0} \quad \text{for } i \in I_0^+,$$

$$\sum_{j=2}^{n+1} e_{kj} x_j^* - e_{k0} \geqslant x_1^* \quad \text{for } k \in I_0^-$$

or, in the equivalent form,

$$e_i x^* \geqslant e_{i0} \text{ for } i \in I_0^+, \quad e_i x^* \geqslant e_{i0} \text{ for } i \in I_0^-.$$

This completes the proof.

PROPOSITION 2. *If $x_2^*, \ldots, x_{n+1}^*$ is a feasible solution to P_1', then there exists an x_1^* such that $x_1^*, x_2^*, \ldots, x_{n+1}^*$ is a feasible solution to P_0.*

Proof. The proposition follows from the equivalence of P_1' and P_1, Lemma 1, Remark 2 and the equivalence of P_0' and P_0.

PROPOSITION 3. *Any optimal solution to P_0 is an optimal solution to P_1'.*

Proof. Suppose $x_1^*, x_2^*, \ldots, x_{n+1}^*$ is an optimal solution to P_0 but not to P_1'. Then there exists a feasible solution $x_2', \ldots, x_{n+1}'$ to P_1' such that $x_{n+1}' < x_{n+1}^*$. By Proposition 2, there exists an x_1' such that $x_1', x_2', \ldots, x_{n+1}'$ is a feasible solution to P_0. This and $x_{n+1}' < x_{n+1}^*$ is contradictory to the assumption.

PROPOSITION 4. *If $x_2^*, \ldots, x_{n+1}^*$ is an optimal solution to P_1', then there exists an x_1^* such that $x_1^*, x_2^*, \ldots, x_{n+1}^*$ is an optimal solution to P_0.*

Proof. By Proposition 2, we can choose an x_1^* such that $x_1^*, x_2^*, \ldots, x_{n+1}^*$ is a feasible solution to P_0. If $x_1^*, x_2^*, \ldots, x_{n+1}^*$ were not an optimal solution to P_0, then there would exist $x_1', x_2', \ldots, x_{n+1}'$ satisfying the constraints of P_0 such that $x_{n+1}' < x_{n+1}^*$ which would mean, by Proposition 1, that $x_2^*, \ldots, x_{n+1}^*$ were not an optimal solution to P_1'. This contradiction completes the proof.

Remark 4. At this stage we make several comments concerning P_1'.

1. Each (e_i^1, e_{i0}^1) is a non-negative linear combination of (d_0, d_{00}), $(d_1, d_{10}), \ldots, (d_{m+n}, d_{m+n,0})$ since (d_i^1, d_{i0}^1) is equal either to (e_i, e_{i0}) or to $(e_i + e_k, e_{i0} + e_{k0})$, and (e_i, e_{i0}) and (e_i^1, e_{i0}^1) are found by (2.3) and (2.7), respectively. Moreover, not all coefficients expressing (e_i^1, e_{i0}^1) as a linear combination of those vectors are equal to zero.

2. The set I_1 is finite.

3. The first element of each vector e_i^1 is zero, and the second one is either 0 or 1 or -1.

Now we can obtain Problem P_2 from P_1' by adding each inequality with $e_{i2}^1 = 1$ to each inequality with $e_{i2}^1 = -1$ and leaving each inequality with $e_{i2}^1 = 0$ without change. By formulas similar to (2.3) and (2.7), we can find vectors e_{i1}^2 and state Problem P_2'. By repeating the same procedure, we can obtain the following sequence of Problems: $P_0, P_0', P_1, P_1', \ldots, P_n, P_n'$.

Remark 5. We state the following properties of P_n':

1. Each (e_i^n, e_{i0}^n) is a non-negative linear combination of (d_0, d_{00}), $(d_1, d_{10}), \ldots, (d_{m+n,0})$. Moreover, not all coefficients expressing (e_i^n, e_{i0}^n) as a linear combination of those vectors are equal to zero.

2. The number of constraints of P'_n is finite, i.e. I_n is a finite set.

3. For each $i \in I_n$, $e^n_{i1} = e^n_{i2} = \ldots = e^n_{in} = 0$ and $e^n_{i,n+1}$ is equal either to 1 or to 0.

Remark 5 can be deduced, by induction, from Remark 4. The last statement of Remark 5 is justified additionally (see (2.2)) by $d_{0,n+1} = 1$, $d_{i,n+1} = 0$ for $i = 1, \ldots, m+n$. Hence we cannot get a negative $e^n_{i,n+1}$ as a non-negative linear combination of $d_{0,n+1}, d_{1,n+1}, \ldots, d_{m+n,n+1}$.

According to Remark 5, Problem P'_n can be written as follows:

PROBLEM P'_n. Minimize x_{n+1} subject to

$$x_{n+1} \geqslant e^n_{i0} \quad \text{for } i \in I^+_n, \tag{2.8}$$

$$0 \geqslant e^n_{i0} \quad \text{for } i \in I^0_n. \tag{2.9}$$

COROLLARY 1. *If $x^*_1, x^*_2, \ldots, x^*_{n+1}$ is a feasible (an optimal) solution to P_0, then x^*_{n+1} is a feasible (an optimal) solution to P'_n.*

COROLLARY 2. *If x^*_{n+1} is a feasible (an optimal) solution to P'_n, then there exist $x^*_1, \ldots, x^*_n$ such that $x^*_1, \ldots, x^*_n, x^*_{n+1}$ is a feasible (an optimal) solution to P_0.*

Corollary 1 follows, by induction, from Propositions 1 and 3, and Corollary 2 — from Propositions 2 and 4.

Observe that, in P'_n, inequalities (2.9) are "responsible" for the consistence of P'_n, and (2.8) — for the objective function to be bounded. Thus, Corollaries 1 and 2, the equivalence of P_0 and P and Remark 1 imply the following corollaries:

COROLLARY 3. *Problem P is consistent if and only if all elements of the set $\{e^n_{i0} \mid i \in I^0_n\}$ are non-positive.*

COROLLARY 4. *If P is consistent, then the objective function of P is bounded from below if and only if $I^+_n \neq \emptyset$. If P is consistent and its objective function is bounded from below, then both P'_n and P have their optimal solutions x^*_{n+1} and $x^*_1, \ldots, x^*_n$, respectively. Moreover,*

$$x^*_{n+1} = \sum_{j=1}^{n} c_j x^*_j.$$

Notice that an optimal solution x^*_{n+1} to P'_n is the maximum element of the finite set $\{e^n_{i0} \mid i \in I^+_n\}$. Corresponding $x^*_n, x^*_{n-1}, \ldots, x^*_1$ can be found by solving constraints of $P'_{n-1}, P'_{n-2}, \ldots, P'_0$, respectively.

PROPOSITION 5. *If $y^k_0, y^k_1, \ldots, y^k_{m+n}$ are non-negative numbers satisfying*

$$(e^n_k, e^n_{k0}) = \sum_{i=0}^{m+n} y^k_i (d_i, d_{i0}), \tag{2.10}$$

then

$$\sum_{i=1}^{m} y_i^k b_i = e_{k0} \quad \text{for } k \in I_n, \tag{2.11}$$

$$\sum_{i=1}^{m} y_i^k a_{ij} \leqslant c_j \quad \text{for } j = 1, \ldots, n \text{ and } k \in I_n^+, \tag{2.12}$$

$$\sum_{i=1}^{m} y_i^k a_{ij} \leqslant 0 \quad \text{for } j = 1, \ldots, n \text{ and } k \in I_n^0. \tag{2.13}$$

Proof. (2.11) follows from (2.2) and (2.10).

If $k \in I_n^+$, then $e_k^n = (0, 1)$. Thus, by (2.2) and (2.10), we have

$$0 = -y_0^k c_j + \sum_{i=1}^{m} y_i^k a_{ij} + y_{m+j}^k, \quad j = 1, \ldots, n,$$

$$1 = y_0^k.$$

Since $y_{m+j}^k \geqslant 0$ for $j = 1, \ldots, n$ (see Remark 5), the latter equalities imply (2.12).

If $k \in I_n^0$, then $e_k^n = (0, 0)$ and in the above-given equalities we have $y_0^k = 0$ (instead of $y_0^k = 1$) which implies (2.13).

3. Duality theorem.

LEMMA 2. *For any $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ which satisfy the constraints of* P *and* D, *respectively, we have*

$$\sum_{j=1}^{n} c_j x_j \geqslant \sum_{i=1}^{m} y_i b_i.$$

Proof. By utilizing the constraints of P and D, we get

$$\sum_{j=1}^{n} c_j x_j \geqslant \sum_{i=1}^{n} \sum_{j=1}^{n} y_i a_{ij} x_j \geqslant \sum_{i=1}^{m} y_i b_i.$$

THEOREM 1. *If* P *and* D *are consistent, then the objective function of* P *attains its minimum value M_{P}, and the objective function of* D *attains its maximum value M_{D}. The objective function of* P *attains M_{P} if and only if the objective function of* D *attains M_{D}. Moreover, in both cases, $M_{\mathrm{P}} = M_{\mathrm{D}}$.*

Proof. Since P and D are consistent, by Lemma 2, the objective function of P is bounded. Hence, by Corollary 4, there exist the optimal solutions $x_1^*, x_2^*, \ldots, x_n^*$ and x_{n+1}^* to P and P_n', respectively. Moreover,

$$x_{n+1}^* = \sum_{j=1}^{n} c_j x_j^*. \tag{3.1}$$

The finite system (2.8) yields lower bounds for x_{n+1} in P'_n. Since x^*_{n+1} is a minimum value of x_{n+1} and I^+_n is finite, there exists a $k \in I^+_n$ such that

$$x^*_{n+1} = \max_{i \in I^+_n} e^n_{i0} = e^n_{k0}. \tag{3.2}$$

On the other hand, by Remark 5, the vector (e^n_k, e^n_{k0}) is a non-negative linear combination of (d_0, d_{00}), $(d_1, d_{10}), \ldots, (d_{m+n}, d_{m+n,0})$. In other words, there exist non-negative $y^k_0, y^k_1, \ldots, y^k_{m+n}$ satisfying (2.10) and, by Proposition 5, (2.11) and (2.12). This means that $y^k_1, \ldots, y^k_m$ is a feasible solution to D which, by (3.1), (3.2) and (2.11), satisfies

$$\sum_{j=1}^{n} c_j x^*_j = \sum_{i=1}^{m} y^k_i b_i.$$

Hence, by Lemma 2, $y^k_1, \ldots, y^k_m$ is an optimal solution to D. This and the above-given equality proves the first part of the theorem.

Suppose that M_{P} is attained at the point $x^*_1, x^*_2, \ldots, x^*_n$. Then, by Corollary 4, Problem P'_n has an optimal solution satisfying (3.1). By the above reasoning, we can prove that $M_{\mathrm{P}} = M_{\mathrm{D}}$. The proof is completed since D can be rewritten in a form of P and *vice versa.*

It should be pointed out that the proof of Theorem 1 does not involve the Farkas-Minkowski lemma. This lemma can be easily proved as a corollary to Theorem 1.

4. Extreme points and extreme rays. Let U denote a convex polytope defined by

$$\sum_{i=1}^{m} y_i a_{ij} \leqslant c_j, \quad j = 1, \ldots, n, \qquad y_i \geqslant 0, \quad i = 1, \ldots, m, \tag{4.1}$$

and let U^0 denote a convex cone defined by

$$\sum_{i=1}^{m} y_i a_{ij} \leqslant 0, \quad j = 1, \ldots, n, \qquad y_i \geqslant 0, \quad i = 1, \ldots, m. \tag{4.2}$$

The aim of this section is to show that the procedure of Section 2 produces all extreme points of U and the extreme rays generating all edges of U^0.

PROPOSITION 6. *If $y^k_0, y^k_1, \ldots, y^k_{m+n}$ are non-negative numbers satisfying (2.10), then $(y^k_1, \ldots, y^k_m)$ is an element of U or U^0 whenever $k \in I^+_n$ or $k \in I^0_n$, respectively.*

The proof follows by (2.12) and (2.13).

The system (4.1) can be rewritten as

$$yp_j \leqslant q_j, \quad j = 1, \ldots, n+m, \tag{4.3}$$

where p_j denotes the j-th column of the matrix $P = (A, -I_m)$, q_j — the j-th element of the $(n+m)$-dimensional vector $q = (c, 0)$, and y — the row vector $(y_1, \ldots, y_m)$.

Let $H_j = \{y \mid yp_j \leqslant q_j\}$ and $E_j = \{y \mid yp_j = q_j\}$ for $j = 1, \ldots, n+m$. Then

$$U = \bigcap_{j=1}^{n+m} H_j.$$

LEMMA 3. *If $y^* = (y_1^*, \ldots, y_m^*)$ is an extreme point of U, then there exist a scalar b_0 and a vector b such that $y^* b = b_0$, and $yb < b_0$ for $y \in U - \{y^*\}$.*

Proof. Suppose that s of the planes $E_1, E_2, \ldots, E_{n+m}$, i.e. $E_1, E_2, \ldots, E_s$, pass through y^*. Since y^* is an extreme point of U (see [1], p. 57), we have

$$\bigcap_{j=1}^{s} E_j = \{y^*\}. \tag{4.4}$$

Since

$$U = \bigcap_{j=1}^{n+m} H_j \subset \bigcap_{j=1}^{s} H_j,$$

$y \in U$ implies $yp_j \leqslant q_j$ for $j = 1, \ldots, s$. By (4.4), $y^* p_j = q_j$ for $j = 1, \ldots, s$ and there exists a $t \in \{1, \ldots, s\}$ such that $yp_t < q_t$ for any $y \in U - \{y^*\}$. To complete the proof, take

$$b = \frac{1}{s} \sum_{j=1}^{s} p_j \quad \text{and} \quad b_0 = \frac{1}{s} \sum_{j=1}^{s} q_j.$$

THEOREM 2. *If $y^* = (y_1^*, \ldots, y_m^*)$ is an extreme point of U (defined by (4.1)), then there exists a $k \in I_n^+$ such that $y^k = (y_1^k, \ldots, y_m^k)$ is equal to y^*, where $y_0^k, y_1^k, \ldots, y_{m+n}^k$ denote coefficients satisfying*

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i,$$

obtained by the procedure transforming P_0 *into* P_n'.

Proof. Problem D with the vector b given in Lemma 3 has a unique optimal solution y^*, and the maximum value of the objective function is b_0 (defined also by Lemma 3). Then, by Theorem 1, the corresponding Problem P has the optimal solution $x_1^*, \ldots, x_n^*$ and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} y_i^* b_i = b_0.$$

Thus, by the reasoning of Section 2 (Corollary 4), the corresponding Problem P_n' has the optimal solution $x_{n+1}^* = b_0$. Since the procedure transforming P_0 into P_0' does not depend on b, there exists a $k \in I_n^+$ such that $e_{k0}^n = x_{n+1}^* = b_0$ and

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i, \quad \text{where } y^k = (y_1^k, \ldots, y_m^k),$$

is the optimal solution to D. Problem D, however, has a unique optimal solution y^*. Thus $y^k = y^*$.

COROLLARY 5. *For a convex polytope U to be non-empty it is necessary and sufficient that I_n^+ be non-empty.*

Proof. The sufficiency is obvious since, by Remark 5, for each $k \in I_n^+$, there exist non-negative numbers $y_0^k, y_1^k, \ldots, y_{m+n}^k$ satisfying (2.10) and thus $(y_1^k, \ldots, y_m^k) \in U$.

To prove the necessity assume that $I_n^+ = \emptyset$ and $U \neq \emptyset$. Then the system

$$\sum_{i=1}^{m} y_i a_{ij} + y_{m+j} = c_j, \quad j = 1, \ldots, n,$$

has a non-negative solution. Hence (see [10], p. 376) it has a basic non-negative solution $y_1^*, y_2^*, \ldots, y_{m+n}^*$ and, consequently, $(y_1^*, y_2^*, \ldots, y_m^*)$ is an extreme point of U. By Theorem 2, $I_n^+ \neq \emptyset$ which is contradictory to $I_n^+ = \emptyset$. This completes the proof.

LEMMA 4. *Let $(y_0^k, y_1^k, \ldots, y_m^k)$ denote a subvector of $(y_0^k, y_1^k, \ldots, y_{m+n}^k)$, where $y_0^k, y_1^k, \ldots, y_{m+n}^k$ denote coefficients satisfying*

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i$$

and obtained by the procedure transforming P *into* P_n'.

If $k \in I_n$, then $(y_0^k, y_1^k, \ldots, y_m^k)$ is a non-zero vector.

Proof. Suppose $y_i^k = 0$, $i = 0, 1, \ldots, m$. By the proof of Proposition 5, $y_0^k = 0$ implies $k \in I_n^0$. Then, by (2.2),

$$e_k^n = \sum_{i=m+1}^{m+n} y_i^k d_i = \sum_{i=m+1}^{m+n} y_i^k (\delta_{i-m}, 0).$$

However, e_k^n is a zero vector which implies $y_i^k = 0$ for $i = m+1, \ldots \ldots, m+n$ and, consequently, $y_i^k = 0$ for $i = 0, \ldots, m+n$. This is contradictory to Remark 5.

For any vector y', we write $D_{y'}^+ = \{y \mid y = \lambda y', \lambda \geqslant 0\}$.

A vector y is said to be an *extreme ray of a convex cone* if D_y^+ is an edge of that cone.

In what follows we often use the following proposition (see [10], p. 400):

PROPOSITION 7. *A vector y is an extreme ray of a convex cone if and only if it cannot be represented as a positive linear combination of two non-collinear (i.e. linearly independent) vectors of that cone.*

LEMMA 5. *Let $y^k = (y_1^k, \ldots, y_m^k)$ denote a subvector of $(y_0^k, y_1^k, \ldots, y_{m+n}^k)$, where $y_0^k, y_1^k, \ldots, y_{m+n}^k$ are coefficients satisfying*

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i,$$

obtained by the procedure transforming P_0 into P_n'. Let $\overline{U}$ be a convex hull of $\{D_{y^1}^+, D_{y^2}^+, \ldots, D_{y^p}^+\}$, where $\{y^k \mid k = 1, \ldots, p\} = \{y^k \mid k \in I_n^0\}$.

If there does not exist $k \in I_n^0$ such that y^k belong to an edge $D_{y^*}^+$ of U^0, then $y^* \notin \overline{U}$.

Proof. If $I_n^0 = \emptyset$, then $\overline{U} = \emptyset$ and the lemma is obvious.

Assume that $I_n^0 \neq \emptyset$ and suppose, to the contrary, that $y^* \in \overline{U}$. Since $\overline{U}$ is a closed convex cone generated by $y^1, \ldots, y^p$ (see [1], p. 58-60), there exist non-negative numbers $\lambda_1, \ldots, \lambda_p$ such that

$$y^* = \sum_{k=1}^{p} \lambda_k y^k.$$

Moreover, since y^* is a non-zero vector (because $D_{y^*}^+$ is an edge of U^0), at least one of λ's is positive. Hence there exists a q $(1 \leqslant q \leqslant p)$ such that

$$y = \sum_{k=1}^{q} \lambda_k y^k \quad \text{and} \quad \lambda_k > 0,\ k = 1, \ldots, q.$$

We show that, without loss of generality, we can assume that any two of the vectors $y^1, \ldots, y^q$ are non-collinear. Suppose, for instance, $y^2 = \mu y^1$. Since $\{1, 2\} \subset \{1, 2, \ldots, p\} = I_n^0$, it follows, by the proof of Proposition 5, that $y_0^1 = y_0^2 = 0$ and, by Lemma 4, y^1, y^2 are non-zero vectors. Moreover, by Remark 5, y^1, y^2 are non-negative. Hence $\mu > 0$ and we can write

$$y^* = \lambda y^1 + \sum_{k=3}^{q} \lambda_k y^k,$$

where $\lambda = \lambda_1 + \mu\lambda_2 > 0$ (since λ_1, λ_2 and μ are positive).

Suppose $q = 1$. Then $y^* = \lambda_1 y^1$ and, since $\lambda_1 > 0$, we have $y^1 \in D_{y^*}^+$, which is contradictory to the assumption. Hence $q \geqslant 2$.

Suppose $\lambda_1 y^1$ and $\sum_{k=2}^{q} \lambda_k y^k$ are collinear. Then, since

$$y^* = \lambda_1 y^1 + \sum_{k=2}^{q} \lambda_k y^k, \tag{4.5}$$

both these vectors are collinear with y^*. In particular, y^1 is collinear with y^* which implies that $y^1 \in D^+_{y^*}$, a contradiction to the assumption.

Finally, if $\lambda_1 y^1$ and $\sum_{k=2}^{q} \lambda_k y^k$ are non-collinear, then y^* can be represented, by (4.5), as a positive linear combination of these two non-collinear vectors which both belong to U^0 (since, by Proposition 6, $\{y^1, \ldots, y^q\} \subset U^0$). Hence, by Proposition 7, y^* is not an extreme ray of U^0 and, consequently, $D^+_{y^*}$ is not an edge of U^0. This contradiction completes the proof.

THEOREM 3. *If a convex polytope U (defined by* (4.1)) *is not empty and if $y^* = (y_1^*, \ldots, y_m^*)$ is an extreme ray of U^0 (defined by* (4.2)), *then there exists a $k \in I_n^0$ such that $y^k = (y_1^k, \ldots, y_m^k)$ belongs to an edge $D^+_{y^*}$, where $y_0^k, y_1^k, \ldots, y_{m+n}^k$ denote coefficients satisfying*

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i,$$

obtained by the procedure transforming P_0 *into* P_0'.

Proof. First, notice that the assumption $U \neq \emptyset$ yields, by Corollary 5, $I_n^+ \neq \emptyset$.

Now assume, to the contrary, that k satisfying the hypothesis does not exist and consider two cases: $I_n^0 = \emptyset$ and $I_n^0 \neq \emptyset$.

In the case $I_n^0 = \emptyset$, Corollary 3 yields that, for any vector b, Problem P is consistent. Furthermore, Corollary 4 implies that it has an optimal solution since $I_n^+ \neq \emptyset$. Hence, by Theorem 1, Problem D has an optimal solution for any vector b.

On the other hand, $y + \lambda y^* \in U$ whenever $y \in U$ and $\lambda \geqslant 0$. Moreover, since y^* as an extreme ray is a non-zero vector, we have

$$\lim_{\lambda \to +\infty} \sum_{i=1}^{m} (y_i + \lambda y_i^*) y_i^* = +\infty$$

which means that, for $b = y^*$, the objective function of Problem D is unbounded from above. This is contradictory to the statement that Problem D has an optimal solution for any vector b.

In the case $I_n^0 \neq \emptyset$, we have $y^* \notin \overline{U}$, where $\overline{U}$ denotes the closed convex cone defined in Lemma 5. By the Second Separation Theorem (see [1], p. 55), there exists a vector $b = (b_1, \ldots, b_m)$ such that, for any vector $y = (y_1, \ldots, y_m) \in \overline{U}$,

$$\sum_{i=1}^{m} y_i^* b_i > \sum_{i=1}^{m} y_i b_i. \tag{4.6}$$

In particular, since $0 \in \overline{U}$, therefore

$$\sum_{i=1}^{m} y_i^* b_i > 0. \tag{4.7}$$

Consequently, for any $y \in \overline{U}$,

$$\sum_{i=1}^{m} y_i b_i \leqslant 0 \tag{4.8}$$

since, if for some $y' \in \overline{U}$ we had

$$\sum_{i=1}^{m} y_i' b_i > 0,$$

then there would exist a positive λ' such that $\lambda' y' \in \overline{U}$ and

$$\sum_{i=1}^{m} y_i^* b_i \leqslant \sum_{i=1}^{m} \lambda' y_i' b_i,$$

which is contradictory to (4.6). In particular, since $y^k \in \overline{U}$ for $k \in I_n^0$, (4.8) implies

$$\sum_{i=1}^{m} y_i^k b_i \leqslant 0 \quad \text{for } k \in I_n^0.$$

This and (2.11) yield, by Corollary 3, that Problem P with b as the right-hand side vector is consistent. Furthermore, Corollary 4 implies that it has an optimal solution and, by Theorem 1, Problem D with b as a vector of the objective function has an optimal solution.

On the other hand, $y + \lambda y^* \in U$ whenever $y \in U$ and $\lambda \geqslant 0$. Moreover, by (4.7),

$$\lim_{\lambda \to +\infty} \sum_{i=1}^{m} (y_i + \lambda y_i^*) b_i = +\infty$$

which means that the objective function of Problem D is unbounded.

In both cases, $I_n^0 = \emptyset$ and $I_n^0 \neq \emptyset$, we have obtained contradictions. Hence the proof is completed.

Remark 6. The hypothesis of Theorem 3 can be proved also in the case $U = \emptyset$.

COROLLARY 6. *For a non-empty convex polytope U to be unbounded it is necessary and sufficient that I_n^0 be non-empty.*

Proof. By Lemma 4, $(y_0^k, \ldots, y_m^k)$ is a non-zero vector whenever $k \in I_n$. Since, by the proof of Proposition 5, $y_0^k = 0$ for $k \in I_n^0$, therefore $k \in I_n^0$ implies that $y^k = (y_1^k, \ldots, y_m^k) \neq 0$. Thus, if $I_n^0 \neq \emptyset$, we have

$$\{y + \lambda y^k \mid y \in U, \lambda \geqslant 0\} \subset U \quad \text{and} \quad \lim_{\lambda \to +\infty} \|y + \lambda y^k\| = +\infty$$

which means that U is unbounded.

Assume $I_n^0 = \emptyset$. A convex cone U^0 is equal to the convex hull of the union of its edges (see [1], p. 60). If U^0 had at least one edge, say $D_{y^*}^+$, then, by Theorem 3, I_n^0 would not be empty. Hence $I_n^0 = \emptyset$ implies that $U^0 = \{0\}$. This (see [10], Theorem B.7) implies that the convex polytope U is equal to the convex hull of a finite number of points. Hence U is bounded which completes the proof.

To summarize, we denote by Y^+ and Y^0 finite sets of $(y_1^k, \ldots, y_m^k)$ from the expressions

$$e_k^n = \sum_{i=0}^{m+n} y_i^k d_i$$

corresponding to $\{e_k^n \mid k \in I_n^+\}$ and $\{e_k^n \mid k \in I_n^0\}$, respectively, and consider the following three cases:

Case 1. $I_n^+ = \emptyset$. By Corollary 5, the convex polytope U is empty.

Case 2. $I_n^+ \neq \emptyset$ and $I_n^0 = \emptyset$. By Corollary 5, the convex polytope U is non-empty and, by Corollary 6, it is bounded. In other words, U is a convex polyhedron. By Theorem 2, a set of extreme points of U is a subset of Y^+.

Case 3. $I_n^+ \neq \emptyset$ and $I_n^0 \neq \emptyset$. By Corollaries 5 and 6, the convex polytope U is non-empty and unbounded. By Theorem 2, a set of extreme points of U is a subset of Y^+. By Theorem 3, a set of edges of U^0 is a subset of $\{D_y^+ \mid y \in Y^0\}$.

5. Numerical examples. We precede numerical examples by the following remark:

Remark 7. Let (E, Y) be a matrix obtained from a matrix (D, I) by the use of elementary transformations. Let

$$(e_k, y^k) = (e_{k1}, \ldots, e_{kN}, y_1^k, \ldots, y_M^k), \quad \text{where } k = 1, \ldots, K,$$

denote the k-th row of the matrix (E, Y) and let

$$(d_i, \delta_i) = (d_{i1}, \ldots, d_{iN}, \delta_{i1}, \ldots, \delta_{iM}), \quad \text{where } i = 1, \ldots, M,$$

denote the i-th row of the matrix (D, I). Then

$$e_k = \sum_{i=1}^{M} y_i^k d_i$$

since, clearly,

$$y^k = \sum_{i=1}^{M} y_i^k \delta_i.$$

Example 1. Consider the following problem:

PROBLEM D. Maximize $b_1y_1+b_2y_2$ subject to

$$2y_1+y_2 \leqslant 1, \quad y_1+2y_2 \leqslant 1, \quad y_1 \geqslant 0, \quad y_2 \geqslant 0,$$

where b_1 and b_2 are unknown constants.

The dual problem to D is of the following form:

PROBLEM P. Minimize x_1+x_2 subject to

$$2x_1+x_2 \geqslant b_1, \quad x_1+2x_2 \geqslant b_2, \quad x_1 \geqslant 0, \quad x_2 \geqslant 0.$$

This problem is equivalent to the following

PROBLEM P_0. Minimize x_3 subject to

$$-x_1-x_1+x_3 \geqslant 0, \quad 2x_1+x_2 \geqslant b_1,$$

$$x_1+2x_2 \geqslant b_2, \quad x_1 \geqslant 0, \quad x_2 \geqslant 0.$$

We transform Problem P_0 into the equivalent Problem P_2' by elementary transformations of the following matrix:

$$\begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first three columns of this matrix form the left-hand side matrix of the constraints of Problem P_0. The right-hand side vector is omitted. Notice that the above-given matrix has the form of the matrix (D, I) from Remark 7.

Adding the first row to the second row divided by 2, to the third row, to the fourth row and leaving the fifth row without any change, we get

$$\begin{pmatrix} 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix corresponds to Problem P_1.

Multiplying the first row by 2 and adding it and the third row to the second and the fourth rows, we obtain the following matrix corresponding to Problem P_2:

$$\begin{pmatrix} 0 & 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Finally, dividing the successive rows of this matrix by 3, 2, 2 and 1, respectively, we get the matrix corresponding to P_2':

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Observe that I_2^+ contains 4 elements and I_2^0 is empty. Hence, the convex polytope U, defined by the system of constraints of Problem D, is a non-empty convex polyhedron. The set of extreme points of U is a subset of $\{(\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, 0), (0, \frac{1}{2}), (0, 0)\}$. It can be easily shown, however, that each of the four above-mentioned points is an extreme point of U.

For instance, the second row of the above-given matrix reads, by Remark 7,

$$(0, 0) = 1(-1, -1) + \tfrac{1}{2}(2, 1) + 0(1, 2) + 0(1, 0) + \tfrac{1}{2}(0, 1)$$

or, equivalently,

$$\tfrac{1}{2}(2, 1) + 0(1, 2) + 0(1, 0) + \tfrac{1}{2}(0, 1) = (1, 1).$$

This means that $(\frac{1}{2}, 0) \in U$. Since $(2, 1)$ and $(0, 1)$ are linearly independent vectors, $(\frac{1}{2}, 0)$ is an extreme point of U.

Example 2. We consider the following problem:

PROBLEM D. Maximize $b_1 y_1 + b_2 y_2$ subject to

$$y_1 - y_2 \geqslant 1, \quad y_1 - y_2 \geqslant 2, \quad y_1 \geqslant 0, \quad y_2 \geqslant 0.$$

One can easily state that the matrix of the corresponding Problem P_0 is of the form

$$\begin{pmatrix} -1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transformed matrix corresponding to Problem P_2' is of the form

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Observe that both I_2^+ and I_2^0 contain 2 elements. Hence, the convex polytope U, defined by the system of constraints of Problem D, is non-empty and unbounded. The set of extreme points of U is a subset of $\{(1, 0), (0, 0)\}$ (it can be easily shown that both points are extreme).

By Theorem 3, the convex cone U^0 associated with U has at most two edges. It can be shown, however, that both $y^1 = (1, 1)$ and $y^2 = (0, 1)$ are extreme rays. Thus, $D^+_{y^1}$ and $D^+_{y^2}$ are the only edges of U^0.

Example 3. Now we consider the following problem:

PROBLEM D. Maximize $b_1y_1 + b_2y_2$ subject to

$$0y_1 + y_2 \leqslant -1, \quad y_1 \geqslant 0, \quad y_2 \geqslant 0.$$

We transform the following matrix corresponding to Problems P_0 and P_0':

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since I_0^- defined by (2.4) is empty, the matrix corresponding to Problem P_1 (and P_1') reduces to

$$(0 \quad 0 \quad 1 \quad 0 \quad 0).$$

Observe that I_1^+ is empty which, by Corollary 6, means that the convex polytope U, defined by the system of constraints of Problem D, is empty. The set I_1^0 contains one element. The corresponding $(y_1, y_2) = (1, 0)$ generates the only edge of the convex cone U^0 defined by $0y_1 + y_2 \leqslant 0$, $y_1 \geqslant 0$ and $y_2 \geqslant 0$.

6. Some properties of the polyhedral convex cone. The described method can be used for linear programming problems with a parametric objective function. The way of using the method in such problems does not have to be explained. However, since the method produces all extreme points and all edges, it can be handled only in "moderately large" problems. On the other hand, it is evident that a problem with no assumptions concerning the vector of the objective function cannot be solved without producing all extreme points and all edges. Clearly, if some extreme point, say y^*, is not produced, then, by Lemma 3, for some vector b of the objective function, the unique optimal solution y^* cannot be found. Similarly, if some edge, say $D^+_{y^*}$, is not produced, then (see the proof of Theorem 3) we can find, for some b, the optimal solution to the problem, despite the fact that the objective function is unbounded.

It should be pointed out, in general, that the method produces also elements of the convex polytope which are not its extreme points and elements of the corresponding convex cone which are not its extreme rays. A very simple procedure which excludes all "redundant" points is given at the end of this section.

Now we give some properties of the convex cone

$$(6.1)\quad C = \left\{(y_0, y_1, \ldots, y_M) \;\middle|\; \sum_{i=0}^{M} y_i h_i = 0,\ y_i \geqslant 0 \text{ for } i = 0, \ldots, M\right\},$$

where $h_0, \ldots, h_M$ are n-dimensional vectors.

Write $\operatorname{sgn} y = (\operatorname{sgn} y_0, \ldots, \operatorname{sgn} y_M)$.

PROPOSITION 8. *If y and y' are non-collinear, non-zero vectors belonging to C such that*

$$(6.2)\qquad \operatorname{sgn} y - \operatorname{sgn} y' \geqslant 0,$$

then y is not an extreme ray of C.

Proof. Since $y \neq 0$, at least one of its elements is positive. We can assume, without loss of generality, that

$$(6.3)\qquad y_i > 0, \quad i = 0, \ldots, p,$$

$$(6.4)\qquad y_i = 0, \quad i = p+1, \ldots, M.$$

By (6.2) and (6.4), we have

$$(6.5)\qquad y_1' = 0, \quad i = p+1, \ldots, M.$$

Since y' is a non-zero vector, it follows from (6.5) that at least one of $y_0', \ldots, y_p'$ is positive. Moreover, $y' \in C$ implies $y_i' \geqslant 0$, $i = 0, \ldots, p$. Hence, there exists a positive λ such that

$$(6.6)\qquad y_i - \lambda y_i' \geqslant 0, \quad i = 0, \ldots, M,$$

and at least one of $y_0 - \lambda y_0', \ldots, y_p - \lambda y_p'$ is equal zero. Let

$$(6.7)\qquad y_0 - \lambda y_0' = 0.$$

Obviously,

$$(6.8)\qquad y = \tfrac{1}{2}(y + \lambda y') + \tfrac{1}{2}(y - \lambda y').$$

It follows from $y \in C$, $y' \in C$, $\lambda > 0$ and (6.6) that $y + \lambda y' \in C$ and $y - \lambda y' \in C$. We show that $y + \lambda y'$ and $y - \lambda y'$ are non-collinear.

Consider then

$$(6.9)\qquad a_1(y + \lambda y') + a_2(y - \lambda y') = 0.$$

Since $y_0 > 0$, $y_0' \geqslant 0$ and $\lambda > 0$, therefore $y_0 + \lambda y_0' > 0$. This, (6.7) and (6.9) imply $a_1 = 0$. Furthermore, since y and y' are non-collinear, $y - \lambda y' \neq 0$. This and $a_2(y - \lambda y') = 0$ imply $a_2 = 0$. Thus (6.9) implies $a_1 = a_2 = 0$. Hence $y + \lambda y'$ and $y - \lambda y'$ are non-collinear. Then y which can be represented, by (6.8), as a positive linear combination of two non-collinear elements of C is not an extreme ray of C (see Proposition 7).

COROLLARY 7. *If y and y' are non-collinear, non-zero vectors belonging to C such that* $\operatorname{sgn} y - \operatorname{sgn} y' = 0$, *then neither y nor y' is an extreme ray of C.*

Proof. The hypothesis follows from Proposition 8, since $\operatorname{sgn} y - \operatorname{sgn} y' \geqslant 0$ and $\operatorname{sgn} y' - \operatorname{sgn} y \geqslant 0$.

COROLLARY 8. *If y and y' are non-zero vectors belonging to C such that* $\operatorname{sgn} y - \operatorname{sgn} y'$ *is a non-negative, non-zero vector, then y is not an extreme ray of C.*

Proof. The assumption concerning $\operatorname{sgn} y - \operatorname{sgn} y'$ yields the existence of p and q such that $y_p > 0$, $y'_p > 0$, $y_q > 0$ and $y'_q = 0$. Thus y and y' are non-collinear. Proposition 8 completes the proof.

COROLLARY 9. *Any $y \in C$ with at least $n+2$ of its positive elements is not an extreme ray of C.*

Proof. Suppose $y_i > 0$, $i = 0, \ldots, p$, where $p > n$. Then

$$\sum_{i=1}^{p} y_i h_i = -y_0 h_0, \quad y_i > 0,\ i = 1, \ldots, p. \tag{6.10}$$

Since $h_0, \ldots, h_{n+1}$ are n-dimensional vectors and $p > n$, it follows that $y_1, \ldots, y_p$ is a non-basic solution of (6.10). Hence (see [10], p. 376) there exists a basic non-negative solution $y'_1, \ldots, y'_p$, where at least $p-n$ of $y'_1, \ldots, y'_p$, say $y'_{n+1}, \ldots, y_p$, are equal to 0. Obviously, $(y_0, y'_1, \ldots, y'_m, 0, \ldots, 0) \in C$. Since $y_0 > 0$, $y_p > 0$ and $y'_p = 0$, we can apply Corollary 8 (taking $(y_0, y'_1, \ldots, y'_n, 0, \ldots, 0)$ as y') to state that y is not an extreme ray of C.

Assume now that C defined by (6.1) is generated by $\{y^k \mid k \in K\}$, i.e.

$$C = \Big\{ y \mid y = \sum_{k \in K} \lambda_k y^k,\ \lambda_k \geqslant 0 \text{ for } k \in K \Big\}. \tag{6.11}$$

PROPOSITION 9. *If y^r is a non-zero vector belonging to $\{y^k \mid k \in K\}$ such that, for any $k \in K - \{r\}$,* $\operatorname{sgn} y^r - \operatorname{sgn} y^k$ *is not a non-negative vector, then y^r is an extreme ray of C.*

Proof. By Proposition 7, it is enough to show that the conditions

$$y^r = a_1 y' + a_2 y'', \quad y' \in C,\ y'' \in C,\ a_1 > 0,\ a_2 > 0,$$

imply that y' and y'' are collinear.

It follows from $y' \in C$ and $y'' \in C$ that

$$y' = \sum_{k \in K} \lambda'_k y^k, \quad y'' = \sum_{k \in K} \lambda''_k y^k,$$

where

$$\lambda'_k \geqslant 0, \quad \lambda''_k \geqslant 0 \quad \text{for } k \in K. \tag{6.12}$$

Hence

$$y^r = \sum_{k \in K} \lambda_k y^k, \tag{6.13}$$

where

$$\lambda_k = a_1 \lambda_k' + a_2 \lambda_k'' \geqslant 0. \tag{6.14}$$

Since $y^k \in C$ for $k \in K$, from (6.1) and (6.11) we have

$$y^k \geqslant 0 \quad \text{for } k \in K. \tag{6.15}$$

Let k be any element of $K - \{r\}$. It follows from the assumption concerning $\operatorname{sgn} y^r - \operatorname{sgn} y^k$ and (6.15) that there exists an i_k such that $y^r_{i_k} = 0$ and $y^k_{i_k} > 0$. This, (6.13), (6.14) and (6.15) imply $\lambda_k = 0$ for $k \in K - \{r\}$. Then, by $a_1 > 0$, $a_2 > 0$, (6.12) and (6.14), we get $\lambda_k' = \lambda_k'' = 0$ for $k \in K - \{r\}$ which means that $y' = \lambda_r' y^r$ and $y'' = \lambda_r'' y_r$. Hence y' and y'' are collinear. This completes the proof.

COROLLARY 10. *Assume*

$$C = \Big\{y \,\Big|\, \sum_{i=0}^{M} y_i h_i = 0,\ y_i \geqslant 0 \ for\ i = 0, 1, \ldots, M\Big\}$$
$$= \Big\{y \mid y = \sum_{k \in K} \lambda_k y^k,\ \lambda_k \geqslant 0 \ for\ k \in K\Big\},$$

where any two vectors from the set $\{y^k \mid k \in K\}$ are non-collinear. Then, for $y^r \in \{y^k \mid k \in K\}$ to be an extreme ray of C, it is necessary and sufficient that, for any $k \in K - \{r\}$, $\operatorname{sgn} y^r - \operatorname{sgn} y^k$ *be not a non-negative vector.*

Proof. This corollary is an immediate consequence of Propositions 8 and 9.

Remark 8. It has been assumed in Corollary 10 that C is a convex cone in R^{M+1}. One can easily observe that the same proof can be conducted under the assumption that C is a convex cone in a generalized finite-sequence space as defined by Charnes, Cooper and Kortanek [3].

We illustrate the above-given properties by the following example:

Example 4. Let $M = 4$, $n = 2$ and consider the convex cone C of form (6.1) which is generated by the following vectors:

$$y^1 = (1, 1, 0, 0, 0), \quad y^2 = (1, 0, 1, 0, 0), \quad y^3 = (3, 1, 2, 0, 0),$$
$$y^4 = (0, 2, 6, 4, 0), \quad y^5 = (0, 3, 9, 6, 0), \quad y^6 = (0, 1, 0, 2, 3),$$
$$y^7 = (2, 4, 6, 4, 0), \quad y^8 = (1, 2, 7, 4, 0), \quad y^9 = (0, 3, 6, 4, 3).$$

The vectors y^7, y^8 and y^9 can be excluded by Corollary 9 (y^7 and y^8 can be excluded also by Corollary 7). The vector y^3 can be excluded by Corollary 8, since $\operatorname{sgn} y^3 - \operatorname{sgn} y^1$ is a non-negative, non-zero vector. Since

$\operatorname{sgn} y^4 - \operatorname{sgn} y^5 = 0$, and y^4 and y^5 are collinear, only one of them, say y^5, can be excluded. None of the remaining vectors, i.e. y^1, y^2 and y^6, can be excluded. All these vectors are extreme rays of C by Proposition 9.

Assume now that $\{y^k \mid k = 1, \ldots, p\}$ generates the convex cone C defined by (6.1). The procedure of selecting the smallest subset L of $\{1, \ldots, p\}$ such that $\{y^k \mid k \in L\}$ generates C can be conducted in the following p steps:

Step r $(r = 1, \ldots, p)$. Given $K_{r-1} \subset \{1, \ldots, p\}$ such that $\{y^k \mid k \in K_{r-1}\}$, where $y^k \neq 0$ for $k \in K_{r-1}$, generates C. If there exists an $s \in K_{r-1} - \{r\}$ such that

$$\operatorname{sgn} y^r - \operatorname{sgn} y^s \geqslant 0, \tag{6.16}$$

take $K_r = K_{r-1} - \{r\}$. Otherwise, take $K_r = K_{r-1}$.

In step 1 we take $K_0 = \{1, \ldots, p\}$. One can take as K_0 any subset extracted from $\{1, \ldots, p\}$ by the use of Corollary 9.

Obviously, if (6.16) holds, then either y^r is collinear with y^s or y^r is not an extreme ray (see Proposition 8). In both cases y^r can be represented as a non-negative linear combination of vectors $\{y^k \mid k \in K_{r-1} - \{r\}\}$. Hence y^r can be excluded since cones C and C' generated by $\{y^k \mid k \in K_{r-1}\}$ and $\{y^k \mid k \in K_{r-1} - \{r\}\}$, respectively, are equal.

In the other case, Proposition 9 implies that y^r is an extreme ray of the convex cone generated by $\{y^k \mid k \in K_{r-1}\}$ and it is non-collinear with any y^s, $s \in K_{r-1} - \{r\}$. Hence $y^r \notin C'$ and, since $y^r \in C$, $C \neq C'$. Therefore, y^r cannot be excluded. Clearly, $L = K_p$ is the required set.

7. Extreme points and extreme rays as extreme rays of some polyhedral convex cone. Take in (6.1) $M = m+n$ and

$$h_i = \begin{cases} -c = (-c_1, \ldots, -c_n), & i = 0, \\ a_i = (a_{i1}, \ldots, a_{in}), & i = 1, \ldots, m, \\ \delta_{i-m}, & \end{cases} \tag{7.1}$$

where δ_k denotes the k-th n-dimensional unit vector.

In this section it will be understood that C is a convex cone defined by (6.1) and (7.1). We establish some connexions between C, U defined by (4.1), U^0 defined by (4.2) and $\tilde{U}$ defined by

$$\tilde{U} = \Big\{(y_0, \ldots, y_m) \;\Big|\; -y_0 c + \sum_{i=1}^{m} y_i a_i \leqslant 0,\ y_i \geqslant 0 \text{ for } i = 0, \ldots, m\Big\}.$$

The following propositions are obvious:

Proposition 10. *$(y_0, \ldots, y_m) \in \tilde{U}$ if and only if $(y_0, \ldots, y_{m+n}) \in C$, where*

$$y_{m+j} = y_0 c_j - \sum_{i=1}^{m} y_i a_{ij}, \quad i = 1, \ldots, n. \tag{7.2}$$

PROPOSITION 11. $(0, y_1, \ldots, y_m) \in \tilde{U}$ *if and only if* $(y_1, \ldots, y_m) \in U^0$.

PROPOSITION 12. $(1, y_1, \ldots, y_m) \in \tilde{U}$ *if and only if* $(y_1, \ldots, y_m) \in U$.

The following proposition follows from Propositions 10 and 11:

PROPOSITION 13. $(y_1, \ldots, y_m)$ *is an extreme ray of* U^0 *if and only if* $(0, y_1, \ldots, y_{m+n})$ *is an extreme ray of* C *where* $y_{m+1}, \ldots, y_{m+n}$ *are defined by* (7.2) *and* $y_0 = 0$.

PROPOSITION 14. $(y_1, \ldots, y_m)$ *is an extreme point of* U *if and only if* $(1, y_1, \ldots, y_{m+n})$ *is an extreme ray of* C *where* $y_{m+1}, \ldots, y_{m+n}$ *are defined by* (7.2) *and* $y_0 = 1$.

Proof. If $(y_1, \ldots, y_m)$ is not an extreme point of U, then it can be represented as

$$(y_1, \ldots, y_m) = \lambda(y_1', \ldots, y_m') + (1-\lambda)(y_1'', \ldots, y_m''), \tag{7.3}$$

where $0 < \lambda < 1$, and $(y_1', \ldots, y_m')$ and $(y_1'', \ldots, y_m'')$ are two distinct points of U. We can assume, without loss of generality, that $y_1' \neq y_1''$.

Take $y_0 = y_0' = y_0'' = 1$ and define, by (7.2), y_{m+j}, y_{m+j}' and y_{m+j}'' for $j = 1, \ldots, n$. By Propositions 12 and 10, $(1, y_1, \ldots, y_{m+n})$, $(1, y_1', \ldots, y_{m+n}')$ and $(1, y_1'', \ldots, y_{m+n}'')$ belong to C. It follows from (7.3) that

$$(1, y_1, \ldots, y_{m+n}) = \lambda(1, y_1', \ldots, y_{m+n}') + (1-\lambda)(1, y_1'', \ldots, y_{m+n}''). \tag{7.4}$$

Since $y_0' = y_0'' = 1$ and $y_1' \neq y_1''$, it follows from (7.4) that $(1, y_1, \ldots, y_m)$ can be represented as a positive linear combination of two non-collinear vectors of C. Hence, by Proposition 7, $(1, y_1, \ldots, y_{m+n})$ is not an extreme ray of C.

If $(1, y_1, \ldots, y_{m+n})$ is not an extreme ray of C, then

$$(1, y_1, \ldots, y_{m+n}) = \lambda_1(z_0', z_1', \ldots, z_{m+n}') + \lambda_2(z_0'', z_1'', \ldots, z_{m+n}''), \tag{7.5}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $(z_0', \ldots, z_{m+n}')$ and $(z_0'', \ldots, z_{m+n}'')$ are non-collinear elements of C. Clearly, $z_0' \geqslant 0$ and $z_0'' \geqslant 0$. Since $1 = \lambda_1 z_0' + \lambda_2 z_0''$, at least one of z_0' and z_0'' is positive.

If both z_0' and z_0'' are positive, then introducing

$$\begin{gathered} y_i' = \frac{1}{z_0'}, \quad y_i'' = \frac{1}{z_0''} \quad \text{for } i = 0, \ldots, m+n, \\ \lambda = \lambda_1 z_0', \end{gathered} \tag{7.6}$$

we get (7.4), where $0 < \lambda < 1$, and $(1, y_1', \ldots, y_{m+n}')$ and $(1, y_1'', \ldots, y_{m+n}'')$ are two elements of C. This implies (7.3), where, by Proposition 10 and 12, $(y_1', \ldots, y_{m+n}')$ and $(y_1'', \ldots, y_{m+n}'')$ are two elements of U. If those were $(y_1', \ldots, y_m') = (y_1'', \ldots, y_m'')$, then, by (7.2) and (7.6), $(z_0', \ldots, z_{m+n}')$ and $(z_0'', \ldots, z_{m+n}'')$ would be collinear, a contradiction to the assumption. Hence $(y_1', \ldots, y_m') \neq (y_1'', \ldots, y_m'')$. This and (7.3) imply that $(y_1, \ldots, y_m)$ is not an extreme point.

Assume $z'_0 = 0$. It follows from $(0, z'_1, \ldots, z'_{m+n}) \in C$ that

$$\sum_{i=1}^{m} z'_1 a_i + \sum_{i=m+1}^{m+n} z'_i \delta_{i-m} = 0.$$

Let $I' = \{i \mid z'_i > 0\}$ and let $\{h_i \mid i \in I'\}$ be the corresponding subset of the set

$$\{h_1, \ldots, h_{m+n}\} = \{a_1, \ldots, a_m, \delta_1, \ldots, \delta_n\}.$$

Since $(0, z'_1, \ldots, z'_{m+n})$ and $\{z''_0, z''_1, \ldots, z''_{m+n}\}$ are non-collinear, $(z'_1, \ldots, z'_{m+n}) \neq 0$ and $I' \neq \emptyset$. Then we can write

$$\sum_{i \in I'} z'_i h_i = 0,$$

where

(7.7) $$z'_i > 0 \quad \text{for } i \in I'.$$

Since $\lambda_1 > 0$, $\lambda_2 > 0$ and vectors appearing in (7.5) are non-negative, $z'_i > 0$ implies $y_i > 0$. In other words, $I' \subset I = \{i \mid y_i > 0, 1 \leqslant i \leqslant m+n\}$, Taking

(7.8) $$a_i = \begin{cases} z'_i & \text{for } i \in I', \\ 0 & \text{for } i \in I - I', \end{cases}$$

we have then

$$\sum_{i \in I} a_i h_i = 0,$$

where, by (7.8), (7.7) and $I' \neq \emptyset$, not all a being equal zero. This means that vectors h_i corresponding to positive y_i are linearly dependent. Hence $(y_1, \ldots, y_{m+n})$ is not an extreme point of the set

$$\left\{(y_1, \ldots, y_{m+n}) \,\middle|\, \sum_{i=1}^{m+n} y_i h_i = c,\ y_i \geqslant 0 \text{ for } i = 1, \ldots, m+n\right\}$$

and, consequently, $(y_1, \ldots, y_m)$ is not an extreme point of U.

The proof is completed.

By the procedure transforming Problem P_0 into Problem P'_n conducted in the way shown in Section 5, we obtain the finite set of vectors generating the convex cone C defined by (6.1) and (7.1). By the procedure given at the end of Section 6, we can extract the smallest subset $\{y^k \mid k \in L\}$ of that set which also generates C. For any element $y^k = (y_0^k, y_1^k, \ldots, y_{m+n}^k)$, there are only two possible cases: either $y_0^k = 1$ or $y_0^k = 0$. Let

$$L_1 = \{k \mid k \in L, y_0^k = 1\} \quad \text{and} \quad L_0 = \{k \mid k \in L, y_0^k = 0\}.$$

Then, by Proposition 14, $\{(y_1^k, \ldots, y_m^k) \mid k \in L_1\}$ is the set of all extreme points of U and, by Proposition 13, $\{(y_1^k, \ldots, y_m^k) \mid k \in L_0\}$ is the smallest set of vectors generating U^0.

The following remarks might be useful in conducting the procedure transforming P_0 into P'_n:

Remark 9. Given the matrix (E^q, Y^q) corresponding to Problem P'_q (obtained in the way shown in Section 5). The l-th row (e_l^q, y_l^q) can be excluded whenever y_l^q has at least $n+2$ positive elements.

To justify this remark consider any row, say (e_k^n, y^k), of the matrix (E^n, Y^n) corresponding to Problem P'_n. By analogy to Remark 5, e_k^n is a non-negative linear combination of $\{e_i^q \mid i \in I_q\}$ which can be written as

$$e_k^n = \sum_{i \in I_q} z_i e_i^q,$$

where z's are non-negative. Consequently,

$$y_k = \sum_{i \in I_q} z_i y_i^q. \tag{7.9}$$

Suppose that y_l^q has at least $n+2$ positive elements. Since numbers z are non-negative and y_i^q are non-negative vectors (by analogy to Remark 5), by (7.9), obviously, any vector y^k such that $z_l > 0$ has at least $n+2$ positive elements. Hence the row (e_k^n, y^k) can be excluded by the use of Corollary 9.

Remark 10. Let n_0, n_1, n_{-1} denote the numbers of elements of the sets I_0^0, I_0^+, I_0^- defined by (2.4). Since the variables $x_1, \ldots, x_n$ can be reordered during transforming P_0 into P'_n, it can be recommended that such a variable x_t $(t \leqslant n)$ which has the smallest $n_0 + n_1 n_{-1}$ be considered as x_1.

This remark is justified by the fact that Problem P_1 has $n_0 + n_1 n_{-1}$ constraints. Clearly, Remark 10 can be applied at any step, i.e. during the transformation of any Problem P_j into Problem P'_j. However, it cannot be proved that the use of Remark 10 leads to the smallest set of constraints in Problem P'_n.

8. Some other forms of Problem D. In the above-given reasoning we have assumed that all constraints of Problem D are inequalities and that all y's are restricted to non-negative values. Obviously, any linear programming problem can easily be rewritten as a problem of that form. However, a more reasonable procedure of treating the problem can be recommended. The idea of that procedure will be clear from the following two cases:

Case 1. Suppose that the n-th constraint of Problem D has the form of an equation with $a_{mn} \neq 0$. In this case y_m can be obtained from the n-th constraint as a linear affine function of $y_1, \ldots, y_{m-1}$. By substituting that expression into the other constraints, we obtain an equivalent problem.

Case 2. Suppose that y_m is not restricted to non-negative values and that $a_{m1} \neq 0$. Then, in Problem P (which is dual to D), the m-th constraint has the form of an equation in which x_1 appears. In this case Problem P_0 can be transformed into Problem P_1 by obtaining x_1 from the m-th constraint as a linear affine function of $x_2, \ldots, x_n$ and substituting that expression into the other constraints.

The procedure of selecting the set of all extreme points of U and the smallest set of extreme rays of U^0 can be conducted in the following way.

In Case 1, two above-mentioned sets corresponding to the reduced problem can be found by using the procedure given in Section 7. Then, for any $(y_1, \ldots, y_{m-1})$ belonging to one of those sets, y_m can be found from the expression representing y_m as a linear affine function of $y_1, \ldots, y_{m-1}$. This gives the required sets corresponding to the initial problem.

In Case 2, it can happen that $y_m^r < 0$ for some r. However, y_m could be written as $y_m = y_m' - y_m''$, where y_m' and y_m'' were both non-negative and at least one of them was zero. Hence we can substitute y^k by $y_m'^k$ and $y_m''^k$, where

(i) $y_m^k = y_m'^k - y_m''^k$,

(ii) $y_m'^k \geqslant 0$ and $y_m''^k \geqslant 0$,

(iii) at least one of $y_m'^k$ and $y_m''^k$ is zero.

Then the procedure of Section 7 can be applied by considering $(y_1^k, \ldots, y_{m-1}^k, y_m'^k, y_m''^k)$ instead of $(y_1^k, \ldots, y_{m-1}^k, y_m^k)$.

Remark 11. In Problem P_0, the constraint corresponding to $i = 0$ can, obviously, be replaced by the equation

$$-\sum_{j=1}^{n} c_j x_j + x_{n+1} = 0.$$

References

[1] C. Berge and A. Ghouila-Houri, *Programming, games and transportation networks*, Wiley, New York 1965.

[2] A. Charnes and W. W. Cooper, *Management models and industrial applications of linear programming*, Wiley, New York 1961.

[3] — and K. Kortanek, *Duality in semiinfinite programs and some works of Haar and Caratheodory*, Manag. Sci. 9 (1963), p. 209-228.

[4] J. B. J. Fourier, *Solution d'une question particulière du calcul des inégalités.* 1826, and extracts from *Histoire de l'Académie — 1823, 1824, Oeuvres II*, p. 317-328.

[5] M. Frank and P. Wolfe, *An algorithm for quadratic programming*, Naval Res. Logist. Quart. 3 (1956), p. 95-110.
[6] S. I. Gass and T. L. Saaty, *Parametric objective function* (Part 2) — *Generalization*, Opns. Res. 3 (1955), p. 395-401.
[7] J. E. Kelley, Jr., *Parametric programming and the primal-dual algorithm*, ibidem 7 (1959), p. 327-334,
[8] T. S. Motzkin, *Beiträge zur Theorie der linearen Ungleichungen*, Jerusalem 1936 (Doctoral Thesis, University of Zurich).
[9] T. L. Saaty and S. I. Gass, *Parametric objective function* (Part 1), Opns. Res. 2 (1954), p. 316-319.
[10] M. Simmonard, *Linear programming*, Prentice-Hall, Englewood Cliffs 1966.
[11] E. Simons, *A note on parametric linear programming*, Manag. Sci. 8 (1963), p. 355-358.
[12] H. M. Wagner, *Parametric programming and the primal-dual algorithm*, Opns. Res. 7 (1959), p. 327-334.

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PUNKTY EKSTREMALNE WIELOŚCIENNEGO ZBIORU WYPUKŁEGO I PROMIENIE EKSTREMALNE ODPOWIEDNIEGO STOŻKA WYPUKŁEGO

STRESZCZENIE

W pracy podano metodę znalezienia wszystkich punktów ekstremalnych wielościennego zbioru wypukłego i promieni ekstremalnych, generujących wszystkie krawędzie stożka wypukłego odpowiadającego temu zbiorowi.

Metoda jest oparta na przekształceniach problemów programowania liniowego, opisanych w rozdziale 2. Za pomocą tych przekształceń uzyskuje się inny dowód twierdzenia o dualności. Dowód tego twierdzenia uzasadnia metodę i jest podany w rozdziale 3.

Metoda pozwala na rozwiązanie problemu programowania liniowego z parametryczną funkcją celu i może być zastosowana praktycznie do rozwiązywania „niewielkich" problemów programowania wypukłego z liniowymi warunkami ograniczającymi.