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ON CERTAIN IMPROPER INTEGRALS

0. Summary. The functions F and G determined by integrals (1) and (2) are considered in the paper. Series expansions as well as asymptotic formulae for these functions are derived.

Integrals (14) and (15) are approximately calculated and expressed in terms of the functions F and G. The error caused by the approximation is evaluated. The application of the integrals considered is illustrated by examples.

1. Fundamental equations. In the sequel, the integrals

$$F(x, y) = \int_0^\infty \frac{e^{-ux}\cos uy}{u+a}\, du, \tag{1}$$

$$G(x, y) = \int_0^\infty \frac{e^{-ux}\sin uy}{u+a}\, du \tag{2}$$

will be considered with $a = \exp(i\pi/4)$, real y and $x > 0$. Integrals (1) and (2) represent functions F and G of real variables x and y.

The function F is even, whereas the function G is odd with respect to y, i. e. $F(x, -y) = F(x, y)$ and $G(x, -y) = -G(x, y)$.

It is easy to show that the functions F and G satisfy the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

2. Series expansions. Substitution of

$$\cos uy = \frac{1}{2}(e^{iuy} + e^{-iuy})$$

into (1) gives

$$F(x, y) = \frac{1}{2}\left[\int_0^\infty \frac{e^{-uz}\, du}{u+a} + \int_0^\infty \frac{e^{-uz^*}\, du}{u+a}\right], \tag{3}$$

where

(4) $$z = x + iy = re^{i\theta}, \quad z^* = x - iy = re^{i\theta}$$

and

(5) $$r = \sqrt{x^2 + y^2}, \quad \theta = \operatorname{arctg} \frac{y}{x}.$$

The integrals which appear in (3) can be expressed in terms of the exponential integral function, giving

(6) $$F(x, y) = \frac{1}{2}[-e^{az} Ei(-az) - e^{az^*} Ei(-az^*)],$$

according to the formula No. 3.352.4 in [1].

The exponential function as well as the exponential integral function which appear in (6) are complex. These functions can be represented in the form

$$e^{az} = A_1 + iA_2, \quad -Ei(-az) = B_1 + iB_2$$

with A_1, A_2, B_1, B_2 determined later. Hence,

(7) $$-e^{az} Ei(-az) = (A_1 + iA_2)(B_1 + iB_2).$$

Inserting $z = re^{i\theta}$ into the expression

$$e^{az} = \sum_{k=0}^{\infty} \frac{a^k z^k}{k!},$$

we obtain

(8) $$A_1 = \sum_{k=0}^{\infty} \frac{a^k r^k \cos k\theta}{k!}, \quad A_2 = \sum_{k=1}^{\infty} \frac{a^k r^k \sin k\theta}{k!}$$

if Euler's identity $e^{ik\theta} = \cos k\theta + i \sin k\theta$ is applied. Using $a^{k+4} = -a^k$, equations (8) become

(9) $$\begin{aligned} A_1 &= 1 + ac_1 + a^2 c_2 + a^3 c_3 + a^4 c_4, \\ A_2 &= as_1 + a^2 s_2 + a^3 s_3 + a^4 s_4, \end{aligned}$$

where

$$c_m = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+4k} \cos(m+4k)\theta}{(m+4k)!},$$

$$s_m = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+4k} \sin(m+4k)\theta}{(m+4k)!}$$

for $m = 1, 2, 3, 4$. Substitution of

$$a = \frac{1}{\sqrt{2}}(1+i), \quad a^2 = i,$$

$$a^3 = \frac{1}{\sqrt{2}}(-1+i), \quad a^4 = -1$$

into (9) gives

$$A_1 = \left[1 - c_4 + \frac{1}{\sqrt{2}}(c_1 - c_3)\right] + i\left[c_2 + \frac{1}{\sqrt{2}}(c_1 + c_3)\right],$$

$$A_2 = \left[-s_4 + \frac{1}{\sqrt{2}}(s_1 - s_3)\right] + i\left[s_2 + \frac{1}{\sqrt{2}}(s_1 + s_3)\right].$$

The functions B_1, B_2 associated with $-Ei(-az)$ are derived in a similar way if the series [5]

$$Ei(z) = C + \ln(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}$$

is used, where $C = 0{,}5772\ldots$ is Euler's constant. We obtain

$$B_1 = \left[-C - \ln r + c_4' + \frac{1}{\sqrt{2}}(c_1' - c_3')\right] + i\left[-\frac{\pi}{4} - c_2' + \frac{1}{\sqrt{2}}(c_1' + c_3')\right],$$

$$B_2 = \left[-\theta + s_4' + \frac{1}{\sqrt{2}}(s_1' - s_3')\right] + i\left[-s_2' + \frac{1}{\sqrt{2}}(s_1' + s_3')\right],$$

where

$$c_m' = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+4k}\cos(m+4k)\theta}{(m+4k)(m+4k)!},$$

$$s_m' = \sum_{k=0}^{\infty} (-1)^k \frac{r^{m+4k}\sin(m+4k)\theta}{(m+4k)(m+4k)!}$$

for $m = 1, 2, 3, 4$.

The relationships for e^{az^*} and $-Ei(-az^*)$ are derived if θ is replaced by $-\theta$, giving

$$-e^{az^*}Ei(-az^*) = (A_1 - iA_2)(B_1 - iB_2), \tag{10}$$

since A_1 and B_1 are even, whereas A_2 and B_2 are odd with respect to θ. Inserting (7) and (10) into (6), we obtain

$$F(x, y) = A_1B_1 - A_2B_2.$$

Now we consider the function G. Substitution of

$$\sin uy = \frac{1}{2i}(e^{iuy} - e^{-iuy})$$

into (2) gives

$$G(x, y) = \frac{1}{2i}[e^{az}Ei(-az) - e^{az^*}Ei(-az^*)]. \tag{11}$$

Inserting (7) and (10) into (11), we have

$$G(x, y) = -(A_1 B_2 + A_2 B_1).$$

For small r we obtain

$$F(x, y) = -0{,}577 + \ln\frac{1}{r} + O(r),$$

$$G(x, y) = \theta + O(r).$$

From the relationships derived the numerical values of the functions F and G can be computed using digital computers. The tables of these functions are given in [2] and [3].

3. Asymptotic formulae. The asymptotic formulae of the functions F and G are obtained from the asymptotic formula for the integral exponential function [5]:

$$Ei(z) \sim \frac{e^z}{z}\sum_{k=0}^{\infty}\frac{k!}{z^k}.$$

We have

$$-e^{az}Ei(-az) \sim \sum_{k=0}^{\infty}(-1)^k\frac{k!}{(az)^{k+1}},$$

whence

$$-e^{az}Ei(-az) \sim \sum_{k=0}^{\infty}(-1)^k\frac{k!\exp[-i(k+1)\theta]}{(ar)^{k+1}} \tag{12}$$

and

$$-e^{az^*}Ei(-az^*) \sim \sum_{k=0}^{\infty}(-1)^k\frac{k!\exp[i(k+1)\theta]}{(ar)^{k+1}}, \tag{13}$$

where r and θ are taken from (5).

Substitution of (12) and (13) into (6) and (11) gives

$$F(x,y) \sim \left[\frac{\sqrt{2}\cos\theta}{2r} - \frac{\sqrt{2}\cos 3\theta}{r^3} + \frac{6\cos 4\theta}{r^4} - \frac{12\sqrt{2}\cos 5\theta}{r^5} + \dots\right] +$$

$$+ i\left[-\frac{\sqrt{2}\cos\theta}{2r} + \frac{\cos 2\theta}{r^2} - \frac{\sqrt{2}\cos 3\theta}{r^3} + \frac{12\sqrt{2}\cos 5\theta}{r^5} + \dots\right],$$

$$G(x,y) \sim \left[\frac{\sqrt{2}\sin\theta}{2r} - \frac{\sqrt{2}\sin 3\theta}{r^3} + \frac{6\sin 4\theta}{r^4} - \frac{12\sqrt{2}\sin 5\theta}{r^5} + \dots\right] +$$

$$+ i\left[-\frac{\sqrt{2}\sin\theta}{2r} + \frac{\sin 2\theta}{r^2} - \frac{\sqrt{2}\sin 3\theta}{r^3} + \frac{12\sqrt{2}\sin 5\theta}{r^5} + \dots\right].$$

4. Approximate evaluation of two integrals. Now we consider the integrals

(14) $$I_1 = \int_0^\infty \frac{e^{-ux}\cos uy}{\mu u + \sqrt{u^2+i}}\, du,$$

(15) $$I_2 = \int_0^\infty \frac{e^{-ux}\sin uy}{\mu u + \sqrt{u^2+i}}\, du,$$

where y is real, $x > 0$ and $\mu > 0$. It is possible to express these integrals in terms of the function F or G if $x \geqslant 20$ and $\mu \geqslant 50$.

In order to evaluate the last integrals, we neglect the term u^2 involved in the root. Then the error resulting becomes

$$\varepsilon = \left|\int_0^\infty \frac{e^{-ux}\cos uy}{\mu u + a}\, du - I_1\right|,$$

where $a = \exp(i\pi/4)$. Hence,

(16) $$\varepsilon \leqslant \int_0^\infty \frac{u^2 e^{-ux}}{\left|(\mu u + a)(\mu u + \sqrt{u^2+i})(\sqrt{u^2+i} + a)\right|}\, du.$$

From

$$|\mu u + a| = \left|\mu u + \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right| = \sqrt{(\mu u)^2 + \sqrt{2}\mu u + 1}$$

we obtain

$$|\mu u + a| \geqslant \sqrt{(\mu u)^2 + 1}$$

if $u \geqslant 0$. Similarly, from

$$\sqrt{u^2+i} = \sqrt{\frac{\sqrt{u^4+1}+u^2}{2}} + i\sqrt{\frac{\sqrt{u^4+1}-u^2}{2}}$$

we obtain

$$\left|\mu u+\sqrt{u^2+i}\,\right| \geqslant \sqrt{(\mu u)^2+1}$$

if $u \geqslant 0$ and $\left|\sqrt{u^2+i}+a\right| \geqslant 1$ for each u. Therefore the expression (16) becomes

$$\varepsilon \leqslant \int\limits_0^\infty \frac{u^2 e^{-ux}}{\mu^2 u^2+1}\,du \leqslant \frac{1}{\mu^2}\int\limits_0^\infty e^{-ux}du,$$

whence

$$\varepsilon \leqslant \frac{1}{\mu^2 x}.$$

For $\mu \geqslant 50$ and $x \geqslant 20$ we obtain $\varepsilon \leqslant 2\cdot 10^{-5}$.

Thus we have

$$I_1 = \int\limits_0^\infty \frac{e^{-ux}\cos uy}{\mu u+a}\,du, \tag{17}$$

approximately. It is easy to show that the result derived is also valid for the integral I_2. Therefore we obtain

$$I_2 = \int\limits_0^\infty \frac{e^{-ux}\sin uy}{\mu u+a}\,du, \tag{18}$$

approximately, with the same error. The error resulting in the both cases is less than $2\cdot 10^{-5}$.

Introducing a new variable $v = \mu u$ into (17) and (18), we derive the approximate relationships

$$\begin{aligned} \int\limits_0^\infty \frac{e^{-ux}\cos uy}{\mu u+\sqrt{u^2+i}}\,du &= \frac{1}{\mu}F\left(\frac{x}{\mu},\frac{y}{\mu}\right), \\ \int\limits_0^\infty \frac{e^{-ux}\sin uy}{\mu u+\sqrt{u^2+i}}\,du &= \frac{1}{\mu}G\left(\frac{x}{\mu},\frac{y}{\mu}\right). \end{aligned} \tag{19}$$

5. Applications. The integrals considered occur in problems concerning the electromagnetic field in a semi-infinite ferromagnetic body due to alternating currents in parallel conductors which are placed in a non-

-conducting medium above the boundary surface of the body, if its permeability is assumed to be constant ([2], [3], [4], [6]). The examples which illustrate the application of the integrals are:

1. The unit-length external impedance of the conductor l placed above the boundary surface of a ferromagnetic body is given by

$$Z_{el} = \frac{i\omega\mu_0}{\pi}\left[\mu_r \int_0^\infty \frac{e^{-2h_l ku}du}{\mu_r u + \sqrt{u^2+i}} + \frac{1}{2}\ln\frac{2h_l}{r_l}\right],$$

whereas the unit-length mutual impedance of the conductors l and m is

$$Z_{lm} = \frac{i\omega\mu_0}{\pi}\left[\mu_r \int_0^\infty \frac{e^{-(h_l+h_m)ku}\cos ka_{lm}}{\mu_r u + \sqrt{u^2+i}}\,du + \frac{1}{2}\ln\sqrt{\frac{(h_l+h_m)^2+a_{lm}^2}{(h_l-h_m)^2+a_{lm}^2}}\right],$$

in accordance with [2]. The notation accepted in the last equations is: ω — angular frequency, μ_r — relative permeability of ferromagnetic medium, h — distance from conductor l to boundary surface, r_l — radius of conductor l, a_{lm} — horizontal distance between conductors l and m, $k = \sqrt{\omega\mu_0\mu_r\gamma}$, μ_0 — permeability of vacuum, γ — conductivity of ferromagnetic medium.

The external and mutual impedances become

$$Z_{el} = \frac{i\omega\mu_0}{\pi}\left[F\left(\frac{2k}{\mu_r}h_l, 0\right) + \frac{1}{2}\ln\frac{2h_l}{r_l}\right],$$

$$Z_{lm} = \frac{i\omega\mu_0}{\pi}\left\{F\left[\frac{k}{\mu_r}(h_l+h_m), \frac{k}{\mu_r}a_{lm}\right] + \frac{1}{2}\ln\sqrt{\frac{(h_l+h_m)^2+a_{lm}^2}{(h_l-h_m)^2+a_{lm}^2}}\right\},$$

approximately, if (19) is used. The external and mutual impedances occur in relationships for eddy-current loss in the ferromagnetic body [2]. The last equations are valid for $\mu_r \geqslant 50$ and $2hk \geqslant 20$, where h is the minimum value of h_l and h_m.

2. The electric intensity at the boundary surface of the ferromagnetic body is [2]

$$E = -\frac{i\omega\mu_0\mu_r I}{\pi}\int_0^\infty \frac{e^{-h_l ku}\cos kyu}{\mu_r u + \sqrt{u^2+i}}\,du,$$

where I is the complex current in the conductor l and y is the horizontal distance from the conductor l to the point under consideration. Using (19), the electric intensity becomes

$$E = -\frac{i\omega\mu_0 I}{\pi}F\left(\frac{k}{\mu_r}h_l, \frac{k}{\mu_r}y\right),$$

approximately.

An approximate relationship for the tangential component of the magnetic intensity at the boundary surface takes the form (cf. [2])

$$H = -\frac{kIe^{i\pi/4}}{\pi\mu_r} F\left(\frac{k}{\mu_r} h_l, \frac{k}{\mu_r} y\right).$$

From the last expressions, the density of complex power on the boundary surface can be determined; as result we obtain

$$\Pi = \frac{\omega\mu_0 k |I|^2 e^{i\pi/4}}{\pi^2 \mu_r} \left| F\left(\frac{k}{\mu_r} h_l, \frac{k}{\mu_r} y\right)\right|^2.$$

3. The function G appears in relationships for the electric and magnetic intensities when the current flows along a ribbon conductor which is placed in the non-conducting medium parallel to the boundary surface (cf. [3]).

References

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O PEWNYCH CAŁKACH NIEWŁAŚCIWYCH

STRESZCZENIE

W pracy rozpatrzono funkcje F i G określone przy pomocy całek (1) i (2) oraz wyznaczono szeregi i wzory asymptotyczne dla tych funkcji.

Obliczono w sposób przybliżony całki (14) i (15), wyrażając je w zależności od funkcji F i G. Oszacowano błąd popełniony w wyniku przyjętych przybliżeń. Zastosowanie rozpatrywanych całek zilustrowano przykładami.