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COMPARISON OF FIXED PRECISION ESTIMATION SCHEMES IN BERNOULLI TRIALS

Two-stage procedures for unbiased estimation of the parameter p in Bernoulli trials are compared. At each stage, one can either conduct a fixed number of trials and count successes, or conduct enough trials to achieve a fixed number of successes. All combinations of these two methods are considered and the expected number of trials is found. If p is near 1 and a high degree of precision is desired, it is found that a wait-count procedure is optimum. If p is near $1/2$, the count-count procedure is preferable. These results apply when the first stage is designed optimally. The case of non-optimal first stage is also studied. Any of the two-stage procedures achieves a considerable saving in observations as compared to one-stage procedures based on binomial or negative binomial sampling, if p is near 1. The results of the paper* apply either to bounded-variance point estimates or to fixed length confidence intervals.

1. Introduction. Consider a sequence of independent trials, each resulting in success or failure with probabilities p and q, respectively. It is desired to estimate p by an unbiased statistic whose variance is less than a predetermined number v. A two-stage procedure for this problem was considered by Birnbaum and Healy [1], and Wasan [3] studied the asymptotic properties of some one-stage procedures.

In this paper, various two-stage estimation schemes are compared. At each stage, one can either conduct a fixed number of trials, counting successes, or one can use inverse sampling, waiting for a prescribed number of successes. Two-stage procedures which are combinations of these two schemes are compared as to their expected number of trials, assuming that a high degree of precision (small v) is desired. Connell and Mikulski [2] used this approach for the Poisson process.

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If the problem is to find an interval estimate for p having fixed length $2d$ and confidence $1-a$, then all the results of the paper apply by setting $v = d^2 a$.

It will also be shown that the two-stage procedures considered here are more efficient than one-stage procedures if p is near 1 and v is small.

2. Description of the procedures. In this section, $[u]$ is the greatest integer not greater than u.

Two one-stage procedures are defined here for reference. The *one-stage count procedure* is to conduct $n = [1/(4v)]+1$ trials and estimate p by X/n, where X is the number of successes. The *one-stage wait procedure* is to conduct trials until $r = 3+[4/(27v)]$ successes have occurred. If X is the total number of trials required, p is estimated by $(r-1)/(X-1)$. These methods lead to unbiased estimates whose variance is bounded by v. This is obvious for the one-stage count procedure; for the one-stage wait procedure one uses an accurate upper bound on $\operatorname{Var}[(r-1)/(X-1)]$, which is derived in Section 5.

The *count-count procedure* was first proposed by Birnbaum and Healy [1]. At the first stage, one conducts n_1 trials, observing X_1 successes. At the second stage, one conducts

$$N_1 = \left[\frac{(X_1+1)(n_1-X_1+1)}{v(n_1+1)(n_1+2)}\right]+1$$

trials. Then, if Y_1 successes occur at the second stage, p is estimated by $\hat{p}_1 = Y_1/N_1$.

In the *wait-count procedure*, one samples until $r_2 > 2$ successes occur. If this requires X_1 trials, then one conducts

$$N_2 = \left[\frac{r_2(X_2+1-r_2)}{v(X_2-1)(X_2-2)}\right]+1$$

trials at the second stage, observing Y_2 successes. Then p is estimated by $\hat{p}_2 = Y_2/N_2$.

The first stage of the *count-wait procedure* is to conduct n_3 trials, observing X_3 successes. At the second stage, one conducts trials until

$$R_3 = \left[\frac{(X_3+1)(X_3+2)(n_3+1-X_3)}{v(n_3+1)(n_3+2)(n_3+3)}\right]+3$$

successes occur. Assuming Y_3 trials are needed to obtain R successes, the estimator is $\hat{p}_3 = (R_3-1)/(Y_3-1)$.

In the *wait-wait procedure*, the first stage is to sample until $r_4 > 3$ successes occur. If X_4 trials are required for the first stage, one computes

$$R_4 = \left[\frac{r_4(r_4+1)(X_4-r_4+1)}{v(X_4-1)(X_4-2)(X_4-3)}\right]+3.$$

Then, at the second stage, one samples until R_4 successes are obtained. This will require Y_4 trials, and the estimate is then $\hat{p}_4 = (R_4-1)/(Y_4-1)$.

3. Efficiency results. Table 1 compares the procedures. It shows the total number of trials T_i, $\mathrm{E}(T_i)$, the optimal first stage, and the minimum of $\mathrm{E}(T_i)$ for $i = 1, \ldots, 4$. These minima are approximate values, obtained by ignoring terms of order less than n^{-1} or r^{-1} in $\mathrm{E}(T_i)$. Expected sample sizes for the one-stage procedures are given for comparison purposes.

Using the Cramer-Rao bound for unbiased sequential estimators, developed by Wolfowitz [4], Birnbaum and Healy [1] considered $pq/v\mathrm{E}_p(T)$ as a measure of efficiency. The approximate lower bounds on $\mathrm{E}(T)$ given in Table 1, therefore, lead to approximate bounds on the efficiency. Here, only situations with $p \geqslant 1/2$ are considered. Clearly, negative binomial sampling is inefficient if p is near 0. If it is known *a priori* that this is the case, then one should interchange "success" and "failure". With $p \geqslant 1/2$, it can be seen that it is always preferable to count at the first stage. If $p > 0.6$, count-wait is preferable; if $0.5 \leqslant p \leqslant 0.6$, count-count is preferable. As $p \to 1$, the two-stage procedures behave similarly, and they are better than the one-stage procedures.

If one has no prior knowledge of p, any of the "wait" procedures may be unsatisfactory. In this situation, the stopping rule of Wasan [3] might be used instead of negative binomial sampling. One stops if either r successes or r failures are observed. Wasan shows that, asymptotically, this rule is equivalent to negative binomial sampling if $p > 1/2$, and that the unbiased estimator associated with this rule has asymptotic variance p^2q/r if $p > 1/2$. If $p < 1/2$, the roles of success and failure and of p and q are reversed in Wasan's rule. One might conjecture that using Wasan's rule in place of negative binomial sampling might lead to an improved two stage rule, at least if v is small.

4. Some practical considerations. In the previous section it was shown that the optimal values at the first stage depend on p. If one guesses p to be about p_0 ($p_0 > 1/2$) and one uses the first-stage parameters suggested by Table 1, one obtains $\mathrm{E}(T_i)$ as shown in Table 2 (ignoring higher order terms).

Each $\mathrm{E}(T) = pq/v + O(1/\sqrt{v})$. By comparing the second terms, one obtains the following result (considering only $p \geqslant 1/2$):

(A) The count-count procedure is better than the count-wait procedure if p is sufficiently close to 1 and p_0 is near $1/2$.

(B) For any p_0 and for any p, the wait-count procedure is worse than the count-count procedure.

(C) For p_0 near $1/2$ and p near 1, the wait-wait procedure is the best.

(D) For $1/2 < p_0 < 3/5$, and p near 1, the wait-count procedure is better than the count-wait procedure.

Procedure	T	E(T)	Optimal first stage	Approximate min E(T)
Count-count	$n_1+\dfrac{(X_1+1)(N_1-X_1+1)}{v(n_1+1)(n_1+2)}$	$n_1+\dfrac{n_1+1+n_1(n_1-1)pq}{v(n_1+1)(n_1+2)}$	$n_1 \doteq \dfrac{\lvert 2p-1\rvert}{\sqrt{v}}$	$\dfrac{2\lvert 2p-1\rvert}{\sqrt{v}}+\dfrac{pq}{v}$
Wait-count	$X_2+\dfrac{r_2(X_2+1-r_2)}{vX_2(X_2-1)}$	$\dfrac{r_2}{p}+\dfrac{r_2pq}{v(r_2-1)}+\dfrac{r_2p^2}{(r_2-1)(r_2-2)v}$	$r_2 \doteq \dfrac{p}{\sqrt{v}}$	$\dfrac{2}{\sqrt{v}}+\dfrac{pq}{v}$
Count-wait	n_3+X_3	$n_3+\dfrac{2}{p}+$ $+\dfrac{n_3^3p^2q+n_3^2p(1+3q^2)-2n_3pq+2}{vp(n_3+1)(n_3+2)(n_3+3)}$	$n_3 \doteq \dfrac{\lvert 3p-2\rvert}{\sqrt{v}}$	$\dfrac{2\lvert 3p-2\rvert}{\sqrt{v}}+\dfrac{pq}{v}$
Wait-wait	X_1+X_2	$\dfrac{r_4+2}{p}+\dfrac{r_4(r_4+1)pq}{v(r_4-1)(r_4-2)}+$ $+\dfrac{r_4(r_4+1)p^2}{v(r_4-1)(r_4-2)(r_4-3)}$	$r_4 \doteq \dfrac{p\sqrt{4-3p}}{\sqrt{v}}$	$\dfrac{2\sqrt{4-3p}}{\sqrt{v}}+\dfrac{pq}{v}$
One-stage count	$\dfrac{1}{4v}$	$\dfrac{1}{4v}$		$\dfrac{1}{4v}$

TABLE 2

Procedure	First stage	Approximate E (T)
Count-count	$\dfrac{\lvert 2p_0-1\rvert}{\sqrt{v}}$	$\dfrac{pq}{v}+\dfrac{(2p-1)^2+(2p_0-1)^2}{\lvert 2p-1\rvert\sqrt{v}}$
Wait-count	$\dfrac{p_0}{\sqrt{v}}$	$\dfrac{pq}{v}+\dfrac{1}{\sqrt{v}}\left(\dfrac{p_0}{p}+\dfrac{p}{p_0}\right)$
Count-wait	$\dfrac{\lvert 3p_0-2\rvert}{\sqrt{v}}$	$\dfrac{pq}{v}+\dfrac{(2-3p)^2+(2-3p_0)^2}{\lvert 3p-2\rvert\sqrt{v}}$
Wait-wait	$\dfrac{p_0\sqrt{4-3p_0}}{\sqrt{v}}$	$\dfrac{pq}{v}+\dfrac{p_0^2\sqrt{(4-3p)(4-3p_0)}+p^2(4-3p)}{pp_0\sqrt{v}\sqrt{4-3p_0}}$

Conclusions (A), (C) and (D) demonstrate that if our guessed value p_0 is highly inaccurate, the procedure suggested in Section 3 may no longer be the best one. It should be noted, however, that choosing the wrong procedure only leads to excess observations on the order of $v^{-1/2}$. Since $\mathrm{E}(T)=pq/v+O(v^{-1/2})$, these excess observations will be small, relative to the total expected number. Also, if p_0 is a reasonably accurate guess of p, it seems unlikely that $\mathrm{E}(T)$ will be far from its minimum.

It is also worth noting that if p is near $1/2$ or $2/3$, then two-stage procedures are wasteful. For p near $1/2$, the one-stage count procedure is nearly optimum, while the one-stage wait procedure is optimum for p near $2/3$.

5. Calculations. The total number of trials needed for a two-stage procedure will be denoted by T_i $(i=1,\ldots,4)$. Throughout this section and the next, the integer correction in the definitions of the second stage is ignored.

If the second stage involves negative binomial sampling, one employs the inequality

$$\mathrm{Var}\left(\frac{r-1}{X-1}\right)\leqslant\frac{p^2q}{r-2},\tag{1}$$

where X is a negative binomial variable with parameters r and p. This is derived by

$$\mathrm{E}\left[\frac{(r-1)^2}{(X-1)^2}\right]=\mathrm{E}\left[\frac{(r-1)^2}{(X-1)(X-2)}\right]-\mathrm{E}\left[\frac{(r-1)^2}{(X-1)^2(X-2)}\right].$$

The first expectation is $(r-1)p^2/(r-2)$, which follows from direct calculation. The second may be expressed, after some calculations, as

$$\frac{p^2(r-1)\mathrm{E}[(X'+1)^{-1}]}{r-2},$$

where X' is a negative binomial with parameters $r-2$ and p. Jensen's inequality is used to bound $\mathrm{E}[(X'+1)^{-1}]$. Combining the results, one has

$$\operatorname{Var}\left[\frac{r-1}{X-1}\right] \leqslant \frac{p^2 q}{r-2+p} \leqslant \frac{p^2 q}{r-2}.$$

Although inequality (1) is not sharp unless $p = 1$, the error in replacing the variance by its upper bound is smaller than $2p^2q/r(r-2)$. This follows from the Cramer-Rao lower bound for the negative binomial. Inequality (1) was also used to construct the one-stage wait procedure of Section 2.

In calculating variances of estimators we notice that all estimators considered here are conditionally unbiased, given the outcome of the first stage. Therefore, one can compute their variances by first finding the conditional variance and then taking its expectation.

For the count-count procedure, $\mathrm{E}(T_1)$ has been computed by Birnbaum and Healy [1], and they provide tables of the optimum n_1 for various p.

For the wait-count procedure,

$$\mathrm{E}\left[\frac{(X_2-1)(X_2-2)}{X_2-r_2+1}\right] = \frac{p^{r_2}}{q^{r_2-3}}\frac{d^2}{dq^2}\sum_{x=r_2}^{\infty}\frac{q^{x-1}(x-1)!}{(x+1-r_2)!(r_2-1)!}$$
$$= \frac{r_2(r_2-1)-4(r_2-1)p+2p^2-(r_2-2)(r_2-3)p^{r+1}}{(r_2-1)pq} \leqslant \frac{r_2}{pq},$$

since the numerator is a decreasing function of p. Now, by the above calculation,

$$\operatorname{Var}(\hat{p}_2) = \mathrm{E}\left[\operatorname{Var}\left\{\frac{Y_2}{N_2}\,\middle|\, X_2\right\}\right] = \mathrm{E}\left[\frac{pq}{N_2}\right] \leqslant v.$$

It is straightforward to evaluate $\mathrm{E}(T_2)$.

In the count-wait procedure,

$$\mathrm{E}[(R_3-2)^{-1}] = \frac{v(1-p^{n_3+3}-(n_3+3)pq^{n_3+2}-q^{n_3+3})}{p^2q} \leqslant \frac{v}{p^2q},$$

since the numerator is $\mathrm{P}[2 \leqslant Z \leqslant n_3+2]$, where Z is a binomial with parameters n_3+3 and p. Hence

$$\operatorname{Var}(\hat{p}_3) = \mathrm{E}\left[\operatorname{Var}\left\{\frac{R_3-1}{Y_3-1}\,\middle|\, X_1\right\}\right] \leqslant \mathrm{E}\left[\frac{p^2q}{R_3-2}\,\middle|\, X_1\right] \leqslant v.$$

The first inequality holds by (1).

$\mathrm{E}(T_3)$ is found by combining moments of the binomial distribution.

The calculations in the wait-wait procedure are carried out similarly:

$$\mathrm{E}[(R_4-2)^{-1}]$$
$$= \frac{v\big((r_4+1)r_4(r_4-1)-7r_4(r_4-1)p+2(r_4-1)p^2-6p^3-(r_4-2)(r_4-3)(r_4-4)p^{r_4+2}\big)}{p^2q(r_4+1)r_4(r_4-1)}.$$

The numerator of this expression is a decreasing function for $0<p<1$, so $\mathrm{E}[(R_4-2)^{-1}] \leqslant v/p^2q$. Now $\mathrm{Var}(\hat{p}_4) \leqslant v$, as in the count-wait procedure.

6. Application to confidence intervals. The estimation procedure of Section 2 can be used to devise fixed-width confidence intervals, in conjunction with Chebyshev's inequality. If $\hat{p}$ is any bounded-variance unbiased estimate of p, by Chebyshev's inequality one has

$$\mathrm{P}[|\hat{p}-p| \leqslant d] \geqslant 1-\frac{\mathrm{Var}(\hat{p})}{d^2} \geqslant 1-\frac{v}{d^2}.$$

Hence a confidence interval with width $2d$ and with confidence level $1-a$ is $(\hat{p}-d, \hat{p}+d)$ if $v = d^2 a$. Any of the procedures discussed in Section 2 may be used to set up fixed-width confidence intervals. The efficiency results of Section 3 imply that the count-count or count-wait procedure is preferable. However, the resulting confidence intervals are conservative because of the crudity of the Chebyshev inequality.

References

[1] A. Birnbaum and W. C. Healy, *Estimates with prescribed variance based on two-stage sampling*, Ann. Math. Statist. 31 (1960), p. 662-676.

[2] T. L. Connell and P. W. Mikulski, *On comparison of waiting time versus counting technique in construction of confidence intervals for a parameter of a Poisson process*, Technometrics 13 (1971), p. 699-703.

[3] M. T. Wasan, *Sequential optimum procedures for unbiased estimation of a binomial parameter*, ibidem 6 (1964), p. 259-271.

[4] J. Wolfowitz, *The efficiency of sequential estimates and Wald's equation for sequential processes*, Ann. Math. Statist. 18 (1974), p. 215-230.

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PORÓWNANIE ESTYMACJI O USTALONEJ PRECYZJI W PRÓBACH BERNOULLIEGO

STRESZCZENIE

Praca zawiera porównanie metod dwustopniowych w nieobciążonej estymacji parametru p w próbach Bernoulliego. W każdym stopniu można ustalić albo liczbę prób, albo też liczbę sukcesów. Porównanie jest przeprowadzone ze względu na oczekiwaną długość badania. Rezultaty stosują się do tych przypadków, w których żądana jest ustalona wariancja nieobciążonego estymatora lub też przedział ufności o ustalonej długości.