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CHARACTERIZATION OF OPTIMAL POLYNOMIAL AND RATIONAL STARTING APPROXIMATIONS

1. Introduction. Let X be a compact metric space and let $C(X)$ denote the space of real functions continuous on X, normed by

$$\|f\| = \max\{w(x)|f(x)|: x \in X\},$$

where w is a positive weighted function from $C(X)$. We assume that Φ is a continuous mapping from the open subset $K \subset C(X)$ into $C(X)$. Let M be an arbitrary fixed non-empty subset of K. The following three definitions from [3] will be useful:

Definition 1. An element $p \in M$ is an *optimal starting approximation* in M for $g \in \Phi(K)$ if

$$\|g-\Phi(p)\| \leqslant \|g-\Phi(q)\| \quad \text{for all } q \in M.$$

Definition 2. The operator Φ is called *pointwise strictly monotone* at $f \in K$ if, for each $h, g \in K$,

$$|\Phi(f)(x_0)-\Phi(h)(x_0)| < |\Phi(f)(x_0)-\Phi(g)(x_0)| \quad \text{for each } x_0 \in X,$$

where either $g(x_0) < h(x_0) \leqslant f(x_0)$ or $f(x_0) \leqslant h(x_0) < g(x_0)$.

Definition 3. The operator Φ is said to be *pointwise fixed* at $f \in K$ if $h \in K$ with $h(x_0) = f(x_0)$ for $x_0 \in X$ implies

$$\Phi(h)(x_0) = \Phi(f)(x_0).$$

Meinardus and Taylor [3] proved the Alternation Theorem characterizing the optimal polynomial starting approximation. Gibson [1] proved this theorem for the optimal rational starting approximation.

In this paper, assuming additionally that

$$(*) \qquad g(z) = h(z) \text{ implies } \Phi(g)(z) = \Phi(h)(z) \text{ for all } g, h \in M,$$

we obtain theorems characterizing the optimal polynomial and rational starting approximations being dual to the Kolmogorov criterion.

We note that assumption $(*)$ is only needed in the proofs of the sufficiency of Lemma 1, Theorems 1 and 2, but not in the proofs of the necessity.

2. Main results. First we prove the following lemma:

LEMMA 1. *Let $f, g, h \in K \subset C(X)$ be arbitrarily fixed and let the operator Φ be pointwise strictly monotone and pointwise fixed at f. Let*

$$D = \{x\colon\ w(x)|\Phi(f)(x)-\Phi(g)(x)| = \|\Phi(f)-\Phi(g)\|\}.$$

Then, for $x \in D$, the inequality $(f-g)(x)h(x) > 0$ (or $(f-g)(x)h(x) < 0$) holds if and only if there exists a real number $\lambda \neq 0$ such that

$$\|\Phi(f)-\Phi(g+\lambda h)\| < \|\Phi(f)-\Phi(g)\|.$$

Proof. Necessity. Let us suppose that

$$(f-g)(x)h(x) > 0 \quad \text{for each } x \in D.$$

Since the set D is closed and K is the open set, we can choose $\lambda_1 > 0$ such that, for $0 < \lambda < \lambda_1$, we have

$$g+\lambda h \in K \quad \text{and} \quad 0 < \lambda \sup\{|h(x)|\colon\ x \in D\} \leqslant \inf\{|(f-g)(x)|\colon\ x \in D\}.$$

Therefore, by the inequality $(f-g)(x)h(x) > 0$, $x \in D$, for those λ and for each $x \in D$ we have either

$$g(x) < g(x)+\lambda h(x) \leqslant f(x)$$

or

$$f(x) \leqslant g(x)+\lambda h(x) < g(x).$$

Now, since Φ is pointwise strictly monotone at f, for $0 < \lambda < \lambda_1$ and for all $x \in D$ we obtain

$$w(x)|\Phi(f)(x)-\Phi(g+\lambda h)(x)| < w(x)|\Phi(f)(x)-\Phi(g)(x)| = \|\Phi(f)-\Phi(g)\|.$$

If $D = X$, then the proof is completed.

Otherwise, from the continuity of the function $w|\Phi(f)-\Phi(g+\lambda h)|$ it follows that there exists an open set $U \supset D$ such that the above inequality is true for $x \in U$. Let $V = X \setminus U$. Obviously, V is a closed set. Let us write

$$\delta = \sup\{w(x)|\Phi(f)(x)-\Phi(g)(x)|\colon\ x \in V\}.$$

Since $V \cap D$ is an empty set, we have $\|\Phi(f)-\Phi(g)\| > \delta$. From the continuity of Φ it follows that there exists $\lambda_2 > 0$ such that for $0 < \lambda < \lambda_2 \leqslant \lambda_1$ we have

$$\|\Phi(g)-\Phi(g+\lambda h)\| < \|\Phi(f)-\Phi(g)\|-\delta.$$

Hence for $x \in V$ and $0 < \lambda < \lambda_2$ we obtain

$$\begin{aligned} w(x)|\Phi(f)(x)-\Phi(g+\lambda h)(x)| &\leqslant \\ &\leqslant w(x)|\Phi(f)(x)-\Phi(g)(x)|+w(x)|\Phi(g)(x)-\Phi(g+\lambda h)(x)| \\ &< \delta+\|\Phi(f)-\Phi(g)\|-\delta = \|\Phi(f)-\Phi(g)\|. \end{aligned}$$

Finally, for $0 < \lambda < \lambda_2$ we have

$$\|\Phi(f) - \Phi(g+\lambda h)\| < \|\Phi(f) - \Phi(g)\|.$$

The proof in the case of $(f-g)(x)h(x) < 0$ for all $x \in D$ is analogous.

Sufficiency. Suppose, on the contrary, that there exist $z, y \in D$ such that

$$(f-g)(z)h(z) \leqslant 0 \quad \text{and} \quad (f-g)(y)h(y) > 0.$$

First, we assume that $(f-g)(z)h(z) = 0$. If $h(z) = 0$, then, by (*), the proof is evident. If $f(z) = g(z)$, then it follows from the fact that Φ is pointwise fixed at f that $\Phi(f)(z) = \Phi(g)(z)$, and hence the proof is completed.

In the second case we assume that $(f-g)(z)h(z) < 0$. Then for an arbitrary $\lambda > 0$ we have either

$$g(z) + \lambda h(z) < g(z) \leqslant f(z)$$

or

$$f(z) \leqslant g(z) < g(z) + \lambda h(z).$$

Hence and from the pointwise strictly monotonicity of Φ at f, for every $0 < \lambda < \lambda_1$ we obtain

$$w(z)|\Phi(f)(z) - \Phi(g+\lambda h)(z)| > w(z)|\Phi(f)(z) - \Phi(g)(z)| = \|\Phi(f) - \Phi(g)\|,$$

where λ_1 is the greatest real number such that $g + \lambda_1 h \in K$. On the other hand, there exists λ_2, $0 < \lambda_2 \leqslant \lambda_1$, such that for $0 < \lambda < \lambda_2$ we have

$$w(y)|\Phi(f)(y) - \Phi(g+\lambda h)(y)| < \|\Phi(f) - \Phi(g)\|.$$

By the above inequalities, the proof of the lemma is completed.

Note that without assumption (*) the sufficiency part of Lemma 1 may be stated as follows:

If there exists a real number $\lambda \neq 0$ such that

$$\|\Phi(f) - \Phi(g+\lambda h)\| < \|\Phi(f) - \Phi(g)\|,$$

then, for $x \in D$, the inequality $(f-g)(x)h(x) \geqslant 0$ (or $(f-g)(x)h(x) \leqslant 0$) holds.

The proof of this is analogous as that of Lemma 1 and, therefore, is omitted.

Definition 4 (Kolmogorov and Fomine [2], p. 125). The open set $K \subset C(X)$ is said to be an *open field* if it contains a non-empty kernel $J(K)$ defined by

$$J(K) = \{f \in K\colon \forall g \in C(X)\ \exists \varepsilon > 0\ (|t| < \varepsilon \text{ implies } f + tg \in K)\}.$$

THEOREM 1. *Let Φ: $K\to C(X)$, where K is an open field in $C(X)$, be a continuous operator. Let V be an arbitrary subspace of $C(X)$ and let $M = K\cap V$ be a non-empty set. Additionally, suppose that Φ is pointwise strictly monotone and pointwise fixed at $f\in K\setminus M$. Then $p\in M$ is the optimal starting polynomial approximation for $\Phi(f)$ in M if and only if there exists no function $g\in V$ such that*

$$\big(f(x)-p(x)\big)g(x) > 0$$

for every $x\in D = \{x\colon w(x)|\Phi(f)(x)-\Phi(p)(x)| = \|\Phi(f)-\Phi(p)\|\}$.

Proof. Since K is an open field, the proof of the necessity follows immediately from Lemma 1.

For the proof of the sufficiency we suppose, on the contrary, that $h\in M$ is a better starting approximation in M for $\Phi(f)$ than p. Then

$$\big\|\Phi(f)-\Phi\big(p+(h-p)\big)\big\| < \|\Phi(f)-\Phi(p)\|$$

and Lemma 1 imply

$$(f-p)(x)(h-p)(x) > 0 \quad \text{for all } x\in D.$$

Thus, we may set $g = h-p$, which completes the proof.

Without assumption (*) the sufficiency of this theorem may be formulated as follows:

If there exists no function $g\in V$ such that

$$\big(f(x)-p(x)\big)g(x) \geqslant 0 \quad \textit{for every } x\in D,$$

then $p\in M$ is the optimal polynomial starting approximation for $\Phi(f)$ in M.

Now, let us assume that two subspaces P and Q are fixed in $C(X)$. We denote by R the family of functions $r = p/q$, where $p\in P$, $q\in Q$ and $q(x) > 0$ on X. For a fixed $r\in R$, let $P+rQ$ be the subspace of $C(X)$ such that

$$P+rQ = \{p+rq\colon p\in P \text{ and } q\in Q\}.$$

THEOREM 2. *Let Φ: $K\to C(X)$, where K is an open field in $C(X)$, be a continuous operator and let $M = K\cap R$ be a non-empty set. Additionally, we assume that Φ is pointwise strictly monotone and pointwise fixed at $f\in K\setminus M$. Then $r\in M$ is the optimal rational starting approximation for $\Phi(f)$ in M if and only if there exists no function $g\in P+rQ$ such that*

$$\big(f(x)-r(x)\big)g(x) > 0$$

for every $x\in D = \{x\colon w(x)|\Phi(f)(x)-\Phi(r)(x)| = \|\Phi(f)-\Phi(r)\|\}$.

Proof. For the proof of the necessity we suppose that the function $g = p_0+rq_0\in P+rQ$, where $r = p/q$ agrees on D in sign with $f-r$. Then the function $(f-r)g/(q-\lambda q_0)$ is positive on D for a positive number λ such that $(q-\lambda q_0)(x) > 0$ on X. Hence and by Lemma 1 there exists a positive real number λ_2 such that for $0 < \lambda < \lambda_2$ the inequality

$$\|\Phi(f)-\Phi(r+\lambda g/(q-\lambda q_0))\| < \|\Phi(f)-\Phi(r)\|$$

holds. Hence and from Definition 4 it follows that there exists a $\lambda > 0$ such that the function

$$r+\lambda g/(q-\lambda q_0) = (p+\lambda p_0)/(q-\lambda q_0) \in M$$

approximates f better than r.

The proof of the sufficiency is analogical as that of Theorem 1.

Without assumption (*) the sufficiency part of Theorem 2 may be stated in an analogous way as in the polynomial approximation.

References

[1] J. B. Gibson, *Optimal rational starting approximations*, J. Approximation Theory 12 (1974), p. 182-198.

[2] A. N. Kolmogorov and S. V. Fomine, *Eléments de la théorie des fonctions et de l'analyse fonctionnelle*, Mir, Moscow 1974. (In Russian: А. Н. Колмогоров и С. В. Фомин, *Элементы теории функций и функционального анализа*, Москва 1972.)

[3] G. Meinardus and G. D. Taylor, *Optimal starting approximations for iterative schemas*, J. Approximation Theory 9 (1973), p. 1-19.

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CHARAKTERYZACJA OPTYMALNYCH STARTOWYCH WIELOMIANOWYCH I WYMIERNYCH APROKSYMACJI

STRESZCZENIE

W niniejszej pracy udowodniono twierdzenia charakteryzujące optymalną startową wielomianową i wymierną aproksymację, dualne w przypadku klasycznej jednostajnej aproksymacji do twierdzeń typu kryterium Kołmogorowa.