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A NUMERICAL METHOD FOR SOLVING THE ABEL INTEGRAL EQUATION

1. Introduction. Many papers (see, e.g., [3], [6] and their references) deal with numerical methods of solving the Abel integral equation

$$\int_0^t \frac{f(s)\,ds}{\sqrt{t^2-s^2}} = g(t) \quad \text{for } t > 0. \tag{1.1}$$

In the present paper we give a method based on the well-known inversion formula (see, for example, [4])

$$f(s) = \frac{2}{\pi}\frac{d}{ds}\int_0^s \frac{tg(t)\,dt}{\sqrt{s^2-t^2}} \quad \text{for } s > 0 \tag{1.2}$$

and using polynomial interpolating spline functions [1]. Usually, the function $g(t)$ is known only at a finite number of points $t \in \Delta = \{t_1, t_2, \ldots, t_n\}$, where $0 = t_1 < t_2 < \ldots < t_n$. Therefore, instead of equation (1.1) we solve here

$$\int_0^t \frac{f_\Delta(s)\,ds}{\sqrt{t^2-s^2}} = g_\Delta(t) \quad \text{for } t \in T = (t_1, t_n\rangle, \tag{1.3}$$

where $g_\Delta(t)$ is a spline function of degree $m = 2k-1$, $1 \leqslant k \leqslant n$, interpolating the function $g(t)$ on the network Δ. In virtue of (1.2) the solution of (1.3) is of the form

$$f_\Delta(s) = \frac{2}{\pi}\frac{d}{ds}\int_0^s \frac{tg_\Delta(t)\,dt}{\sqrt{s^2-t^2}} \quad \text{for } s \in T. \tag{1.4}$$

In the sequel we suppose that the function g has an absolutely continuous $(k-1)$-st derivative and the k-th derivative integrable with its square. These assumptions with a suitable chosen boundary conditions guarantee (see [5]) that the spline function $g_\Delta(t)$ exists and is uniquely determined.

The function $g_\Delta(t)$ can be represented in the form (see [5])

$$g_\Delta(t) = \sum_{i=0}^{m} a_i t^i + \sum_{j=1}^{n} \beta_j \theta(t, t_j)(t-t_j)^m, \tag{1.5}$$

where a_i $(i = 0, 1, \ldots, m)$ and β_j $(j = 1, 2, \ldots, n)$ are given real numbers and

$$\theta(t, t_j) = \begin{cases} 0 & \text{for } t < t_j, \\ 1 & \text{for } t \geqslant t_j. \end{cases}$$

Using (1.4) and (1.5) we give in Section 2 some analytical formulae for the solution $f_\Delta(s)$ of equation (1.3) which approximates the solution $f(s)$ of the original equation (1.1).

2. Numerical method. It is known [7] that

$$\begin{cases} I_0 = \int \dfrac{dx}{\sqrt{R}} = \dfrac{-1}{\sqrt{-c}} \arcsin \dfrac{2cx+b}{\sqrt{-D}} \quad \text{for } c < 0,\ D < 0, \\ I_1 = \int \dfrac{x\,dx}{\sqrt{R}} = \dfrac{\sqrt{R}}{c} - \dfrac{b}{2c} \int \dfrac{dx}{\sqrt{R}}, \\ I_m = \int \dfrac{x^m}{\sqrt{R}}\,dx = \dfrac{1}{mc} x^{m-1}\sqrt{R} - \dfrac{2m-1}{2mc} I_{m-1} - \dfrac{(m-1)a}{mc} I_{m-2}, \end{cases} \tag{2.1}$$

where $m = 2, 3, \ldots, R = a+bx+cx^2$, and $D = 4ac-b^2$.

Replacing $g_\Delta(t)$ in (1.4) by the right-hand side of (1.5) we obtain

$$f_\Delta(s) = \frac{2}{\pi} \sum_{i=0}^{m} a_i \frac{d}{ds} \int_0^s \frac{t^{i+1}}{\sqrt{s^2-t^2}}\,dt + \frac{2}{\pi} \sum_{j=1}^{n} \beta_j \frac{d}{ds} \int_0^s \frac{t\,\theta(t, t_j)(t-t_j)^m}{\sqrt{s^2-t^2}}\,dt,$$

and writing

$$a_i(s) = \frac{2}{\pi} \frac{d}{ds} \int_0^s \frac{t^{i+1}}{\sqrt{s^2-t^2}}\,dt,$$

$$b_j(s) = \frac{2}{\pi} \frac{d}{ds} \int_0^s \frac{t\,\theta(t, t_j)(t-t_j)^m}{\sqrt{s^2-t^2}}\,dt$$

we have

$$f_\Delta(s) = \sum_{i=0}^{m} a_i a_i(s) + \sum_{j=1}^{n} \beta_j b_j(s). \tag{2.2}$$

Now we evaluate $a_i(s)$ $(i = 0, 1, \ldots, m)$ and $b_j(s)$ $(j = 1, 2, \ldots, n)$.

Let us write

$$(-1)!! \equiv 0!! \equiv 1,$$

$$i!! = \begin{cases} 1\cdot 3\cdot \ldots \cdot i & \text{for } i \text{ odd}, \\ 2\cdot 4\cdot \ldots \cdot i & \text{for } i \text{ even}. \end{cases}$$

By induction and by (2.1) we may prove that

$$K_i(s) \equiv \int_0^s \frac{t^i}{\sqrt{s^2-t^2}}\,dt = \begin{cases} \dfrac{\pi}{2}\,\dfrac{(i-1)!!}{i!!}\,s^i & \text{for } i = 0, 2, \ldots, \\ \dfrac{(i-1)!!}{i!!}\,s^i & \text{for } i = 1, 3, \ldots, \end{cases}$$

and for $j = 1, 2, \ldots, n$ we have

$$L_{ij}(s) \equiv \int_{t_j}^s \frac{t^i}{\sqrt{s^2-t^2}}\,dt$$

$$= \begin{cases} \dfrac{(i-1)!!}{i!!}\sqrt{s^2-t_j^2}\displaystyle\sum_{k=0}^{(i-1)/2}\frac{(2k-1)!!}{(2k)!!}\,t_j^{2k}s^{i-2k-1} & \text{for } i = 1, 3, \ldots, \\ \dfrac{(i-1)!!}{i!!}\left\{\left[\dfrac{\pi}{2}+\arcsin\left(-\dfrac{t_j}{s}\right)\right]s^i + \sqrt{s^2-t_j^2}\displaystyle\sum_{k=1}^{i/2}\frac{(2k-2)!!}{(2k-1)!!}\,t_j^{2k-1}s^{i-2k}\right\} & \\ & \text{for } i = 0, 2, \ldots \end{cases}$$

Differentiating these formulae we obtain

$$\frac{d}{ds}K_i(s) = \begin{cases} \dfrac{(i-1)!!}{(i-2)!!}\,s^{i-1}\,\dfrac{\pi}{2} & \text{for } i = 2, 4, \ldots, \\ \dfrac{(i-1)!!}{(i-2)!!}\,s^{i-1} & \text{for } i = 1, 3, \ldots, \end{cases}$$

$$\frac{d}{ds}L_{ij}(s)$$

$$= \begin{cases} \dfrac{(i-1)!!}{(i-2)!!}\,\dfrac{s}{\sqrt{s^2-t_j^2}}\left[s^{i-1} - \displaystyle\sum_{k=1}^{(i-1)/2}\frac{(2k-3)!!}{(2k)!!}\,t_j^{2k}s^{i-2k-1}\right] & \text{for } i = 1, 3, \ldots, \\ \dfrac{(i-1)!!}{(i-2)!!}\left\{s^{i-1}\left[\dfrac{\pi}{2}+\arcsin\left(-\dfrac{t_j}{s}\right)\right] + \right. & \\ \left. \quad + \dfrac{st_j}{\sqrt{s^2-t_j^2}}\left[s^{i-2} - \displaystyle\sum_{k=2}^{i/2}\frac{(2k-4)!!}{(2k-1)!!}\,t_j^{2k-2}s^{i-2k}\right]\right\} & \text{for } i = 2, 4, \ldots \end{cases}$$

Hence

$$(2.3)\qquad a_i(s) = \begin{cases} \dfrac{i!!}{(i-1)!!}\,\dfrac{2}{\pi}\,s^i & \text{for } i = 0, 2, \ldots, m-1, \\ \dfrac{i!!}{(i-1)!!}\,s^i & \text{for } i = 1, 3, \ldots, m. \end{cases}$$

If $s \geqslant t_j > 0$, then

$$b_j(s) = \frac{2}{\pi}\frac{d}{ds}\int_{t_j}^{s}\frac{t(t-t_j)^m dt}{\sqrt{s^2-t^2}}.$$

For $s < t_j$ we have $\theta(s, t_j) = 0$ and thus $b_j(s) = 0$. Hence for $j = 1, 2, \ldots, n$ we obtain

$$b_j(s) = \frac{2}{\pi}\theta(s, t_j)\frac{d}{ds}\int_{t_j}^{s}\frac{t(t-t_j)^m dt}{\sqrt{s^2-t^2}}$$

$$= \frac{2}{\pi}\theta(s, t_j)\sum_{i=0}^{m}\binom{m}{i}(-t_j)^{m-i}\frac{d}{ds}\int_{t_j}^{s}\frac{t^{i+1}}{\sqrt{s^2-t^2}}dt.$$

Finally, for $j = 1, 2, \ldots, n$ we obtain

$$(2.4)\qquad b_j(s) = \frac{2}{\pi}\theta(s, t_j)\sum_{i=0}^{m}\binom{m}{i}(-t_j)^{m-i}C_{ij}(s),$$

where

$$(2.5)\qquad C_{ij}(s)$$

$$= \begin{cases} \dfrac{i!!}{(i-1)!!}\,\dfrac{s}{\sqrt{s^2-t_j^2}}\left[s^i - \displaystyle\sum_{k=1}^{i/2}\frac{(2k-3)!!}{(2k)!!}t_j^{2k}s^{i-2k}\right] & \text{for } i = 0, 2, \ldots, m-1, \\ \dfrac{i!!}{(i-1)!!}\left\{s^i\left[\dfrac{\pi}{2} + \arcsin\left(-\dfrac{t_j}{s}\right)\right] + \right. \\ \left. + \dfrac{st_j}{\sqrt{s^2-t_j^2}}\left[s^{i-1} - \displaystyle\sum_{k=2}^{(i+1)/2}\frac{(2k-4)!!}{(2k-1)!!}t_j^{2k-2}s^{i-2k+1}\right]\right\} & \text{for } i = 1, 3, \ldots, m. \end{cases}$$

Thus, it is proved that the solution of equation (1.3) may be obtained from formulae (2.2)-(2.5). Since $b_j(t_r) = 0$ for $r = 1, 2, \ldots, j-1$, the solution $f_\Delta(s)$ of equation (1.3) for $s \in \Delta$ may be expressed in the form

$$f(t_r) = \sum_{i=0}^{m}\alpha_i a_i(t_r) + \sum_{j=1}^{r-1}\beta_j b_j(t_r) \qquad \text{for } r = 1, 2, \ldots, n,$$

where

$$a_i(t_r) = \begin{cases} \dfrac{2}{\pi}\dfrac{i!!}{(i-1)!!}t_r^i & \text{for } i = 0, 2, \ldots, m-1, \\ \dfrac{i!!}{(i-1)!!}t_r^i & \text{for } i = 1, 3, \ldots, m, \end{cases}$$

$$b_j(t_r) = \frac{2}{\pi}\sum_{i=0}^{m}\binom{m}{i}(-t_j)^{m-i}C_{ij}(t_r)$$

and

$$C_{ij}(t_r) = \begin{cases} t_r^i\dfrac{i!!}{(i-1)!!}\dfrac{1}{\sqrt{1-(t_j/t_r)^2}}\left[1-\displaystyle\sum_{k=1}^{i/2}\frac{(2k-3)!!}{(2k)!!}\left(\frac{t_j}{t_r}\right)^{2k}\right] \\ \qquad\qquad\qquad\qquad \text{for } i = 0, 2, \ldots, m-1, \\ t_r^i\dfrac{i!!}{(i-1)!!}\left\{\dfrac{\pi}{2}+\arcsin\left(-\dfrac{t_j}{t_r}\right)+\right. \\ \qquad \left.+\dfrac{1}{\sqrt{-1+(t_r/t_j)^2}}\left[1-\displaystyle\sum_{k=2}^{(i+1)/2}\frac{(2k-4)!!}{(2k-1)!!}\left(\frac{t_j}{t_r}\right)^{2k-2}\right]\right\} \\ \qquad\qquad\qquad\qquad \text{for } i = 1, 3, \ldots, m. \end{cases}$$

It is remarkable that the above-described method is exact for all spline functions g_Δ of degree not greater than m and that the obtained solution $f_\Delta(s)$ is not a spline function.

For this reason the present method is not equivalent to the existing methods (as described for $m = 1$ in [3]), where the solution of the integral equation is chosen from the class of spline functions. The proposed method requires to solve a system of $m+n+1$ equations with the m-diagonal (m-band) matrix $[b_{ij}]_{1\leqslant i,j\leqslant m+n+1}$ (since $b_{ij} = 0$ for $j \geqslant i+(m+1)/2$ and $i \geqslant j+(m+1)/2$) and then to use formulae (2.2)-(2.5); if a method of the type as in [3] is applied, it is necessary to solve a system with the matrix $[a_{ij}]_{1\leqslant i,j\leqslant n+m+1}$, where $a_{ij} = 0$ only for $j \geqslant i+(m+1)/2$.

3. Numerical examples. The proposed method was tested on the Odra 1204 computer for the following three equations for which we knew the exact solutions:

$$\int_0^t \frac{f(s)\,ds}{\sqrt{t^2-s^2}} = t^4, \quad f(s) = \frac{16}{3\pi}s^4, \tag{3.1}$$

$$\int_0^t \frac{f(s)\,ds}{\sqrt{t^2-s^2}} = t^5, \quad f(s) = \frac{15}{8}s^5, \tag{3.2}$$

$$(3.3)\qquad \int_0^t \frac{f(s)\,ds}{\sqrt{t^2-s^2}} = t^6, \quad f(s) = \frac{32}{5\pi}s^6.$$

It is known [1] that the interpolating cubic spline function $g_\Delta(t)$ can be given in the form

$$g_\Delta(t) = \sum_{k=0}^{m} A_{ik}t^k \quad \text{for } t \in \langle t_i, t_{i+1}\rangle,\ i = 1, 2, \ldots, n-1.$$

Since $t_1 = 0$, we can obtain

$$a_k = A_{1k} \quad \text{for } k = 0, 1, \ldots, m$$

and

$$\beta_k = \frac{d_k - \sum_{j=1}^{k-1}(t_{k+1}-t_j)\beta_j}{t_{k+1}-t_k},$$

where

$$d_k = \frac{A_{k,m-1}-a_{m-1}}{m} + (A_{k,m}-a_m)t_{k+1} \quad \text{for } k = 1, 2, \ldots, n-1.$$

To calculate the coefficients A_{ik} we used numerically stable methods described in [1] or [2]. Since the proposed method of determination of β_k for $k = 1, 2, \ldots, n-1$ is equivalent to solving a system of equations having a matrix with only non-zero lower triangle $[t_i-t_j]_{2\leqslant i\leqslant n, 1\leqslant j\leqslant i-1}$, it is also numerically stable ([8], p. 227-231). Note that the equality

$$\theta(t, t_n)(t-t_n)^m = 0 \quad \text{for } t \in T$$

implies that the coefficient β_n is unessential in the formula for $f_\Delta(t)$ and, therefore, it need not be determined.

All calculations were performed in single precision with 37-bit floating-point mantissa only for the case $m = 3$ using equidistant nodes $t_k = t_{k-1}+h$. The coefficients a_i and β_j in (1.5) were calculated as described above. In these calculations the exact values of the derivatives $g'(t_1)$ and $g'(t_n)$ were used. In Table 1 we confront the relative errors $(f(s)-f_\Delta(s))/f(s)$ for two different values of h at five points of the interval $\langle 0, 10\rangle$.

TABLE 1. Relative errors $(f(s)-f_\Delta(s))/f(s)$

s	Example (3.1)		Example (3.2)		Example (3.3)	
	$h = 0.05$	$h = 0.02$	$h = 0.05$	$h = 0.02$	$h = 0.05$	$h = 0.02$
2.0	$1.1_{10}-7$	$1.9_{10}-8$	$5.3_{10}-7$	$8.5_{10}-8$	$1.5_{10}-6$	$2.3_{10}-7$
4.0	$5.7_{10}-8$	$-4.8_{10}-10$	$2.7_{10}-7$	$-1.7_{10}-9$	$7.1_{10}-7$	$-3.9_{10}-9$
6.0	$-1.6_{10}-9$	$-7.9_{10}-10$	$-6.6_{10}-9$	$-1.1_{10}-9$	$-1.8_{10}-8$	$-1.7_{10}-9$
8.0	$-7.9_{10}-10$	$-5.7_{10}-10$	$-2.9_{10}-9$	$-9.4_{10}-10$	$-7.2_{10}-9$	$-1.4_{10}-9$
10.0	$-7.3_{10}-10$	$-6.5_{10}-10$	$-1.7_{10}-9$	$-9.7_{10}-10$	$-4.0_{10}-9$	$-8.1_{10}-10$

References

[1] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The theory of splines and their applications*, Academic Press, New York 1967.
[2] P. M. Anselone and P. J. Laurent, *A general method for the construction of interpolating or smoothing spline-functions*, Num. Math. 12 (1968), p. 66-82.
[3] K. E. Atkinson, *The numerical solution of an Abel integral equation by a product trapezoidal method*, SIAM J. Numer. Anal. 11 (1974), p. 97-101.
[4] M. Böcher, *An introduction to the study of integral equations*, 2nd ed., Cambridge University Press, London 1914.
[5] C. de Boor, *Best approximation properties of spline functions of odd degree*, J. Math. Mech. 12 (1963), p. 747-750.
[6] H. Brunner, *The numerical solution of a class of Abel integral equations by piecewise polynomials*, J. Comp. Physics 12 (1973), p. 412-416.
[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York 1965.
[8] J. H. Wilkinson, *The algebraic eigenvalue problem*, Russian ed., Moscow 1970.

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PEWNA NUMERYCZNA METODA ROZWIĄZYWANIA RÓWNANIA CAŁKOWEGO ABELA

STRESZCZENIE

W pracy zostało podane analityczne rozwiązanie $f_\Delta(s)$ równania całkowego Abela (1.3), gdzie $g_\Delta(t)$ jest funkcją sklejaną o węzłach w siatce $\Delta = \{t_1, t_2, \ldots, t_n\}$. Opierając się na nim skonstruowano metodę przybliżonego rozwiązywania równania całkowego Abela (1.1) oraz zilustrowano ją trzema przykładami numerycznymi.
