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IMMEDIATE SERVICE IN A BENEŠ-TYPE $G/G/1$ QUEUEING SYSTEM

1. Introduction. In this paper we consider a class of single-server systems in which consecutive inter-arrival times and also service times are not necessarily independent and identically distributed. Such systems were investigated by Beneš [1] (wirtual waiting time $W(t)$) and Borovkov [2] (actual waiting time W_n of the n-th arriving customer). For the case of the Poisson input and general independent not necessarily indentically distributed service times, Szczotka [6] has found the generating function of the number of units in the system and the stability conditions.

Beneš [1] has given conditions for the existence of the limit

$$\lim_{t\to\infty} \mathrm{P}\{W(t) = 0 \mid W(0) = 0\}$$

and also the value of it. Marcerian and Tauberian methods were used to investigate this limit. Results of Beneš are not sufficiently effective to be used in queueing practice. He has described the queueing systems in terms of total work load $K(t)$ offered up to time t and used an assumption of Tauberian or Marcerian type concerning the functions

$$F(t) = \mathrm{E}\max\big(0, t-K(t)\big), \quad R(t) = \mathrm{P}\{K(t) \leqslant t\},$$
$$P(t) = \mathrm{P}\{W(t) = 0\}$$

and their Fourier transforms.

In this paper we give the local description of the considered queueing systems, i.e. in the independence case in terms of the distributions of service times. Stability conditions are also given. Furthermore, we give conditions for the existence of the limit of the conditional probability $\mathrm{P}\{W(t) \leqslant x \mid W(0) = 0\}$ for t tending to infinity and we give the value of it.

2. Local description of the queueing system. Suppose that

$$\Gamma = \{\tau_j^i;\ i, j \geqslant 1\} \quad \text{and} \quad S = \{s_j^i;\ i, j \geqslant 1\}$$

are infinite matrices of non-negative random variables defined on the probability space $\langle \Omega, \mathscr{F}, \mathrm{P} \rangle$.

Let τ_0 be a proper, non-negative random variable independent of s_j^i and τ_j^i, $i, j \geqslant 1$.

Put

$$\xi_j^i = s_j^i - \tau_j^i, \quad i, j \geqslant 1,$$

$$\xi^i = \{\xi_j^i;\ j \geqslant 1\}, \quad r_i = \min\Big\{n\colon \sum_{j=1}^{n} \xi_j^i \leqslant 0\Big\}, \quad i \geqslant 1.$$

Let us write the following conditions:

a_1. the sequences ξ^i are independent and identically distributed;

a_2. $\mathrm{P}\{\xi_1^i > 0\} > 0$;

a_3. $\mathrm{E}\, r_1 < \infty$;

a_4. $\mathrm{E}(s_k^i \mid r_i \geqslant k) = \mathrm{E}\, s_k^i$ and $\mathrm{E}(\tau_k^i \mid r_i \geqslant k) = \mathrm{E}\, \tau_k^i$, where $\mathrm{E}(\cdot \mid r_i \geqslant k)$ denotes the conditional expectation of $\cdot$ in relation to event $\{r_i \geqslant k\}$.

Let us define the following sequences of random variables:

$$\eta(n) = \max\Big\{k\colon \sum_{i=1}^{k} r_i < n\Big\},$$

$$R_0 = 0, \quad R_i = R_{i-1} + r_i, \quad i \geqslant 1,$$

$$\gamma_n = n - R_{\eta(n)},$$

$$s_n = s_{\gamma_n}^{\eta(n)+1}, \quad \tau_n = \tau_{\gamma_n}^{\eta(n)+1}, \quad n \geqslant 1.$$

The sequence $\{s_{j+1}, \tau_j, j \geqslant 0\}$ generates the single-server queueing system in which s_j is the service time of the j-th unit, and τ_j is the inter--arrival time between the $(j-1)$-st and j-th units.

We assume that conditions a_1-a_3 are satisfied. Then τ_j^i is the time between the arrivals of the j-th and $(j+1)$-st units during the i-th busy period, s_j^i is the service time of the j-th unit during the i-th busy period, and r_i is the number of services during the i-th period.

Condition a_1 implies that the random variables r_i are independent and identically distributed. Condition a_2 implies that the expected value of the busy period is positive. Condition a_4 holds if all random variables s_j^i and τ_j^i, $j \geqslant 1$, are independent.

According to Chow and Robbins [4], we give two groups of conditions for which a_3 holds.

a. *Independent case.* Here we assume that ξ_j^i, $j \geqslant 1$, are independent random variables satisfying

(i) $$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{E}\xi_j^1 = c, \quad -\infty < c < 0,$$

(ii) $$\lim_{n\to\infty} \int\limits_{\{E\xi_n^1-\xi_n^1>n\varepsilon\}} (E\xi_n^1-\xi_n^1) = 0 \quad \text{for every } \varepsilon>0,$$

(iii) $$\int\limits_{\{E\xi_n^1-\xi_n^1<0\}} (E\xi_n^1-\xi_n^1) \geqslant -k > -\infty.$$

Sometimes it is advantageous in practice to express these conditions by moments. One can see that condition (i) and the condition

$$E|\xi_n^1-E\xi_n^1|^a < k \quad \text{for some } a>1$$

imply (ii)-(iii).

b. *Dependent case.* Here we assume that the dependence of ξ_j^i is expressed by the formula

$$E(\xi_j^i \mid \mathscr{F}_{j-1}) = E\xi_j^i,$$

where $E(\cdot\,|\,\mathscr{F}_j)$ denotes the conditional expectation of $\cdot$ in relation to $\mathscr{F}_j$, and $\mathscr{F}_j$ is the σ-field generated by the random variables $(\xi_1, \xi_2, \ldots, \xi_j)$.

Then a sufficient condition for a_3 is condition (i) and

(ii′) $$E(|\xi_n^1-E\xi_n^1|^a \mid \mathscr{F}_{n-1}) \leqslant k \quad \text{for some } a>1.$$

Let $W(t)$ be the virtual waiting time in the considering queueing system, and let

$$T_0 = \tau_0, \quad T_k = T_{k-1}+\tau_k, \quad k\geqslant 1,$$
$$r_0 = 0, \quad t_0 = T_0, \quad t_k = T_{r_k} - T_{r_{k-1}}, \quad k\geqslant 1.$$

Under conditions a_1-a_3 the sequence of random variables $\{t_k, k\geqslant 0\}$ forms a general renewal process. Denote by F the distribution function of the random variable t_k for $k\geqslant 1$.

THEOREM 1. *If conditions a_1-a_3 hold, the distribution function F is aperiodic and*

$$E\sum_{j=1}^{r_1} \tau_j^1 < \infty,$$

then, for any number $x\geqslant 0$, there exists the limit

$$\lim_{t\to\infty} P\{W(t)\leqslant x \mid W(0) = 0\}.$$

Remark. One can see that condition a_4 and the boundedness of the sequence

$$\left\{\frac{1}{n}\sum_{i=1}^{n} E\,\tau_j^1\right\}$$

imply

$$\mathrm{E}\sum_{j=1}^{r_1}\tau_j^1 < \infty.$$

This follows from the equalities

$$\mathrm{E}\sum_{j=1}^{r_1}\tau_j^1 = \sum_{j=1}^{\infty}\mathrm{E}(\tau_j^1 \mid r_1 \geqslant j)\mathrm{P}\{r_1 \geqslant j\} = \sum_{i=1}^{\infty}\mathrm{P}\{r_1 = i\}i\cdot\frac{1}{i}\sum_{j=1}^{i}\mathrm{E}\,\tau_j^1.$$

Since $\mathrm{E}\tau_j^1 \geqslant 0$, we can change the order of summation in the last expression.

Proof. We use the results of Smith [5] to prove this theorem. Notice that $\{T_{r_k}, k \geqslant 0\}$ is the sequence of regeneration points of the process $W(t)$. Let x be a fixed and non-negative number. We define the functions

$$\Phi(t-u) = \mathrm{P}\{W(t) \leqslant x \mid W(0) = 0,\ T_{n_t} = u,\ n_t > 0\}$$

and

$$\Psi(t) = \big(1-F(t)\big)\Phi(t),$$

where n_t is the greatest number k such that $T_{r_1+r_2+\ldots+r_k} < t$. We prove that the function $\Psi(t)$ is of bounded variation in every finite interval $[a, b)$.

Let $S = \{t\colon W(t) \leqslant x\}$, $\mathrm{Fr}(S)$ denote the frontier of S, $I = [a+T_0, b+T_0)$, and $L(I) = n_{b+T_0} - n_{a+T_0}$ be the number of regeneration points of the process $W(t)$ in the interval I.

From condition a_2 we know that the expected value of the length of the renewal period is positive. Hence $\mathrm{E}\,L(I) < \infty$.

Let $y_J = \mathrm{card}\{J \cap \mathrm{Fr}(S)\}$, where J is an interval, and let $\tilde{y}_i = y_{[T_{r_{i-1}}, T_{r_i})}$, where $\tilde{y}_i$ is the number of times the process attains or crosses level x in the i-th busy period. Notice that $\tilde{y}_i \leqslant r_i + 2$. Hence $\mathrm{E}\,\tilde{y}_i < \infty$. In the above-given notation we have

$$y_I \leqslant \sum_{i=n_{a+T_0}}^{L(I)+1} \tilde{y}_i + 2. \tag{1}$$

Notice that the random variables $\tilde{y}_i$, $i \geqslant 1$, are independent, identically distributed and

$$\mathrm{E}\big(\tilde{y}_{n_{a+T_0}+i} \mid L(I) \geqslant i-1\big) = \mathrm{E}\,\tilde{y}_{n_{a+T_0}+i}. \tag{2}$$

From (1) and (2) we obtain

$$\begin{aligned}\mathrm{E}\big(y_I \mid T_0,\ W(0) = 0\big) &\leqslant \mathrm{E}\big(\big(L(I)+1\big) \mid T_0,\ W(0) = 0\big)\mathrm{E}\big(\tilde{y}_i + 2 \mid T_0, W(0) = 0\big)\\ &\leqslant \mathrm{E}\,\big(L(I)+1\big)\mathrm{E}\,(\tilde{y}_i+2) < \infty.\end{aligned}$$

y_J is the non-negative random function of subintervals J of I satisfying

$$y_{I_1}+y_{I_2} = y_{I_1 \cup I_2} \quad \text{for} \quad I_1, I_2 \subset I \text{ and } I_1 \cap I_2 = \emptyset.$$

The random function y_I satisfies the conditions of Smith's lemma ([5], Lemma 2). Hence the function $\Psi(t)$ is of bounded variation in every finite interval. From [5] (formula 3.4.3, p. 15) we have

$$\mathrm{P}\{W(t) \leqslant x \mid W(0) = 0\} = \mathrm{P}\{W(t) \leqslant x,\ T_0 > t \mid W(0) = 0\} + \\ + \int_0^t \Phi(t-u)\big(1-F(t-u)\big)\,dH(u),$$

where $H(u)$ is the renewal function of the general renewal process $\{T_{r_k},\ k \geqslant 0\}$.

Using Smith's theorem ([5], theorem 2, p. 14), we obtain theorem 1. Observe that theorem 1 can be obtained immediately from Smith's theorem if the distributed function of s_1^i is absolutely continuous.

THEOREM 2. *If the assumptions of theorem* 1 *are satisfied, then*

$$\lim \mathrm{P}\{W(t) = 0 \mid W(0) = 0\} = 1-s/\tau, \tag{3}$$

where

$$s = \mathrm{E}\sum_{i=1}^{r_1} s_i \quad \textit{and} \quad \tau = \mathrm{E}\sum_{i=1}^{r_1} \tau_i.$$

Proof. To prove this theorem we use the results of Beneš [1]. Let $n(t)$ be the greatest number k such that $T_k < t$. Put

$$K(t) = \sum_{i=1}^{n(t)} s_i, \quad K(0) = 0.$$

The process $K(t)$ satisfies the requirements of [1]. Beneš has shown ([1], formula 17, p. 42) that

$$0 = \int_0 \mathrm{P}\{K(t) \leqslant u\}\,du - \int_0^t \mathrm{P}\{K(t)-K(u)-t+u \leqslant 0,\ W(u) = 0\}\,du.$$

From [1] (formula 16, p. 15) we have

$$0 = \mathrm{E}\max\big(0, t-K(t)\big) - \int_0^t \mathrm{P}\{W(u) = 0\}\,du + \\ + \int_0^t \mathrm{P}\{K(t)-K(u)-t+u > 0 \mid W(u) = 0\}\,du, \tag{4}$$

$$\mathrm{E}\max\big(0, t-K(t)\big) = \mathrm{E}\big(t-K(t)\big) - \mathrm{E}\big(t-K(t)\big)\chi_{\Omega_t},$$

where χ_{Ω_t} is the indicator of the set Ω_t, $\Omega_t = \{t-K(t) < 0\}$. Write

$$x_k = s_{r_{k-1}+1}+s_{r_{k-1}+2}+\ldots+s_{r_k}, \quad k \geqslant 1.$$

The random variables x_k are non-negative, independent and identically distributed. The following inequalities are satisfied:

$$\sum_{k=1}^{n_t} x_k \leqslant \sum_{k=1}^{n(t)} s_k \leqslant \sum_{k=1}^{n_t+1} x_k.$$

From the ergodic theorem ([5], theorem 7, p. 27) we have

$$\frac{1}{t}\sum_{k=1}^{n_t} x_k \to \frac{s}{\tau} < 1 \quad \text{as } t\to\infty,$$

where $s = \mathrm{E}\,x_1$, and $\tau = \mathrm{E}\,t_1$. Hence

$$t-\sum_{k=1}^{n_t} x_k \to \infty \quad \text{a.e.,}$$

$$t-\sum_{k=1}^{n(t)} s_k \to \infty \quad \text{a.e.}$$

Therefrom, $P(\Omega_t)\to 0$ as $t\to\infty$. By [5] (theorem 8, p. 28), we have

$$\mathrm{E}\left(1-\frac{1}{t}K(t)\right) = 1-\frac{\mathrm{E}\,x_1}{\mathrm{E}\,t_1}+o(1) \quad \text{as } t\to\infty.$$

Notice that the last expression in (4) is $o(t)$. Dividing (4) by t and increasing t to infinity, we have

$$\lim_{t\to\infty}\mathrm{P}\{W(t) = 0 \mid W(0) = 0\} = 1-\frac{\mathrm{E}\,x_1}{\mathrm{E}\,t_1}.$$

In the general case, theorem 2 does not give a simple method of the effective calculation of limit (3). This limit can be estimated basing on the distributions of the sequences $\{s_j^1, \tau_j^1, j \geqslant 1\}$. Let us consider two particular cases.

COROLLARY 1. *If the assumptions of theorem* 2 *and* a_4 *hold and if there exists a number* k *such that* $\mathrm{P}\{r_1 < k-1\} = 0$ *and also if* $\mathrm{E}\,s_i^1 = \mathrm{E}\,s_k^1$ *and* $\mathrm{E}\,\tau_i^1 = \mathrm{E}\,\tau_k^1$ *for* $i \geqslant k$, *then*

$$\lim_{t\to\infty}\mathrm{P}\{W(t) = 0 \mid W(0) = 0\} = 1-\frac{\mathrm{E}\,s_k^1}{\mathrm{E}\,\tau_k^1}.$$

COROLLARY 2. *If the assumptions of theorem* 2 *and* a_4 *hold and if there exists a number* μ *such that* $\mu\mathrm{E}\,s_k^1 = \mathrm{E}\,\tau_k^1$, *then*

$$\lim_{t\to\infty}\mathrm{P}\{W(t) = 0 \mid W(0) = 0\} = 1-1/\mu.$$

Notice that $\mu > 1$. This follows from a theorem of Kolmogorov and Prokhorov [3], from the inequality

$$\sum_{i=1}^{r_1} s_i^1 \leqslant \sum_{i=1}^{r_1} \tau_i^1$$

and from condition a_4. In fact, we have

$$\sum_{k=1}^{\infty} \mathrm{P}\{r_1 = k\} \sum_{i=1}^{k} \mathrm{E}\,\tau_i^1 = \mu \sum_{k=1}^{\infty} \mathrm{P}\{r_1 = k\} \sum_{i=1}^{k} \mathrm{E}\,s_i^1.$$

Since

$$\mathrm{E} \sum_{k=1}^{r_1} \tau_k < \infty \quad \text{and} \quad \mathrm{E} \sum_{k=1}^{r_1} \tau_k \geqslant \mathrm{E} \sum_{k=1}^{r_1} s_k,$$

we obtain $\mu > 1$.

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PRAWDOPODOBIEŃSTWO OBSŁUGI BEZ CZEKANIA W BENEŠOWSKICH SYSTEMACH TYPU $G/G/1$

STRESZCZENIE

W pracy rozważamy klasę systemów masowej obsługi, w których kolejne odstępy między zgłoszeniami, a także czasy obsługi zgłoszeń, nie muszą mieć tych samych rozkładów. Podaliśmy warunki stabilności systemu, warunki istnienia granicy $\mathrm{P}\{W(t) \leqslant x \mid W(0) = 0\}$ dla t dążącego do nieskończoności, gdzie $W(t)$ jest wirtualnym czasem czekania, oraz znaleźliśmy tę granicę w przypadku $x = 0$.