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OPTIMAL STOPPING OF A SEQUENCE OF MAXIMA BETWEEN RANDOM AND FIXED BARRIERS

0. Introduction. Let $X_1, X_2, \ldots, Y_1^k, Y_2^k, \ldots$ $(k = 1, 2, \ldots, K)$ be independent copies of a known continuous distributed random variable. Suppose that we observe the realization of the sequence $\xi_n = \max(X_1, \ldots, X_n)$ only. We want to stop the observation at the moment at which ξ_n exceeds the maximum of unobservable sequences and it does not exceed a fixed constant with maximal probability. The infinite and finite fixed lengths of observation are considered. In Section 1 the precise formulation is given. The above problems can be led to the classical optimal stopping problem for some Markov chain (see, e.g., [3]). The way of this reduction is presented in Section 2. Sections 3 and 4 contain the solutions of these problems. The optimal gains are found. In the finite case, the optimal stopping time and, in the infinite case, the method of calculation of the ε-optimal stopping time are obtained.

Maximizing the probability that at the moment of stopping the given event is attained in the so-called "secretary problem" and "disorder problem" was investigated (cf. [1]–[3]). The problem of optimal stopping the two sequences of maxima when only one is observed was considered by Szajowski in [4]. This paper generalizes a part of Szajowski's results.

1. Formulation of problems. Let K be a fixed positive integer number. Assume that

(1) $X_1, X_2, \ldots, Y_1^k, Y_2^k, \ldots$ $(k = 1, 2, \ldots, K)$ are independent identically distributed random variables with a continuous distribution function F, defined on the probability space $(\Omega, \mathscr{F}, P)$.

Let

$$\xi_n = \max(X_1, \ldots, X_n), \quad \eta_n = \max(Y_1^1, \ldots, Y_n^1, \ldots, Y_1^K, \ldots, Y_n^K),$$
$$\underset{\bullet}{a} = \sup\{x \in \boldsymbol{R}\colon F(x) < A\}, \tag{2}$$

where A is a given number in the interval $(0, 1]$, and $\boldsymbol{R}$ is the set of real numbers. Let $\mathscr{F}_n$ be the σ-field of events generated by $\xi_1, \ldots, \xi_n$, and $\mathscr{T}$ be the set of all Markov moments with respect to the family $(\mathscr{F}_n)_{n=1}^{\infty}$.

Consider the following two problems (P_N) and (P):

(P_N) Find a stopping time $\tau^* \in \mathscr{T}$ such that

$$P(\tau^* \leqslant N,\, a \geqslant \xi_{\tau^*} > \eta_{\tau^*}) = \sup_{\tau \in \mathscr{T}} P(\tau \leqslant N,\, a \geqslant \xi_\tau > \eta_\tau),$$

where N is a fixed positive integer number, and under the assumption that

(3) $$A \neq 1$$

(P) Find a stopping time $\tau^* \in \mathscr{T}$ such that

$$P(\tau^* < \infty,\, a \geqslant \xi_{\tau^*} > \eta_{\tau^*}) = \sup_{\tau \in \mathscr{T}} P(\tau < \infty,\, a \geqslant \xi_\tau > \eta_\tau).$$

2. Reduction of the problems. In this section, for convenience we put $N^* = N$ for the problem (P_N) and $N^* = \infty$ for (P). Write

(4) $$Z_n = P(a \geqslant \xi_n > \eta_n \mid \mathscr{F}_n), \qquad n = 1, 2, \ldots, N^*,$$

for the both problems. From (3) we have $Z_\infty = 0$ a.s. (almost surely) for the problem (P). Thus

$$P(a \geqslant \xi_\tau > \eta_\tau) = \mathrm{E}(Z_\tau) \qquad \text{for } \tau \in \mathscr{T}$$

such that $P(\tau < N^*+1) = 1$ (E denotes the expectation with respect to the distribution P).

Since F is a continuous function, $F(X_n)$ is uniformly distributed on $[0, 1]$ and

$$P\big(X_n > Y_n^k \Leftrightarrow F(X_n) > F(Y_n^k)\big) = 1.$$

Therefore, without loss of generality we may additionally assume

(5) the random variables X_n and Y_n^k ($n = 1, 2, \ldots;\ k = 1, \ldots, K$) are uniformly distributed on $[0, 1]$.

Using (5) we can write (4) in the form

$$Z_n = \begin{cases} \xi_n^{Kn} & \text{if } \xi_n \leqslant a, \\ 0 & \text{if } \xi_n > a. \end{cases}$$

The sequence $\xi = (\xi_n)_{n=0}^{N^*}$ is the Markov chain with respect to $(\mathscr{F}_n)_{n=0}^{N^*}$ (we assume $\xi_0 = 0$ a.s. and $\mathscr{F}_0 = \{\emptyset, \Omega\}$) with the state space $[0, 1]$ and the transition function

(6) $$P_x(B) = P(\xi_{n+1} \in B \mid \xi_n = x) = \begin{cases} x + |B \cap (x, 1]| & \text{if } x \in B, \\ |B \cap (x, 1]| & \text{if } x \notin B \end{cases}$$

for $n = 0, 1, \ldots, N^*-1$, $x \in [0, 1]$, B being a Borel subset of the interval $[0, 1]$, where $|\cdot|$ stands for the Lebesgue measure.

Thus we reduce both initial problems to the optimal stopping of the

Markov chain $\xi = (\xi_n)_{n=0}^{N^*}$ with the reward function f, where

$$f(n, x) = \begin{cases} x^{Kn} & \text{for } x \in [0, a], \\ 0 & \text{for } x \in (a, 1] \end{cases} \tag{7}$$

for $n = 0, 1, \ldots, N^*$. The problem of the optimal stopping of the Markov chain ξ with the reward function $f(n, \cdot)$ consists in calculating

$$v(x) \stackrel{\text{df}}{=} \sup_{\tau \in \mathcal{T}} \mathrm{E}_x f(\tau, \xi_\tau)$$

(E_x denotes the expectation with respect to the distribution P_x, x is the initial state) and in finding a stopping time $\tau^* \in \mathcal{T}$ such that

$$\mathrm{E}_x f(\tau^*, \xi_{\tau^*}) = v(x)$$

(cf. [3]). Put

$$\mathcal{T}_n^{N^*} = \{\tau \in \mathcal{T}\colon n \leqslant \tau \leqslant N^*\},$$

$$v(n, x) = \sup_{\tau \in \mathcal{T}_n^{N^*}} \mathrm{E}_x f(\tau, \xi_\tau) \quad \text{if } \xi_n = x, \tag{8}$$

$$Tf(n, x) = \mathrm{E}_x f(n+1, \xi_1).$$

For solving the problems, first of all we show the following

LEMMA 1. *The equation $Tf(n, x) = f(n, x)$ has in the interval* $(0, a)$ *the unique solution x_n^0 for $n = 1, 2, \ldots, N^*-1$ such that*

$$x_n^0 < x_{n+1}^0 \quad \textit{and} \quad Tf(n, x) \leqslant f(n, x) \textit{ for } x \geqslant x_n^0.$$

Proof. Let $x \in [0, a]$ and $n < N^*$. From (6) we have

$$\mathrm{E}_x f(n+1, \xi_1) = x^{K(n+1)} x + \int_x^a y^{K(n+1)} dy$$
$$= \frac{K(n+1)}{K(n+1)+1} x^{K(n+1)+1} + \frac{a^{K(n+1)+1}}{K(n+1)+1}.$$

Define the function $g(n, x) = Tf(n, x) - f(n, x)$. It is continuous on $[0, a]$, has at most a unique extremum and $g(n, 0) > 0$, $g(n, a) < 0$. Therefore, the equation $g(n, x) = 0$ has the unique root x_n^0. On $[x_n^0, a]$ the function $g(n, x)$ is obviously non-positive. To prove the inequality $x_n^0 < x_{n+1}^0$ note that

$$g(n+1, x) = x^K g(n, x) + q(n, x),$$

where

$$q(n, x) \stackrel{\text{df}}{=} \frac{K}{[K(n+1)+1][K(n+2)+1]} x^{K(n+2)+1}$$
$$- \frac{a^{K(n+1)+1}}{K(n+1)+1} x^K + \frac{a^{K(n+2)+1}}{K(n+2)+1}.$$

Since the function $q(n, x)$ is positive on $[0, a)$, we have

$$g(n+1, x_n^0) = q(n, x_n^0) > 0.$$

This fact and the properties of the function $g(n+1, x)$ yield $x_{n+1}^0 > x_n^0$. The lemma is proved.

3. Solution of the problem (P_N). To solve the problem (P_N) we use the following

LEMMA 2 ([3], [4]). *Let* $\xi = (\xi_n)_{n=0}^N$ *be a homogeneous Markov chain with a state space* E *and let*

$$f\colon \{0, 1, \ldots, N\} \times E \to \boldsymbol{R}$$

be a non-negative bounded function. Then the function $v(n, x)$ *satisfies the equation*

$$v(n, x) = \max\{f(n, x), Tv(n, x)\},$$

and the stopping time

$$\tau_n^* = \min\{n \leqslant k \leqslant N\colon\ v(k, \xi_k) = f(k, \xi_k)\}$$

is optimal in $\mathcal{T}_n^N$. *The optimal gain is* $v(x) = v(0, x)$ *and the optimal stopping time is* τ_0^*.

Now we prove the following theorem giving the solution of (P_N).

THEOREM 1. *Under the assumptions* (1) *and* (5) *there exists a solution of the problem* (P_N) *which is of the form*

$$\tau_N^* = \min\{n \leqslant N\colon\ \xi_n \geqslant x_n\}, \tag{9}$$

where $x_N = 0$ *and* x_n $(n = 1, \ldots, N-1)$ *is the root of the equation*

$$\frac{KN}{KN+n} x^{KN+n} + \varphi_{N-n} - x^{K(N-n)} = 0 \tag{10}$$

in the interval $(0, a)$. *The constant* $\varphi_{N-n} = \varphi_{N-n}(x_{N-n+1})$ *can be obtained recursively by the formula*

$$\begin{aligned}\varphi_{N-n} = {} & \frac{KN}{(KN+n-1)(KN+n)} x_{N-n+1}^{KN+n} - \frac{1}{K(N-n+1)+1} x_{N-n+1}^{K(N-n+1)+1} \\ & + \frac{a^{K(N-n+1)+1}}{K(N-n+1)+1} + \varphi_{N-n+1}\, x_{N-n+1}\end{aligned} \tag{11}$$

for $n = 1, \ldots, N-1$ $(\varphi_N \overset{\mathrm{df}}{=} 0)$. *The optimal gain is*

$$v_N = \frac{1}{K+1}\left(a^{K+1} + Kx_1^{K+1} - Kx_1^{N(K+1)}\right). \tag{12}$$

Proof. It is clear that $v(n, x) = 0$ for $x > a$. Now we assume that $x \leqslant a$. Lemmas 1 and 2 together with (6) and (7) give

$$(13) \qquad v(N-1, x) = \begin{cases} x^{K(N-1)} & \text{for } x_{N-1} \leqslant x \leqslant a, \\ \dfrac{KN}{KN+1} x^{KN+1} + \dfrac{a^{KN+1}}{KN+1} & \text{for } x < x_{N-1}, \end{cases}$$

where $x_{N-1} = x^0_{N-1}$ (defined in Lemma 1). Notice that

$$\frac{a^{KN+1}}{KN+1} = \varphi_{N-1}.$$

Now we show that

$$(14) \qquad v(N-2, x) = \begin{cases} x^{K(N-2)} & \text{for } x_{N-2} \leqslant x \leqslant a, \\ \dfrac{KN}{KN+2} x^{KN+2} + \varphi_{N-2} & \text{for } x < x_{N-2}, \end{cases}$$

where φ_{N-2} is defined by (11), x_{N-2} is the solution of equation (10) for $n = 2$ and $0 < x_{N-2} < x_{N-1}$. Lemma 2 implies

$$v(N-2, x) = \max\{f(N-2, x), Tv(N-2, x)\}.$$

Using (6), (11) and (13) we obtain

$$Tv(N-2, x) = \begin{cases} \dfrac{K(N-1)}{K(N-1)+1} x^{K(N-1)+1} + \dfrac{a^{K(N-1)+1}}{K(N-1)+1} & \text{for } x_{N-1} \leqslant x \leqslant a, \\ \dfrac{KN}{KN+2} x^{KN+2} + \varphi_{N-2} & \text{for } x < x_{N-1}. \end{cases}$$

The function $h(N-2, x) \overset{\text{df}}{=} Tv(N-2, x) - f(N-2, x)$ is continuous on $(0, a)$ and it has the following properties:

(a) $h(N-2, x) < 0$ for $x_{N-1} \leqslant x \leqslant a$;

(b) $h(N-2, x) > g(N-2, x)$ for $x < x_{N-1}$;

(c) for $x < x_{N-1}$ it has at most a unique extremum.

The statement (a) is valid because $h(N-2, x) = g(N-2, x)$ for $x \in (x_{N-1}, a)$, the function $g(N-2, x)$ has on $(0, x_{N-1})$ the root x^0_{N-2}, $x^0_{N-2} < x^0_{N-1} = x_{N-1}$ and $g(N-2, x) < 0$ for $x > x^0_{N-2}$ (cf. Lemma 1). The statement (b) is a consequence of the fact that $Tv(N-2, x) > Tf(N-2, x)$ for $x < x_{N-1}$. From (a), (b) and (c) ((c) is easy to show) we conclude that the function $h(N-2, x)$ has the unique root $x_{N-2} \in (0, x_{N-1})$ and $x^0_{N-2} < x_{N-2} < x_{N-1}$. The form of $v(N-2, x)$ is shown.

Let us assume for induction that we have obtained for $n = 1, 2, \ldots, i-1$ the roots x_{N-n} of equation (10) having the property

$$(15) \qquad x_{N-1} > x_{N-2} > \ldots > x_{N-i+1} > x^0_{N-i+1}$$

and

$$(16)\qquad v(N-n, x) = \begin{cases} x^{K(N-n)} & \text{for } x_{N-n} \leqslant x \leqslant a, \\ \dfrac{KN}{KN+n} x^{KN+n} + \varphi_{N-n} & \text{for } x < x_{N-n}. \end{cases}$$

Now we determine x_{N-i} and $v(N-i, x)$. Using (11) we can write $Tv(N-i, x)$ in the form

$$Tv(N-i, x) = \begin{cases} \dfrac{K(N-i+1)}{K(N-i+1)+1} x^{K(N-i+1)+1} + \dfrac{a^{K(N-i+1)+1}}{K(N-i+1)+1} & \text{for } x_{N-i+1} \leqslant x \leqslant a, \\ \dfrac{KN}{KN+i} x^{KN+i} + \varphi_{N-i} & \text{for } x < x_{N-i+1}. \end{cases}$$

Taking into account the inequality $x_{N-i+1} > x^0_{N-i+1}$ we can show analogously as above that the function

$$h(N-i, x) \overset{\text{df}}{=} Tv(N-i, x) - f(N-i, x)$$

is continuous on $(0, a)$ and it has the following properties which can be proved similarly as for $i = 2$:

(a) $h(N-i, x) < 0$ for $x_{N-i+1} \leqslant x \leqslant a$;

(b) $h(N-i, x) > g(N-i, x)$ for $x < x_{N-i+1}$;

(c) for $x < x_{N-i}$ it has at most a unique extremum.

Hence the equality $h(N-i, x) = 0$ has the unique solution x_{N-i} on $(0, a)$ and $x^0_{N-i} < x_{N-i} < x_{N-i+1}$. This implies that (16) holds for $n = 1, \ldots, N-1$.

The inequalities $x_1 < x_2 < \ldots < x_{N-1}$ and the fact that the reward function (7) is increasing on $[0, a]$ imply that the stopping time (9) is optimal. Assuming $P(\xi_0 = 0) = 1$ we calculate the optimal gain:

$$(17)\qquad v_N = v_N(0, 0) = Tv(0, x)|_{x=0}$$

$$= \frac{1}{K+1}\left[\frac{K}{N(K+1)-1} x_1^{N(K+1)} - x_1^{K+1} + a^{K+1}\right] + x_1 \varphi_1.$$

From (16) for $n = N-1$ we have

$$\varphi_1 = x_1^K - \frac{KN}{N(K+1)-1} x_1^{N(K+1)-1}$$

and the optimal gain (17) leads to (12). The theorem is proved.

4. Solution of the problem (P). To solve the problem (P) we use the following lemma:

LEMMA 3 ([3], [4]). *Let $\xi = (\xi_n)_{n=0}^{\infty}$ be a homogeneous Markov chain with* *a state space E and let*

$$f\colon \{0, 1, 2, \ldots\} \times E \to \mathbf{R}$$

be a non-negative bounded function. Then the function $v(n, x)$ *given by* (8) *satisfies the equations*

$$v(n, x) = \max\{f(n, x), Tv(n, x)\},$$

$$v(n, x) = \lim_{m\to\infty} Q^m f(n, x),$$

where

(18) $$Qf(n, x) \stackrel{\mathrm{df}}{=} \max\{f(n, x), Tf(n, x)\},$$

and the stopping time

$$\tau^*_{n,\varepsilon} = \inf\{m \geqslant n\colon v(m, \xi_m) \leqslant f(m, \xi_m) + \varepsilon\}, \qquad \varepsilon > 0,$$

is ε*-optimal in* $\mathcal{T}_n \stackrel{\mathrm{df}}{=} \mathcal{T}_n^\infty$, *i.e.,*

$$v(n, x) \leqslant \mathrm{E}_x f(\tau^*_{n,\varepsilon}, \xi_{\tau^*_{n,\varepsilon}}) + \varepsilon.$$

If $P(\tau^*_{0,0} < \infty) = 1$, *then the optimal stopping time is* $\tau^*_{0,0}$ *and the optimal gain is* $v(x) = v(0, x)$.

From this lemma we have

THEOREM 2. *Under the assumptions* (1), (3) *and* (5) *there exists a solution of the problem* (P) *which is of the form*

(19) $$\tau^* = \inf\{n\colon \xi_n \geqslant y_n\},$$

where $(y_n)_{n=1}^\infty$ *is a non-decreasing sequence such that*

$$\lim_{n\to\infty} y_n = a$$

and its components fulfil the recursive equalities

(20) $$y_n^{Kn} = \frac{K(n+1)}{K(n+1)+1} y_{n+1}^{K(n+1)+1} + \frac{a^{K(n+1)+1}}{K(n+1)+1}.$$

The value of y_n *can be calculated as the limit*

$$y_n = \lim_{l\to\infty} z_n^l,$$

where z_n^l *is the unique root of the equation*

(21) $$\frac{K(n+l)}{K(n+l)+l}\left[x^{K(n+l)+l} - (z_{n+1}^{l-1})^{K(n+l)+l}\right]$$
$$+ \frac{K(n+1)}{K(n+1)+1}(z_{n+1}^{l-1})^{K(n+1)+1} + \frac{a^{K(n+1)+1}}{K(n+1)+1} - x^{Kn} = 0$$

in the interval $(0, z_{n+1}^{l-1})$ *for* $l = 1, 2, \ldots$ $(z_n^0 \stackrel{\text{df}}{=} a,\ n = 1, 2, \ldots)$. *The optimal gain is*

$$v = \frac{1}{K+1}(a^{K+1} + Ky_1^{K+1}). \tag{22}$$

Proof. We assume that $x \leqslant a$ because $v(n, x) = 0$ for $x > a$. From (6) and (7) we obtain

$$Tf(n, x) = \frac{K(n+1)}{K(n+1)+1} x^{K(n+1)+1} + \frac{a^{K(n+1)+1}}{K(n+1)+1}.$$

Using Lemma 1 we can write (18) in the form

$$Qf(n, x) = \begin{cases} x^{Kn} & \text{for } z_n^1 \leqslant x \leqslant a, \\ \dfrac{K(n+1)}{K(n+1)+1} x^{K(n+1)+1} + \dfrac{a^{K(n+1)+1}}{K(n+1)+1} & \text{for } x < z_n^1, \end{cases}$$

where $z_n^1 = x_n^0$ (defined in Lemma 1). Thus $z_n^1 < z_{n+1}^1$ and z_n^1 is the unique solution of equality (21) for $l = 1$ in $(0, a)$. Since from (18) we have

$$\begin{aligned} Q^m f(n, x) &= \max\{Q^{m-1} f(n, x),\ TQ^{m-1} f(n, x)\} \\ &= \max\{f(n, x),\ TQ^{m-1} f(n, x)\}, \end{aligned} \tag{23}$$

we consider on $[0, a]$ the function

$$g_m(n, x) \stackrel{\text{df}}{=} TQ^{m-1} f(n, x) - f(n, x).$$

Let $m = 2$. Then $g_2(n, x)$ is a continuous function and it has the following properties:

(a) if $x \geqslant z_{n+1}^1 > z_n^1$, then

$$g_2(n, x) = TQf(n, x) - f(n, x) = Tf(n, x) - f(n, x) = g_1(n, x) < 0,$$

because $g_1(n, x) = g(n, x)$ and $z_n^1 = x_n^0$ (defined in Lemma 1);

(b) $g_2(n, x) \geqslant g_1(n, x) \geqslant 0$ for $x \leqslant z_n^1$;

(c) for $x < z_{n+1}^1$ the function $g_2(n, x)$ has at most a unique extremum.

From (a), (b) and (c) we conclude that $g_2(n, x)$ has the unique root z_n^2 and $z_n^1 < z_n^2 < z_{n+1}^1$. Hence $z_{n+1}^1 < z_{n+1}^2$, and therefore $z_n^2 < z_{n+1}^2$. Moreover,

$$Q^2 f(n, x) = \begin{cases} x^{Kn} & \text{for } z_n^2 \leqslant x \leqslant a, \\ TQf(n, x) & \text{for } x < z_n^2. \end{cases}$$

The constant z_n^2 satisfies (21) because for $x < z_n^2 < z_{n+1}^1$ we have

$$\begin{aligned} TQf(n, x) = {} & \frac{K(n+2)}{K(n+2)+2}\left[x^{K(n+2)+2} - (z_{n+1}^1)^{K(n+2)+2}\right] \\ & + \frac{K(n+1)}{K(n+1)+1}(z_{n+1}^1)^{K(n+1)+1} + \frac{a^{K(n+1)+1}}{K(n+1)+1}. \end{aligned}$$

Let us assume for induction that we have obtained for $j = 1, 2, \ldots, l$ the root z_n^j of equation (21), $n = 1, 2, \ldots$, and

(24) $$z_n^1 < z_n^2 < \ldots < z_n^l, \quad z_n^j < z_{n+1}^j,$$

(25) $$Q^l f(n, x) = \begin{cases} x^{Kn} & \text{for } z_n^l \leqslant x \leqslant a, \\ \dfrac{K(n+l)}{K(n+l)+l}[x^{K(n+l)+l} - (z_{n+1}^{l-1})^{K(n+l)+l}] \\ \quad + \dfrac{K(n+1)}{K(n+1)+1}(z_{n+1}^{l-1})^{K(n+1)+1} + \dfrac{a^{K(n+1)+1}}{K(n+1)+1} & \text{for } x < z_n^l. \end{cases}$$

The function $g_{l+1}(n, x)$ has the following properties:

(a) if $x \geqslant z_{n+1}^l > z_n^l$, then

$$g_{l+1}(n, x) = Tf(n, x) - f(n, x) = g_l(n, x) < 0;$$

(b) $g_{l+1}(n, x) > g_l(n, x) \geqslant 0$ for $x \leqslant z_n^l$ because of (23) and

$$Q^{l+1} f(n, x) \geqslant Q^l f(n, x) > f(n, x) \quad \text{for } x < z_n^l;$$

(c) for $x < z_{n+1}^l$ the function $g_{l+1}(n, x)$ has at most a unique extremum.

From these properties the equality $g_{l+1}(n, x) = 0$ has the unique solution z_n^{l+1} and $z_n^l < z_n^{l+1} < z_{n+1}^l$, $n = 1, 2, \ldots$ Hence $z_n^{l+1} < z_{n+1}^{l+1}$ and the statement (25) holds for each $l = 1, 2, \ldots$

Since the sequence $(z_n^l)_{l=1}^{\infty}$ is increasing and bounded, the limit $\lim_{l\to\infty} z_n^l = y_n$ exists. Hence

$$v(n, x) = \lim_{l\to\infty} Q^l f(n, x)$$

and, consequently,

$$v(n, x) = \begin{cases} x^{Kn} & \text{for } y_n \leqslant x \leqslant a, \\ \dfrac{K(n+1)}{K(n+1)+1} y_{n+1}^{K(n+1)+1} + \dfrac{a^{K(n+1)+1}}{K(n+1)+1} & \text{for } x < y_n. \end{cases}$$

The function $v(n, x)$ is continuous at the point $x = y_n$, so y_n fulfils the recurrence relation (20). This yields

$$\left[\frac{a^{K(n+1)+1}}{K(n+1)+1}\right]^{1/Kn} < y_n < [a^{K(n+1)+1}]^{1/Kn}$$

and, obviously,

$$\lim_{n\to\infty} y_n = a.$$

Since the reward function (7) is increasing and, obviously, $y_n \leqslant y_{n+1}$, the optimal Markov moment takes the form (19). It is a stopping time because we assumed that $P(\xi_0 = 0) = 1$ and

$$P(\tau^* < \infty) = P\Big(\bigcup_{n=1}^{\infty} \{\xi_n \geqslant y_n\}\Big) \geqslant P\Big(\bigcup_{n=1}^{\infty} \{\xi_n \geqslant a\}\Big) = \lim_{n\to\infty} P(\xi_n \geqslant a) = 1.$$

It is easy to show that the optimal gain is given by (22). Thus Theorem 2 is proved.

5. Remark about a limit relation between (P_N) **and** (P). Theorem 1 gives another method of calculating the ε-optimal stopping rule for the problem (P). Theorems about convergence of the optimal gains and the optimal stopping times in $\mathscr{T}_0^N$ as N tends to $+\infty$ for the stopping in the classical case imply that the limit of the sequence of the optimal stopping times τ_N^* given by (9) as N tends to $+\infty$ is an optimal stopping time in (P) (cf. [3]).

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